Comparison of the product structures in algebraic and in topological $K$-theory

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Abstract

The compatibility up to sign of the product structures in algebraic $K$-theory and in topological $K$-theory of unital Banach algebras is established in total degree $\leq 2$. This answers a question posed by Milnor.

1 Statement of the theorem and definition of the product structures in $K$-theories

As an application of the computations made in [7], we prove the following result.

1.1 Theorem. Let $A$ and $B$ be two unital Banach algebras. Then the diagram

$$
\begin{array}{ccc}
K^\text{alg}_p(A) \otimes K^\text{alg}_q(B) & \xrightarrow{\phi_p \otimes \phi_q} & K^\text{alg}_{p+q}(A \otimes \mathbb{Z} B) \\
\downarrow & & \downarrow (-1)^{pq} \hat{\phi}_{p+q} \\
K_p(A) \otimes K_q(B) & \otimes & K_{p+q}(A \hat{\otimes} B)
\end{array}
$$

commutes for $p, q \geq 0$ satisfying $p + q \leq 2$. In other words, the external product structures in algebraic and in topological $K$-theory of unital Banach algebras are compatible in total degree $\leq 2$, up to the sign $(-1)^{pq}$. In particular, for commutative unital Banach algebras, the internal product structures are also compatible in the same range and up to the same sign.

Let us explain the notations. For a unital Banach algebras $A$ (always over $\mathbb{C}$), we denote by $GL(A)$ the infinite matrix group with the usual direct limit topology, by $E(A)$ the group of infinite elementary matrices, which coincides with the commutator subgroup $[GL(A), GL(A)]$ of $GL(A)$, and by $St(A)$ the infinite Steinberg group of $A$ with standard generators $(x_{ij}(a))_{i \neq j, a \in A}$. The algebraic and topological $K$-theory groups are defined by:

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• \( K_0^\text{alg}(A) = K_0(A) \) is the projective class group of the underlying ring \( A \);

• \( K_1^\text{alg}(A) := \text{GL}(A)^{ab} = \text{GL}(A)/E(A) \);

• \( K_1(A) := \pi_0(\text{GL}(A)) = \text{GL}(A)/\text{GL}(A)_0 \), where \( \text{GL}(A)_0 \) is the arc component of the identity in \( \text{GL}(A) \);

• \( K_2^\text{alg}(A) := \text{Ker}( \text{St}(A)^{\phi} E(A) ) \), where the map \( \text{St}(A)^{\phi} E(A) \) takes the standard generator \( x_{ij}(a) \) of \( \text{St}(A) \) to the elementary matrix \( e_{ij}(a) \);

• \( K_2(A) := \pi_1(\text{GL}(A)) \).

By Bott periodicity, we have, for any Banach algebra \( A \), \( K_2(A) \cong K_0(A) \). We now depict the canonical and natural maps \( \phi^A_{i,j} = \phi_i : K_{i+1}^\text{alg}(A) \longrightarrow K_i(A) \).

For \( i = 0 \), \( \phi^A_0 \) is merely the identity of \( K_0^\text{alg}(A) \), and the well-known inclusion \( E(A) \subseteq \text{GL}(A)_0 \) allows for defining the map \( \phi^A_1 \) taking, for \( u \in \text{GL}(A) \), the class \( [u] \) in \( K_1^\text{alg}(A) \) to the class \( [u] \) in \( K_1(A) \). Let us now describe \( \phi^A_2 \). Let \( \text{GL}(A)_0 \) be the universal covering space of the topological group \( \text{GL}(A) \). As usual, we see the group \( \text{GL}(A)_0 \) as the set of homotopy classes (rel. to \( \{0, 1\} \)) of paths in \( \text{GL}(A)_0 \) (parameterized by \( t \in [0, 1] \)) emanating from \( \mathbb{I} \), with pointwise multiplication, and the projection \( \text{GL}(A)_0 \to \text{GL}(A)_0 \) is given by evaluation at \( t = 1 \), and has its kernel equal to \( \pi_1(\text{GL}(A)_0) = \pi_1(\text{GL}(A)) = K_2(A) \).

Consider the map \( \text{St}(A) \longrightarrow \text{GL}(A)_0 \) defined on the standard generators of \( \text{St}(A) \) by

\[
\psi : x_{ij}(a) \longmapsto [t \mapsto e_{ij}(t \cdot a)],
\]

where \( a \in A \), \( t \) ranges over \([0, 1]\), and the above brackets designate a homotopy class. One can easily check that the images of the \( x_{ij}(a) \)'s satisfy all the defining relations of \( \text{St}(A) \), consequently, the map \( \psi \) is a well-defined homomorphism.

Now, the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_2^\text{alg}(A) & \longrightarrow & \text{St}(A) & \xrightarrow{\varphi} & E(A) & \longrightarrow & 0
\\
& & \phi^A_2 \downarrow & & \psi \downarrow & & \gamma \\
0 & \longrightarrow & K_2(A) & \longrightarrow & \text{GL}(A)_0 & \longrightarrow & \text{GL}(A)_0 & \longrightarrow & 0
\end{array}
\]

commutes. Therefore, by restriction, \( \psi \) induces a homomorphism \( \phi^A_2 \); explicitly,

\[
\phi^A_2 : K_2^\text{alg}(A) \longrightarrow K_2(A) = \pi_1(\text{GL}(A)_0)
\]

\[
\prod_s x_{ij,s}(a_s) \longmapsto \left[ e^{2\pi it} \mapsto \prod_s e_{ij,s}(t \cdot a_s) \right].
\]

1.2 Remark. Algebraic and topological \( K \)-groups in higher degree \( (p \geq 1) \) can be defined by

\[
K_p^\text{alg}(A) := \pi_p(\text{BGL}^\delta(A)^+) \quad \text{and} \quad K_p(A) := \pi_{p-1}(\text{GL}(A)) \cong \pi_p(\text{BGL}(A)),
\]

where \( \text{GL}^\delta(A) \) stands for \( \text{GL}(A) \) made discrete. (The definition of \( K_p^\text{alg} \) makes sense for any unital ring). The map \( B(Id) : \text{BGL}^\delta(A) \longrightarrow \text{BGL}(A) \) induces at the level of fundamental groups a map taking \( E(A) \subseteq \text{GL}^\delta(A) \) to zero, since
\[ \pi_1(\mathcal{G}(A)) = \pi_0(\mathcal{G}(A)) = \text{GL}(A)/\text{GL}(A)_0 \text{ and } E(R) \subseteq \text{GL}(A)_0. \] Consequently, \( B(Id) \) induces a map \( B(Id)^+ : \mathcal{G}(A)^+ \to \mathcal{G}(A) \). For any \( p \geq 1 \), this allows for defining a canonical and natural map

\[ \phi_p^A := \pi_p(B(Id)^+) : K_p^A(A) \to K_p(A). \]

These definitions extend functorially to the non-unital situation. One can check that for \( p = 1 \) and 2, all these definitions coincide with the ones given above.

For two rings \( A \) and \( B \) (not necessarily unital), the external product in algebraic \( K \)-theory (see [6]) is denoted by

\[ K_p^{alg}(A) \otimes K_q^{alg}(B) \to K_{p+q}^{alg}(A \otimes \mathbb{Z} B). \]

The internal product is defined for \( A \) commutative by composing the external product with the homomorphism \( K_p^{alg}(A \otimes \mathbb{Z} A) \to K_p^{alg}(A) \), induced by the product map \( \mu : A \otimes \mathbb{Z} A \to A \) (which is an ring homomorphism, precisely because \( A \) is commutative). It will be denoted by \( \star_A \) or by \( \star \). Note that this internal product is graded-commutative (see thm. 2.1.12 in [6]).

As noticed by Loday in [6], the internal product he defines at the level of the plus construction (and of spectra) coincides, in total degree \( p + q \leq 2 \), with the product defined case by case by Milnor \emph{only up to sign}. More precisely, both definitions coincide, except for \( p = q = 1 \), where Loday’s product is minus Milnor’s product (see prop. 2.2.3 in [6]): for \( x, y \in K_1^{alg}(A) \) with \( A \) commutative, the formula

\[ x \star_A y = -\{x, y\} \in K_2^{alg}(A) \]

holds, where \( \{x, y\} \) is the Steinberg symbol of \( x \) by \( y \).

For a Banach algebra \( A \) and for \( p \geq 1 \), the \( p \)-fold suspension of \( A \) is defined by \( \mathcal{S}^pA := S(S^{p-1}A) \cong C_0(\mathbb{R}^p) \otimes A \); note that it is not unital if so is \( A \). The \( p \)-fold suspension isomorphism is a natural isomorphism

\[ \sigma^p : K_p(A) \xrightarrow{\cong} K_0(\mathcal{S}^pA). \]

(We also write \( \mathcal{S}^0A := A \) and \( \sigma^0 := Id_{K_0(A)} \). Let \( A \hat{\otimes} B \) be the completed projective tensor product (over \( \mathbb{C} \)) of two Banach algebras \( A \) and \( B \). The equality of functors \( K_0^{alg} = K_0 \) and the suspension isomorphism uniquely define the external cross product

\[ K_p(A) \otimes K_q(B) \to K_{p+q}(A \hat{\otimes} B), \]

in topological \( K \)-theory, by requiring commutativity in the diagram

\[ \begin{array}{ccc}
K_p(A) \otimes K_q(B) & \xrightarrow{\times} & K_{p+q}(A \hat{\otimes} B) \\
\cong & \sigma^p \otimes \sigma^q & \sigma^{p+q} \cong \\
K_0(\mathcal{S}^pA) \otimes K_0(\mathcal{S}^qB) & \xrightarrow{\times} & K_0(\mathcal{S}^{p+q}(A \hat{\otimes} B))
\end{array} \]

with \( \nu : \mathcal{S}^pA \otimes \mathcal{S}^qB \to \mathcal{S}^{p+q}(A \hat{\otimes} B) \) (compare with II.3.26 in [5]). As in the algebraic case, the internal product \( ^\circ \cup \), called
cup product is defined for $A$ commutative by composing with the homomorphism $K_{p+q}(A \otimes A) \to K_{p+q}(A)$, induced by the "completed product map" $\hat{\mu} : A \hat{\otimes} A \to A$ (which is a Banach algebra morphism). Note that the cup product is graded-commutative (compare with propositions II.4.10 and II.5.27 in [5]). Finally, for $p \geq 0$, $\hat{\phi}_p$ denotes the composition

$$K_{p}^{alg}(A \otimes B) \to K_{p}^{alg}(A \otimes \mathbb{C} B) \to K_{p}^{alg}(A \hat{\otimes} B) \xrightarrow{\hat{\phi}_p} K_{p}^{alg}(A \hat{\otimes} B).$$

(Notice that $\nu_a$ in the above diagram is just $\hat{\phi}_0$.) This makes all the notations used in theorem 1.1 meaningful. Note that the statement amounts to the formula

$$\sigma^{p+q} \circ \hat{\phi}_{p+q}(x \ast y) = (-1)^{pq}(\sigma^p \circ \phi_p(x)) \times (\sigma^q \circ \phi_q(y)) \in K_0(S^{p+q}(A \hat{\otimes} B)),$$

for all $x \in K_{p}^{alg}(A)$ and $y \in K_{q}^{alg}(B)$.

Before stating an important corollary of theorem 1.1, for a compact Hausdorff space $X$, we let

$$\theta_a : K_a(C(X)) \overset{\cong}{\to} K^{-a}(X)$$

be the Swan-Serre isomorphism, where $C(X)$ is the commutative unital $C^*$-algebra of continuous complex valued functions on $X$, with the norm of uniform convergence.

1.3 Corollary. For a compact Hausdorff space $X$, the diagram

$$K_{p}^{alg}(C(X)) \otimes K_{q}^{alg}(C(X)) \xrightarrow{\phi_p \otimes \phi_q} K_{p+q}^{alg}(C(X)) \xrightarrow{(-1)^{pq}\hat{\phi}_{p+q}} K_{p+q}^{alg}(C(X))$$

$$\theta_p \otimes \theta_q \xrightarrow{= \theta_{p+q}} K^{-p}(X) \otimes K^{-q}(X) \xrightarrow{=} K^{-p-q}(X)$$

commutes, for $p, q \geq 0$ satisfying $p + q \leq 2$, where the bottom horizontal map is the usual cup product in $K$-theory.

Proof. The product $\mu : C(X) \otimes_{\mathbb{Z}} C(X) \to C(X)$ yields a commutative diagram

$$K_{p+q}^{alg}(C(X) \otimes_{\mathbb{Z}} C(X)) \xrightarrow{K_{p+q}^{alg}(\mu)} K_{p+q}^{alg}(C(X))$$

$$\xrightarrow{\hat{\phi}_{p+q}} K_{p+q}^{alg}(C(X) \hat{\otimes} C(X)) \xrightarrow{\phi_{p+q}} K_{p+q}^{alg}(C(X))$$

Consequently, commutativity of the upper square follows from theorem 1.1. The bottom square commutes, since the Swan-Serre isomorphism is a ring map. \square

1.4 Remark. i) Theorem 1.1 easily extends to the case of non-unital Banach algebras, and corollary 1.3 to the more general situation of Hausdorff locally compact spaces, using the commutative $C^*$-algebra $C_0(X)$.  

4
For the external cross product $K^{-p}(X) \otimes K^{-q}(Y) \xrightarrow{\cong} K^{-(p+q)}(X \times Y)$, the result corresponding to corollary 1.3 obviously holds (for Hausdorff locally compact spaces).

iii) Corollary 1.3 was an open question in Milnor’s book [8] (see p. 67).

For the proof of theorem 1.1, we can assume that $p \leq q$.

This paper is organized as follows. In section 2, we prove theorem 1.1 for $p = 0$. The most difficult case, namely $p = q = 1$, is dealt with in section 3, applying results of [7] (coping with the $C^*$-algebra $C^*\mathbb{Z}^2 \cong C(T^2)$).

2 The cases $p = 0$

By direct computation, we prove theorem 1.1 for $p = 0$.

Recall that the algebraic and the topological $K$-theory groups are Morita invariant: for $i \geq 0$ and $n \geq 1$, there are isomorphisms

$$K_i^a(A) \cong K_i^a(M_n(A))$$

and $K_i(A) \cong K_i(M_n(A))$, induced by the inclusion $A \hookrightarrow M_n(A)$, $a \mapsto \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right)$. In particular, the products being natural, they are compatible with Morita equivalence. We can therefore reduce to the case of idempotent $(1 \times 1)$-matrices and invertible $(1 \times 1)$-matrices. Let $x \in K_n^a(A)$ and $y \in K_q^a(B)$. We have to show that

$$\sigma^q \circ \hat{\phi}_q(x \ast y) = x \times (\sigma^q \circ \hat{\phi}_q(y)) \in K_0^a(\mathbb{Z} \hat{\otimes} B).$$

Let $x$ be the class of an idempotent $\varepsilon \in A$. For $q = 0$, there is nothing to prove. For $q = 1$, suppose that $y$ is the class of an invertible element $u \in B$. By definition of the $\ast$-product (see [8]), one has

$$x \ast y = [\varepsilon \otimes u + (1 - \varepsilon) \otimes 1] \in K_1^a(A \hat{\otimes} B).$$

(The inverse of this matrix is $\varepsilon \otimes u^{-1} + (1 - \varepsilon) \otimes 1$.) The suspension isomorphism is given by

$$\sigma = \sigma^1 : K_1(A) \xrightarrow{\cong} K_0(\mathbb{Z} \hat{\otimes} A), \quad [v] \mapsto [t \mapsto R_t \cdot P \cdot R_t^{-1} - [P]],$$

where $v \in GL_n(A)$, $P := \text{Diag}(\mathbb{1}_n, 0_n)$, and $R_t = R_t(v)$ is a homotopy (i.e. a path) in $GL_n(A)$ from $\mathbb{1}_n$ to the matrix $\text{Diag}(v, v^{-1})$ which, by the Whitehead lemma, belongs to the arc component of $\mathbb{1}_n$ in $GL_n(A)$. The suspension isomorphism is independent of the chosen homotopy. If $R_t$ is a path from $\mathbb{1}_2$ to $\text{Diag}(u, u^{-1})$, then $S_t := \varepsilon \otimes R_t + (1 - \varepsilon) \otimes \mathbb{1}_2$ (tensor product of matrices) is a path from $1 \hat{\otimes} \mathbb{1}_2 = \mathbb{1}_2$ to $\text{Diag}(\varepsilon \otimes u + (1 - \varepsilon) \otimes 1, \varepsilon \otimes u^{-1} + (1 - \varepsilon) \otimes 1)$, so that

$$\sigma \circ \hat{\phi}_1(x \ast y) = [t \mapsto S_t \cdot Q \cdot S_t^{-1}] - [Q],$$

with $Q := \text{Diag}(1 \hat{\otimes} 1, 0 \hat{\otimes} 0)$. On the other hand, letting $P := \text{Diag}(1, 0)$,

$$x \times (\sigma \circ \hat{\phi}_1(y)) = [t \mapsto \varepsilon \hat{\otimes}(R_t \cdot P \cdot R_t^{-1})] - [\varepsilon \hat{\otimes} P] = [t \mapsto \varepsilon \hat{\otimes}(R_t \cdot P \cdot R_t^{-1}) + (1 - \varepsilon) \hat{\otimes} P] - [Q].$$
holds. Now, observe that the matrices \( S_t \cdot Q \cdot S_t^{-1} \) and \( \varepsilon \otimes (R_t \cdot P \cdot R_t^{-1}) + (1 - \varepsilon) \otimes P \) are equal (and not just equivalent). This proves theorem 1.1 for \( p = 0 \) and \( q = 1 \).

2.1 Remark. We deduce from this computation that

\[
\times : K_0(A) \otimes K_1(B) \longrightarrow K_1(A \otimes B), \quad [e] \otimes [u] \longmapsto [\varepsilon \otimes u + (I_n - \varepsilon) \otimes I_n],
\]

provided that \( \varepsilon = \varepsilon^2 \in M_m(A) \) and \( u \in \text{GL}_n(B) \).

Now, let us prove theorem 1.1 for \( p = 0 \) and \( q = 2 \). Let \( x \in K_0^\text{alg}(A) \); using Morita invariance, we can assume that \( x \) is represented by an idempotent \( \varepsilon \in A \).

First, we give explicit formulas for the corresponding products by \( x \) in algebraic and in topological \( K_2 \)-theory. If \( A \) is commutative, following the definition given by Milnor (see [8], p. 67), one easily checks that the product

\[
x \star : K_2^\text{alg}(A) \longrightarrow K_2^\text{alg}(A), \quad y \longmapsto x \star y
\]

is given by the automorphism \( (\gamma_x)_* \) of \( H_2(E(R); \mathbb{Z}) \cong K_2^\text{alg}(A) \) induced by

\[
\gamma_x : E(A) \longrightarrow E(A), \quad E_n(A) \ni X \longmapsto \varepsilon \cdot X + (1 - \varepsilon) \cdot I_n.
\]

We need to express the map \( (\gamma_x)_* \) explicitly on \( K_2^\text{alg}(A) \) considered as the kernel in the universal central extension \( 0 \longrightarrow K_2^\text{alg}(A) \longrightarrow \text{St}(A) \xrightarrow{\varphi} E(A) \longrightarrow 0 \).

Let \( X = \prod_s e_{i_s j_s}(a_s) \in E_n(A) \) (a finite product of elementary matrices). Since \( \varepsilon = \varepsilon^2 \), one has clearly

\[
\varepsilon \cdot X + (1 - \varepsilon) \cdot I_n = \prod_s (\varepsilon \cdot e_{i_s j_s}(a_s) + (1 - \varepsilon) \cdot I_n) = \prod_s e_{i_s j_s}(\varepsilon a_s).
\]

This means that the map \( \gamma_x \) is simply given by \( e_{ij}(a) \longmapsto e_{ij}(\varepsilon a) \). We can therefore lift this map to \( \text{St}(A) \) by defining

\[
\tilde{\gamma}_x : \text{St}(A) \longrightarrow \text{St}(A), \quad x_{ij}(a) \longmapsto x_{ij}(\varepsilon a).
\]

We obtain a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_2^\text{alg}(A) \\
\downarrow \gamma_x & & \downarrow \varphi \\
0 & \longrightarrow & \text{St}(A)
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \tilde{\gamma}_x & & \downarrow \gamma_x \\
0 & \longrightarrow & K_2^\text{alg}(A) \\
\end{array}
\]

This shows that \( (\gamma_x)_* = \tilde{\gamma}_x|_{K_2^\text{alg}(A)} \), and gives a satisfactory description of the product in question, namely

\[
x \star : K_2^\text{alg}(A) \longrightarrow K_2^\text{alg}(A), \quad \prod_s x_{i_s j_s}(a_s) \longmapsto \prod_s x_{i_s j_s}(\varepsilon a_s).
\]

For \( A \) and \( B \) two unital rings, this generalizes to give

\[
x \star : K_2^\text{alg}(B) \longrightarrow K_2^\text{alg}(A \otimes \mathbb{Z} B), \quad \prod_s x_{i_s j_s}(b_s) \longmapsto \prod_s x_{i_s j_s}(\varepsilon \otimes b_s).
\]
Now, for a unital commutative Banach algebra $A$, we would like to describe the product $x \cup : K_2(A) \to K_2(A)$. First, observe that by definition of the cup product and naturality of the suspension isomorphism, the diagram

$$
\begin{array}{ccc}
K_0(A) \times K_2(A) & \to & K_2(A \hat{\otimes} A) \\
\cong & & \cong \\
K_0(A) \times K_1(SA) & \cong & K_1(S(A \hat{\otimes} A)) \\
\cong & & \cong \\
K_0(A) \times K_0(S^2A) & \to & K_0(S^2(A \hat{\otimes} A))
\end{array}
$$

commutes, where $S\hat{\mu}$ is induced by $\hat{\mu} : A \hat{\otimes} A \to A$ and is explicitly given by

$$S\hat{\mu} : S(A \hat{\otimes} A) \to SA, \ (t \mapsto a(t) \hat{\otimes} b(t)) \mapsto (t \mapsto a(t) \cdot b(t)).$$

The map $K_2(A) = \pi_1(\text{GL}(A)) \cong K_1(SA) \cong [e^{2\pi it} \mapsto v(t)] \mapsto [t \mapsto v(t)]$ is the isomorphism indicated above. This explicit description and the one of the product $K_0 \times K_1 \to K_1$ given in remark 2.1, allow for computing

$$x \cup : K_2(A) \to K_2(A)$$

$$[e^{2\pi it} \mapsto \prod_e e_{i,j}(t \cdot a_s)] \mapsto [e^{2\pi it} \mapsto \prod_{\varepsilon} e_{i,j}(t \cdot \varepsilon a_s)].$$

For two unital Banach algebras $A$ and $B$, this generalizes to yield

$$x \times : K_2(B) \to K_2(A \hat{\otimes} B)$$

$$[e^{2\pi it} \mapsto \prod_e e_{i,j}(t \cdot b_s)] \mapsto [e^{2\pi it} \mapsto \prod_{\varepsilon} e_{i,j}(t \cdot \varepsilon b_s)].$$

We are now in position to prove theorem 1.1 for $p = 0$ and $q = 2$. For an element $y = \prod_e x_{i,j}(b_s) \in K_2^{\text{alg}}(B)$, one has $\phi_2(y) = [e^{2\pi it} \mapsto \prod_{\varepsilon} e_{i,j}(t \cdot b_s)]$ (see section 1 for the explicit description of $\phi_2$). For $x = [\varepsilon] \in K_0^{\text{alg}}(A)$, with $\varepsilon = \varepsilon^2 \in A$, we deduce from the above considerations that

$$\phi_2 : K_2^{\text{alg}}(A \hat{\otimes} B) \to K_2^{\text{alg}}(A \hat{\otimes} B)$$

$$\prod_e x_{i,j}(e \otimes b_s) \mapsto \prod_{\varepsilon} x_{i,j}(\varepsilon \otimes b_s) \mapsto [e^{2\pi it} \mapsto \prod_{\varepsilon} e_{i,j}(t \cdot \varepsilon \otimes b_s)] = x \times \phi_2(y),$$

i.e. $\phi_2(x \star y) = x \times \phi_2(y)$, as was to be shown.

### 3 The case $p = q = 1$

In this section, we prove theorem 1.1 for $p = q = 1$. It is the most difficult case, although the difficulty is not conspicuous here, since it is almost completely contained in the lengthy computations of [7].

Here, we use the same notation for an invertible matrix and for its $K_1^{\text{alg}}$-theory class. Roughly speaking, the following lemma tells us that we can restrict to the commutative case and the internal products $\star_A$ and $\cup$.
3.1 Lemma. Let $A$ and $B$ be two unital Banach algebras, and let $x \in \text{GL}_1(A)$ and $y \in \text{GL}_1(B)$ be two invertibles. Consider $C := \langle 1, \hat{x}, \hat{y} \rangle$ the unital Banach sub-algebra of $A \hat{\otimes} B$ generated by $\hat{x} := x \hat{\otimes} 1$ and $\hat{y} := 1 \hat{\otimes} y$. Denote by $i$ the inclusion of $C$ in $A \hat{\otimes} B$, and by $j : A \hat{\otimes} B \to A \hat{\otimes} B$ the canonical map. Then, $C$ is a commutative unital Banach algebra and the following formulas hold:

\begin{align*}
&i) \quad j_*(x \cdot y) = i_*(\hat{x} \cdot_C \hat{y}) \in K_2^\text{alg}(A \hat{\otimes} B); \\
&ii) \quad \phi_1(x) \times \phi_1(y) = i_*(\hat{x} \cup \hat{y}) \in K_2(A \hat{\otimes} B).
\end{align*}

Proof. Recall that the products for algebraic $K_1$-theory are given by

$$x \cdot y = -(x \otimes 1, 1 \otimes y) \quad \text{and} \quad \hat{x} \cdot_C \hat{y} = -(\hat{x}, \hat{y}).$$

Naturality of the Steinberg symbol yields

$$j_*(\{x \otimes 1, 1 \otimes y\}) = \{i_*(\hat{x}), i_*(\hat{y})\} = \{\hat{x}, \hat{y}\},$$

establishing $i)$. Using the suspension isomorphism (for $x$) and remark 2.1, the product $\phi_1(x) \times \phi_1(y)$ equals the homotopy class of the map taking $e^{2\pi it}$ to

$$X_t := (P \cdot (1 - R_t \cdot P \cdot R_t^{-1}) \otimes 1) \cdot (P \otimes y + (1 - P) \otimes 1)^{-1},$$

where $P := \text{Diag}(1, 0)$, and $R_t = R_t(x)$ is a homotopy in $\text{GL}_2(A)$ from $1$ to $\text{Diag}(x, x^{-1})$. Similarly, $\phi_1(x) \cup \phi_1(y)$ is determined by

$$(R_t(\hat{x}) \cdot Q \cdot R_t(\hat{x})^{-1} \cdot \hat{y} + (1 - R_t(\hat{x}) \cdot Q \cdot R_t(\hat{x})^{-1}) \cdot (Q \cdot \hat{y} + (1 - Q) \otimes 1)^{-1},$$

where $Q := \text{Diag}(1 \otimes 1, 0 \otimes 0)$. Since $i_*$ takes this element to $X_t$, $ii)$ follows. □

The final lemma deals with the case of internal products.

3.2 Lemma. Let $A$ be a commutative unital Banach algebra. Then, for two invertibles $x, y \in \text{GL}_1(A)$, one has

$$\phi_2(x \cdot_A y) = -\phi_2([x, y]) = -\phi_1(x) \cup \phi_1(y) \in K_2(A).$$

Proof. The lemma is a consequence of the computations we made to prove the main result in [7]. In fact, proposition 6.1 in loc. cit. is precisely the content of lemma 3.2 for the particular Banach algebra $C^*\mathbb{Z}^2 \cong C(T^2)$ and for the product $a \cdot_C \mathbb{Z}^2 b$, where $a$ and $b$ are prescribed generators of $\mathbb{Z}^2$, viewed as unitaries in $C^*\mathbb{Z}^2$. (Indeed, $\phi_1(a) \cup \phi_1(b)$ is well-known to be the Bott element $\delta$ of $K_2(C^*\mathbb{Z}^2) \cong K^0(T^2)$.) Now, we claim that by naturality and by classical results on the $K$-theory of commutative Banach algebras, the general case follows. To prove this, we first consider the sub-algebra

$$A_\rho := \left\{ (\lambda_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \middle| \sum_{n \in \mathbb{Z}} \rho^{|n|} \cdot |\lambda_n| < \infty \right\}$$

of $\ell^2\mathbb{Z}$, where $\rho \geq 1$ is a real number. In other words, $A_\rho$ is the completion of the algebra $\mathbb{C}[\mathbb{Z}]$ for the norm

$$\left\| \sum_{n \in \mathbb{Z}} \lambda_n \cdot a^n \right\|_\rho := \sum_{n \in \mathbb{Z}} \rho^{|n|} \cdot |\lambda_n|,$$
where $a$ is a prescribed generator of the group $\mathbb{Z}$. So, $A_\rho$ is a unital Banach algebra for this norm, with the following “universal property”: given $u \in \text{GL}_1(A)$, where $A$ is any unital Banach algebra, one has $1 = \|1\|_A \leq \|u\|_A \cdot \|u^{-1}\|_A$, therefore $\rho_u := \max\{\|u^{-1}\|_A, \|u\|_A\}$ is $\geq 1$, and the inequalities

$$\left\|\sum_{n \in \mathbb{Z}} \lambda_n \cdot u^n\right\|_A \leq \sum_{n < 0} |\lambda_n| \cdot \|u^{-1}\|^{|n|}_A + \sum_{n \geq 0} |\lambda_n| \cdot \|u\|^n_A \leq \left\|\sum_{n \in \mathbb{Z}} \lambda_n \cdot a^n\right\|_{\rho_u}$$

imply that the algebra map $\nu_u : C[\mathbb{Z}] \rightarrow A$, $a \mapsto a$ extends uniquely to a unital Banach algebra morphism $\nu_u : A_{\rho_u} \rightarrow A$. Applying this result twice, by the universal property of the projective tensor product of Banach algebras, we obtain a unital Banach algebra morphism

$$\nu_{x, y} : A_{\rho_x} \hat{\otimes} A_{\rho_y} \rightarrow A, \xi \otimes \eta \mapsto \nu_x(\xi) \cdot \nu_y(\eta).$$

It is clear that $\nu_{x, y}(a) = x$ and $\nu_{x, y}(b) = y$, where $a$ and $b$ designate the prescribed generators of $\mathbb{Z}^2$, considered as elements of $\text{GL}_1(A_{\rho_x} \hat{\otimes} A_{\rho_y})$ via the map $\mathbb{Z}[\mathbb{Z}^2] \cong \mathbb{Z}[\mathbb{Z}] \otimes \mathbb{Z}[\mathbb{Z}] \hookrightarrow A_{\rho_x} \hat{\otimes} A_{\rho_y}$.

In our context, the second important feature of the algebra $A_\rho$ is that it is dense in $\ell^1 \mathbb{Z}$ and that the inclusions

$$A_\rho^{\text{incl}} \hookrightarrow \ell^1 \mathbb{Z} \hookrightarrow C^* \mathbb{Z}$$

induce isomorphisms in topological $K$-theory, for any $\rho \geq 1$. For the second inclusion, this follows from the Wiener lemma (see [9], 11.6) and the density theorem (see [3], prop. 3, pp. 285-286), and the first is a consequence of the Oka principle in $K$-theory established by Bost in [2] (see theorem 1.1.1 and example 1.1.3 therein). This also follows from a theorem of Arens, Eidlin and Novodvorski: let $B$ be a commutative unital Banach algebra, and let $\text{Spec}(B)$ be its spectrum (it is a compact Hausdorff space); then, the Gelfand transform

$$\mathcal{G}^B : B \rightarrow C(\text{Spec}(B))$$

is a natural morphism and induces an isomorphism in topological $K$-theory (see [2], thm. 1.3.2). It is clear that $\text{Spec}(\ell^1 \mathbb{Z})$ identifies with the unit circle $S^1$ and is included in $\text{Spec}(A_\rho)$, that correspondingly identifies with the closed crown with radii $\rho^{-1}$ and $\rho$. This inclusion is a homotopy equivalence, hence the isomorphism $\text{incl}_* = (\mathcal{G}_*^{\ell^1 \mathbb{Z}})^{-1} \circ \mathcal{G}_*^{A_\rho} : K_*(A_\rho) \xrightarrow{\cong} K_*(\ell^1 \mathbb{Z})$. Similarly, the inclusions $A_{\rho_x} \hat{\otimes} A_{\rho_y} \hookrightarrow \ell^1 \mathbb{Z} \hat{\otimes} \ell^1 \mathbb{Z} \cong \ell^1 \mathbb{Z}^2 \hookrightarrow C^* \mathbb{Z}^2$ induce isomorphisms

$$K_*(A_{\rho_x} \hat{\otimes} A_{\rho_y}) \cong K_*(\ell^1 \mathbb{Z}^2) \rightarrow K_*(C^* \mathbb{Z}^2),$$

since for two commutative unital Banach algebras $B_1$ and $B_2$, there is a canonical homeomorphism ([4], prop. IV.1.20)

$$\text{Spec}(B_1 \hat{\otimes} B_2) \cong \text{Spec}(B_1) \times \text{Spec}(B_2).$$

We denote $A_{\rho_x} \hat{\otimes} A_{\rho_y}$ simply by $A$. By naturality of the internal $*$-product, of the cup product and of the maps $\phi_1$ and $\phi_2$, we deduce from this argument that

$$\phi_2^*(a \ast_A b) = -\phi^*_1(a) \cup \phi^*_1(b).$$
By naturality, \( \phi_2^A(x \star_A y) = -\phi_1^A(x) \cup \phi_1^A(y) \) holds, as was to be shown. □

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We now prove theorem 1.1 for \( p = q = 1 \). Let \( x \in K_{alg}^1(A) \) and \( y \in K_{alg}^1(B) \). We have to establish that \( \hat{\phi}_2(x \star y) = -\phi_1(x) \times \phi_1(y) \). By Morita invariance of the product, we can assume that \( x \in GL_1(A) \) and \( y \in GL_1(B) \). We have, with the notations of lemma 3.1,

\[
\hat{\phi}_2(x \star y) = \phi_2^{A \otimes B} \circ j_*(x \star y) = \phi_2^{A \otimes B} \circ i_*(\hat{x} \star_C \hat{y}) = i_* \circ \phi_2^C(\hat{x} \star_C \hat{y}) =
- i_*(\phi_1(\hat{x}) \cup \phi_1(\hat{y})) = -\phi_1(x) \times \phi_1(y),
\]

where the first equality follows from the definition of \( \hat{\phi}_2 \), the second from lemma 3.1, the third from naturality of \( \phi_2 \), the fourth from lemma 3.2 for \( C \), and the last one from lemma 3.1 again.

Now, the proof of theorem 1.1 is complete. □

References


