Spectral theorem for multipliers on $L^2_\omega(\mathbb{R})$

Violeta Petkova

Abstract. We study the spectrum $\sigma(M)$ of the multipliers $M$ which commute with the translations on weighted spaces $L^2_\omega(\mathbb{R})$. For operators $M$ in the algebra generated by the convolutions with $\phi \in C_c(\mathbb{R})$ we show that $\mu(\Omega) = \sigma(M)$, where the set $\Omega$ is determined by the spectrum of the shift $S$ and $\mu$ is the symbol of $M$. For the general multipliers $M$ we establish that $\mu(\Omega)$ is included in $\sigma(M)$. A generalization of these results is given for the weighted spaces $L^2_\omega(\mathbb{R}^k)$ where the weight $\omega$ has a special form.

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1. Introduction

In this paper we examine the spectrum of multipliers $M$ on a weighted space $L^2_\omega(\mathbb{R})$. Our approach is based heavily on the existence of symbols for this class of operators and we show that the spectrum $\sigma(M)$ can be expressed by the symbol $\mu$ of $M$ applied to a set $\Omega$ defined by the spectrum of the shift operator $S$. To announce our results we need some definitions.

A weight $\omega$ on $\mathbb{R}$ is a positive measurable function on $\mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} \frac{\omega(x+y)}{\omega(x)} < +\infty, \forall y \in \mathbb{R}.$$

Denote by $L^2_\omega(\mathbb{R})$ the space of measurable functions on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |f(x)|^2 \omega(x)^2 dx < +\infty.$$

Let $C_c(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$ with compact support. For a compact $K$ of $\mathbb{R}$ denote by $C_K(\mathbb{R})$ the subset of functions of $C_c(\mathbb{R})$ with support in $K$. The space $L^2_\omega(\mathbb{R})$ equipped by the norm

$$\|f\| = \left( \int_{\mathbb{R}} |f(x)|^2 \omega(x)^2 dx \right)^{\frac{1}{2}}.$$
is a Banach space and \( C_c(\mathbb{R}) \) is dense in \( L^2_\omega(\mathbb{R}) \). We denote by \( \hat{f} \) or by \( \mathcal{F}(f) \) the usual Fourier transform of \( f \in L^2(\mathbb{R}) \). Denote by \( S_x \) the operator of translation by \( x \) defined on \( L^2_\omega(\mathbb{R}) \) by
\[
(S_x f)(t) = f(t - x), \quad \forall t \in \mathbb{R}.
\]
Let \( S \) (resp. \( S^{-1} \)) be the translation by 1 (resp. -1) on the space \( L^2_\omega(\mathbb{R}) \). Define the set
\[
\Omega = \{ z \in \mathbb{C}, -\ln \rho(S^{-1}) \leq \Im z \leq \ln \rho(S) \},
\]
where \( \rho(A) \) is the spectral radius of \( A \). For \( \phi \in C_c(\mathbb{R}) \) denote by \( M_\phi \) the operator of convolution by \( \phi \) on \( L^2_\omega(\mathbb{R}) \). Let \( \mathcal{A} \) (resp. \( \mathcal{B} \)) be the closed algebra generated by operators \( M_\phi \), for \( \phi \in C_c(\mathbb{R}) \) (resp. \( S_x \), \( x \in \mathbb{R} \)) with respect to the topology of the operator norm. Denote by \( \hat{\mathcal{A}} \) the set of characters of a commutative algebra \( \mathcal{A} \).

**Definition 1.1.** A bounded operator \( M \) on \( L^2_\omega(\mathbb{R}) \) is called a multiplier if
\[
MS_x = S_x M, \quad \forall x \in \mathbb{R}.
\]
We will denote by \( \mathcal{M} \) the algebra of the multipliers on \( L^2_\omega(\mathbb{R}) \). Notice that \( \mathcal{M} \) is a commutative Banach algebra with unit. The commutativity of \( \mathcal{M} \) follows from the fact that for every \( M \in \mathcal{M} \) there exists a distribution \( \mu \) such that \( Mf = \mu * f \) for \( f \in C^\infty_c(\mathbb{R}) \) (see [6]).

**Theorem 1.2.** For \( \phi \in C_c(\mathbb{R}) \), we have
\[
\sigma(M_\phi) = \overline{\hat{\phi}(\Omega)}.
\]
On the other hand, for every \( M \in \mathcal{A} \) we obtain

**Theorem 1.3.** Let \( M \in \mathcal{A} \) and let \( \nu \) be the symbol of \( M \). Then \( \nu \) is a continuous function on \( \Omega \) and we have
\[
\sigma(M_\phi) = \overline{\nu(\Omega)}.
\]

For general multiplier \( M \) with symbol \( \mu \) denote by \( \mu(\Omega) \) the essential range of \( \mu \). We have a weaker result leading to the inclusion of \( \mu(\Omega) \) in the spectrum of \( M \).

**Theorem 1.4.** Let \( M \) be a multiplier on \( L^2_\omega(\mathbb{R}) \) and let \( \mu \) be the symbol of \( M \). Then we have
\[
\overline{\mu(\Omega)} \subset \sigma(M).
\]
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We may characterize the spectrum of $M$ by the set $\mu(\Omega)$. For example, Theorem 1.4 yields the well known inclusion $e^{itS} \subset \sigma(e^{itS})$.

On the other hand for the operators in $\mathcal{M}$ it seems difficult to obtain an analog of (1.2) by using the techniques developed for $C_0$-semi-groups and special Banach algebra (see [4], [3], [5], [1]).

2. Preliminaries

First, we explain the link between the spectrum of $S$ and the set where the symbol of a multiplier is defined. For $a \in \mathbb{R}$, denote by $g_a$ the function $g_a(x) = g(x)e^{ax}$.

In [6], we have established the following theorem.

**Theorem 2.1.** For every $M \in \mathcal{M}$, and for every $a \in I = [-\ln \rho(S^{-1}), \ln \rho(S)]$, we have:

1) $(Mf)_a \in L^2(\mathbb{R})$, $\forall f \in C_c(\mathbb{R})$.
2) There exists $\mu(a) \in L^\infty(\mathbb{R})$ such that
   \[
   \int_{\mathbb{R}} (Mf)(x)e^{ax}e^{-itx}dx = \mu(a)(t)\int_{\mathbb{R}} f(x)e^{ax}e^{-itx}dx, \ a.e., \ \forall f \in C_c(\mathbb{R})
   \]
   i.e.
   \[
   \hat{(Mf)}_a = \hat{\mu(a)}(\hat{f})_a.
   \]
3) If $\mathcal{I} \neq \emptyset$ then the function $\mu(z) = \mu(\Im z)(\Re z)$ is holomorphic on $\hat{\Omega}$.

**Definition 2.2.** Given $M \in \mathcal{M}$, if $\hat{\Omega} \neq \emptyset$, we call symbol of $M$ the function $\mu$ defined by

$\mu(z) = \mu(\Im z)(\Re z)$, $\forall z \in \hat{\Omega}$.

Moreover, if $a = -\ln \rho(S^{-1})$ or $a = \ln \rho(S)$, the symbol $\mu$ is defined for $z = x + ia$ by the same formula for almost all $x \in \mathbb{R}$.

We will say that $a \in \mathbb{R}$ verifies the property (P) if for every $M \in \mathcal{M}$ we have:

1) $(Mf)_a \in L^2(\mathbb{R})$, $\forall f \in C_c(\mathbb{R})$
2) There exists $\mu(a) \in L^\infty(\mathbb{R})$ such that
   \[
   \int_{\mathbb{R}} (Mf)(x)e^{ax}e^{-itx}dx = \mu(a)(t)\int_{\mathbb{R}} f(x)e^{ax}e^{-itx}dx, \ a.e.
   \]

Theorem 2.1 may be extended. Indeed we have the following lemma, which will be useful in our analysis.

**Lemma 2.3.** Let $M \in \mathcal{M}$, $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$ and let $f \in L^2(\mathbb{R})$ be such that $(f)_a \in L^2(\mathbb{R})$. Then we have:

1) $(Mf)_a \in L^2(\mathbb{R})$
2) $(\hat{Mf})_a = \mu(a)(\hat{f})_a$. 

Proof. To establish Lemma 2.3 we use the arguments exploited in the proof of Theorem 2.1 (see [6]). Let $M \in \mathcal{M}$. Then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset C_c(\mathbb{R})$ such that $M$ is the limit of $(M_{\phi_n})_{n \in \mathbb{N}}$ with respect to the strong operator topology and we have $\|M_{\phi_n}\| \leq C\|M\|$, where $C$ is a constant independent of $n$ (see [6], [7]). Let $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$. We will apply the following not trivial property (see [6])

$$\|\hat{\psi}(a)(x)\| \leq \|M\|, \ \forall \psi \in C_c(\mathbb{R}), \ \forall x \in \mathbb{R}. \quad (2.1)$$

Observe that

$$\|\hat{\psi}(a)(x)\| \leq \|M\|, \ \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}.$$ 

and replace $(\hat{\phi_n})_{n \in \mathbb{N}}$ by a suitable subsequence also denoted by $(\hat{\phi_n})_{n \in \mathbb{N}}$ converging in the weak topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$ to a function $\mu(a) \in L^\infty(\mathbb{R})$ such that $\|\mu(a)\| \leq C\|M\|$. We have

$$\lim_{n \to +\infty} \int_{\mathbb{R}} (\hat{\phi_n}(x) - \mu(a)(x)) g(x) \, dx = 0, \ \forall g \in L^1(\mathbb{R}).$$

Fix $f \in L^2(\mathbb{R})$ so that $(f)_a \in L^2(\mathbb{R})$. Then we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}} (\hat{\phi_n}(x)(f_a)(x) - \mu(a)(x)(f)_a(x)) g(x) \, dx = 0$$

for all $g \in L^2(\mathbb{R})$. We conclude that $(\hat{\phi_n}(f)_a)_{n \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R})$ to $\mu(a)(f)_a$.

On the other hand, we have

$$(M_{\hat{\phi_n}f})_a(x) = (\hat{\phi_n}(x)(f)_a(x), \ a.e.$$

and

$$\|(M_{\hat{\phi_n}f})_a\|_{L^2} = (2\pi)^{-1/2} \|(M_{\hat{\phi_n}f})_a\|_{L^2} \leq (2\pi)^{-1/2}\|\hat{\phi_n}\|_{L^\infty} \|f\|_{L^2}$$

$$\leq (2\pi)^{-1/2}C\|M\|\|f\|_{L^2}, \ \forall n \in \mathbb{N}.$$ 

We replace $((M_{\hat{\phi_n}f})_a)_{n \in \mathbb{N}}$ by a suitable subsequence and we suppose that $((M_{\hat{\phi_n}f})_a)_{n \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R})$ to a function $h_a \in L^2(\mathbb{R})$. For $g \in C_c(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} |(M_{\hat{\phi_n}f})_a(x) - (Mf)_a(x)| \, |g(x)| \, dx$$

$$\leq C_{a,g}\|M_{\hat{\phi_n}f} - Mf\|, \ \forall n \in \mathbb{N},$$

where $C_{a,g}$ is a constant depending only on $g$ and $a$. Since $(M_{\phi_n}f)_{n \in \mathbb{N}}$ converges to $Mf$ in $L^2(\mathbb{R})$, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}} (M_{\phi_n}f)_a(x)g(x) \, dx = \int_{\mathbb{R}} (Mf)_a(x)g(x) \, dx, \ \forall g \in C_c(\mathbb{R}).$$
Thus we conclude that \((Mf)_{\alpha} = h_{\alpha}\) and \((Mf)_{\alpha} \in L^2(\mathbb{R})\). Now taking into account that for \(g \in L^2(\mathbb{R})\) we have

\[
\lim_{n \to +\infty} \langle (M_{\phi_n}f)_{\alpha}, \hat{g} \rangle_{L^2} = \lim_{n \to +\infty} \langle (\phi_n)_{\alpha} \hat{(f)}_{\alpha}, \hat{g} \rangle_{L^2} = \langle (\mu_{\alpha}(f))_{\alpha}, \hat{g} \rangle_{L^2},
\]

and also

\[
\lim_{n \to +\infty} \langle (M_{\phi_n}f)_{\alpha}, \hat{g} \rangle_{L^2} = \langle (Mf)_{\alpha}, \hat{g} \rangle_{L^2},
\]

we obtain

\[
(Mf)_{\alpha} = \mu_{\alpha}(f)_{\alpha}.
\]

Consequently, for every \(f \in L^2(\mathbb{R})\) such that \((f)_{\alpha} \in L^2(\mathbb{R})\) we have

\[
(Mf)_{\alpha}(x) = \mu_{\alpha}(x)(f)_{\alpha}(x) \text{ a.e.}
\]

Denote by \(\sigma(A)\) the spectrum of the operator \(A\). First we have the following

**Proposition 2.4.** We have

\[
\sigma(S) = \left\{ z \in \mathbb{C}, \quad \frac{1}{\rho(S^{-1})} \leq |z| \leq \rho(S) \right\}
\]

and the real \(a\) satisfies the property (P) if and only if \(a \in [-\ln \rho(S^{-1}), \ln \rho(S)]\).

Theorem 2.1 implies that if \(a \in [-\ln \rho(S^{-1}), \ln \rho(S)]\), then \(a\) satisfies the property (P).

**Lemma 2.5.** If \(a \in \mathbb{R}\) verifies the property (P), then \(e^{a+ib} \in \sigma(S)\), for all \(b \in \mathbb{R}\).

**Proof.** Let \(\alpha \in \mathbb{C}\) be such that \(e^{\alpha} \notin \sigma(S)\). Then it is clear that \(T = (S - e^{\alpha}I)^{-1}\) is a multiplier. Let \(a \in \mathbb{R}\) verify the property (P). Then there exists \(\nu_{\alpha} \in L^\infty(\mathbb{R})\) such that

\[
(Tf)_{\alpha} = \nu_{\alpha}(f)_{\alpha}, \quad \forall f \in C_c(\mathbb{R}), \text{ a.e.}
\]

Replacing \(f\) by \((S - e^{\alpha}I)g\), for \(g \in C_c(\mathbb{R})\) we get

\[
(g)_{\alpha}(x) = \nu_{\alpha}(x)[(S - e^{\alpha}I)g]_{\alpha}(x), \quad \forall g \in C_c(\mathbb{R}), \text{ a.e.}
\]

and

\[
(g)_{\alpha}(x) = \nu_{\alpha}(x)(g)_{\alpha}(x)[e^{a-ix} - e^{\alpha}], \quad \forall g \in C_c(\mathbb{R}), \text{ a.e.}
\]

Choosing a suitable \(g \in C_c(\mathbb{R})\), we have

\[
\nu_{\alpha}(x)(e^{a-ix} - e^{\alpha}) = 1, \quad \text{a.e.}
\]

Since \(\nu_{\alpha} \in L^\infty(\mathbb{R})\), we obtain that \(\Re \alpha \neq a\) and we conclude that

\[
e^{a+ib} \in \sigma(S), \quad \forall b \in \mathbb{R}.
\]

Taking into account Theorem 2.1 and Lemma 2.5, Proposition 2.4 follows directly.
3. Spectrum of $M \in \mathcal{M}$

In this section we investigate the spectrum of a general multiplier on $L^2_\omega(\mathbb{R})$.

We recall that the symbol $\mu$ of a multiplier $M$ is in $\mathcal{H}^\infty(\hat{\Omega})$ and it is essentially bounded on the boundary $\delta(\Omega)$ of $\Omega$ (see Theorem 2.1).

Proof of Theorem 1.4

Proof. Assume that $\lambda \notin \sigma(M)$. Then $(M - \lambda I)^{-1}$ is a multiplier and for every $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$, we introduce the symbol $\nu(a) \in L^\infty(\mathbb{R})$ so that

$$F\left( ((M - \lambda I)^{-1} f)_a \right)(x) = \nu(a)(x)\hat{f}(x), \forall f \in C_c(\mathbb{R}), \text{ a.e.}$$

Following Lemma 2.3, we can use the above equality for $f = (M - \lambda I)g$ with $g \in C_c(\mathbb{R})$. Indeed, for $g \in C_c(\mathbb{R})$ an application of Theorem 2.1 yields $((M - \lambda I)g)_a \in L^2(\mathbb{R})$ and the assumptions of Lemma 2.3 are fulfilled. We obtain

$$\hat{g}(x) = \nu(a)(x)(\mu(a)(x) - \lambda)\hat{g}(x), \text{ a.e.}$$

Choosing a suitable $g \in C_c(\mathbb{R})$, we get

$$1 = \nu(a)(x)(\mu(a)(x) - \lambda), \text{ a.e.}$$

Given $x \in \mathbb{R}$, satisfying

$$|\mu(a)(x)| \leq ||\mu(a)||_\infty \text{ and } |\nu(a)(x)| \leq ||\nu(a)||_\infty,$$

it is clear that if $\lambda = \mu(a)(x) = \mu(x + ia)$ we obtain a contradiction. For every $x$ for which (3.1) holds and for $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$ we deduce that

$$\mu(a)(x) = \mu(x + ia) \in \sigma(M).$$

According to Theorem 2.1, $\nu$ and $\mu$ are holomorphic on $\hat{\Omega}$ and essentially bounded on $\delta(\Omega)$. Consequently, (3.1) holds for almost every $x$ and the proof is complete. □

Notice that $\nu$ may not be continuous on the boundary of $\Omega$. Indeed, for $\omega = 1$, let $h \in L^\infty(\mathbb{R})$ be a function which is not continuous on $\mathbb{R}$. Define the operator $H$ on $L^2(\mathbb{R})$ by the formula

$$Hf = F^{-1}(hf).$$

Then $H$ is a multiplier on $L^2(\mathbb{R})$, but its symbol $h$ is not continuous on $\Omega = \mathbb{R}$.

Now we present one example. Suppose that $\chi$ is a complex Borel measure such that

$$\int_{-\infty}^{+\infty} ||S_t||d|\chi|(t) < +\infty. \quad (3.2)$$

Then the operator $M_\chi$ defined by the formula

$$M_\chi(f) = \int_{-\infty}^{+\infty} S_t(f)d\chi(t), \forall f \in L^2_\omega(\mathbb{R}),$$
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is obviously a multiplier on $L^2_\omega(\mathbb{R})$. The condition (3.2) implies that

$$\int_{\mathbb{R}} e^{ax} d|\chi|(x) < +\infty, \quad \forall a \in [-\ln \rho(S^{-1}), \ln \rho(S)],$$

so the integral

$$\int_{\mathbb{R}} e^{-iax} d\chi(x)$$

converges for all $\alpha \in \Omega$. Clearly, the symbol of $M_\chi$, defined in Section 2, becomes

$$\hat{\chi}(\alpha) = \int_{\mathbb{R}} e^{-iax} d\chi(x).$$

The results of the previous sections imply

$$\hat{\chi}(\Omega) \subset \sigma(M_\chi). \quad (3.3)$$

The inclusion (3.3) has been established by other methods in [1] and [4]. We see that the inclusion (3.3) may be obtained by the tools of our paper.

4. Spectrum of $M_\phi$

In this section we characterize the spectrum of $M_\phi$, for $\phi \in C_c(\mathbb{R})$ by using the results of the previous one. We recall that $M_\phi$ denotes the operator of convolution $\phi \in C_c(\mathbb{R})$. We recall that $\Omega$ is the set

$$\Omega = \left\{ z \in \mathbb{C}, -\ln \rho(S^{-1}) \leq \Im z \leq \ln \rho(S) \right\}.$$ 

First notice that $0 \in \sigma(M_\phi)$. Indeed, suppose that $M_\phi$ is invertible. Then $M_\phi^{-1}$ is a multiplier and let $\mu$ be its symbol. For $a \in I$, we have

$$\mathcal{F}\left( (M_\phi^{-1}(M_\phi f))_a \right) = (\hat{f})_a, \quad \forall f \in C_c(\mathbb{R}).$$

Then, we get

$$\mu_{(a)}(x)(\hat{\phi})_a(x)(\hat{f})_a(x) = (\hat{f})_a(x), \quad \forall f \in C_c(\mathbb{R}), \text{ a.e.}$$

and

$$\mu_{(a)}(x)(\hat{\phi})_a(x) = 1, \quad \text{a.e.}$$

Taking into account that $\mu_{(a)} \in L^\infty(\mathbb{R})$ and $\lim_{x \to +\infty} (\phi)_a(x) = 0$, we obtain a contradiction. Thus, we conclude that $0 \in \sigma(M_\phi)$. Now, we establish the following

**Lemma 4.1.** Let $\gamma \in \hat{M}$. If there exists $\phi \in C_c(\mathbb{R})$ such that $\gamma(M_\phi) \neq 0$, then there exists $\alpha \in \Omega$ such that

$$\gamma(M_\phi) = \int_{\mathbb{R}} \psi(x)e^{-i\alpha x} dx = \hat{\psi}(\alpha), \quad \forall \psi \in C_c(\mathbb{R}). \quad (4.1)$$
Proof. Let \( \gamma \in \hat{\mathcal{M}} \) and suppose that there exists \( \phi \in C_c(\mathbb{R}) \) such that \( \gamma(M\phi) \neq 0 \). We have

\[
M\phi f = \int_{\mathbb{R}} S_x(f)\phi(x)dx
\]

but we cannot deduce that

\[
\gamma(M\phi) = \int_{\mathbb{R}} \gamma(S_x)\phi(x)dx,
\]

since we do not have the convergence of the Bochner integral \( M\phi f = \int_{\mathbb{R}} S_x(f)\phi(x)dx \) with respect to the operator norm. However, we claim that

\[
\gamma(M\psi) = \int_{\mathbb{R}} \gamma(M\phi \circ S_x)\psi(x)dx, \quad \forall \psi \in C_c(\mathbb{R}). \quad (4.2)
\]

Consider the application

\[
\eta : C_c(\mathbb{R}) \ni \psi \rightarrow \eta(\psi) = \gamma(M\psi)
\]

which is a continuous linear form on \( C_c(\mathbb{R}) \). Here \( C_c(\mathbb{R}) \) is equipped by the topology given by the inductive limit of \( C_K(\mathbb{R}) \), \( K \) being a compact subset of \( \mathbb{R} \). Indeed, if \( K \) is a compact subset of \( \mathbb{R} \) and if \( (\psi_n)_{n \in \mathbb{N}} \subset C_K(\mathbb{R}) \) is a sequence uniformly convergent to \( \psi \in C_K(\mathbb{R}) \), for every \( g \in L^2_\omega(\mathbb{R}) \), we have

\[
\|M\psi_n g - M\psi g\| \leq \int_K \|\psi_n - \psi\|_\infty \sup_{y \in K} \|S_y\|\|g\|dx
\]

and so

\[
\lim_{n \to \infty} \|M\psi_n - M\psi\| = 0.
\]

We deduce that the application

\[
C_c(\mathbb{R}) \ni \psi \rightarrow M\psi \in A
\]

is sequentially continuous and so it is continuous from \( C_c(\mathbb{R}) \) to \( A \). It follows that \( \eta \) is a continuous linear form on \( C_c(\mathbb{R}) \) and there exists a measure \( m \) (see [2], Chapter 3) such that

\[
\eta(\psi) = \int_{\mathbb{R}} \psi(x)dm(x), \quad \forall \psi \in C_c(\mathbb{R})
\]

and hence

\[
\gamma(M\psi) = \int_{\mathbb{R}} \psi(x)dm(x).
\]

This implies that for every \( \psi \in C_c(\mathbb{R}) \) we have

\[
\gamma(M\psi \circ M\phi) = \int_{\mathbb{R}} (\psi \ast \phi)(t)dm(t)
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \psi(x)\phi(t-x)dx \right)dm(t).
\]

Using Fubini theorem, we get

\[
\gamma(M\psi \circ M\phi) = \gamma(M\phi \ast \psi) = \int_{\mathbb{R}} \psi(x) \left( \int_{\mathbb{R}} \phi(t-x)dm(t) \right)dx
\]
= \int_{\mathbb{R}} \psi(x) \gamma(S_x \circ M_\phi) dx

and this yields the claim (4.2). Consequently, we conclude that

\[ \gamma(M_\phi) = \int_{\mathbb{R}} \psi(x) \gamma(S_x) dx, \forall \psi \in C_c(\mathbb{R}). \]  

(4.3)

Since \( \gamma \in \hat{M} \), it is clear that \( \gamma(S_x) \in \sigma(S_x) \) (see [9]). Set

\[ \theta_\gamma(x) = \gamma(S_x) = \frac{\gamma(M_\phi \circ S_x)}{\gamma(M_\phi)}, \forall x \in \mathbb{R}. \]

The application

\[ x \mapsto S_x \circ M_\phi = M_{S_x(\phi)} \]

is continuous from \( \mathbb{R} \) into \( A \) and we deduce that \( \theta_\gamma \) is a continuous morphism from \( \mathbb{R} \) to \( \mathbb{C} \). The function \( \theta_\gamma \) verifies

\[ \theta_\gamma(x + y) = \theta_\gamma(x) \theta_\gamma(y), \forall x, y \in \mathbb{R}. \]

Therefore there exists \( \alpha \in \mathbb{C} \) such that

\[ \theta_\gamma(x) = e^{-i\alpha x}, \forall x \in \mathbb{R}. \]

Applying (4.2) we get

\[ \gamma(M_\phi) = \int_{\mathbb{R}} \psi(x) e^{-i\alpha x} dx = \hat{\psi}(\alpha), \forall \psi \in C_c(\mathbb{R}). \]

Since \( e^{-i\alpha} \in \sigma(S) \), an application of Proposition 2.4 yields

\[ \frac{1}{\rho(S^{-1})} \leq e^{i\alpha} \leq \rho(S) \]

which implies \( \alpha \in \Omega \). This completes the proof of Lemma 4.1. \( \square \)

**Corollary 4.2.** Let \( \phi \in C_c(\mathbb{R}) \). If \( \lambda \in \sigma(M_\phi) \setminus \{0\} \) then there exists \( \alpha \in \Omega \) such that \( \lambda = \hat{\phi}(\alpha) \) and

\[ \sigma(M_\phi) \setminus \{0\} \subset \hat{\phi}(\Omega). \]

**Proof.** Following [9], we have \( \sigma(M_\phi) = \{ \gamma(M_\phi), \gamma \in \hat{M} \} \). Let \( \gamma \in \hat{M} \) be fixed such that \( \lambda = \gamma(M_\phi) \). The Corollary 4.2 follows from the previous lemma. \( \square \)

### 5. Spectrum of \( M \in \mathcal{A} \)

Now we examine the spectrum of multipliers forming a larger class than those of operators \( M_\phi \). We will show that if \( M \in \mathcal{A} \) and \( \nu \) is the symbol of \( M \), then \( \nu \) is a continuous function on \( \Omega \) and we have

\[ \sigma(M) = \overline{\nu(\Omega)}. \]

**Proof of Theorem 1.3.**
Proof. First, we show that \( \nu \) is continuous on \( \Omega \). Let \( (\phi_n)_{n \in \mathbb{N}} \) be a sequence of \( C_c(\mathbb{R}) \) such that \( (M_{\phi_n})_{n \in \mathbb{N}} \) converges to \( M \) with respect to the operator topology. The construction of \( \nu_{(a)} \) for \( a \in [-\ln \rho(S^{-1}), \ln \rho(S)] \), defined in [6] (see also the proof of Lemma 2.3 in Section 2), defines \( \nu_{(a)} \) as the limit of \( (\psi_n)_{a} \) with respect to the weak topology of \( L^2(\mathbb{R}) \), where \( (\psi_n)_{a} \) is a special sequence in \( C_c(\mathbb{R}) \) such that \( (M_{\psi_n})_{n \in \mathbb{N}} \) converges to \( M \) with respect to the strong operator topology. Using the same argument as in [6], we get

\[
\nu_{(a)}(x) = \lim_{n \to \infty} (\phi_n)_{a}(x)
\]

with respect to the weak topology of \( L^2(\mathbb{R}) \). For fixed \( a \in [-\ln \rho(S^{-1}), \ln \rho(S)] \) and \( x \in \mathbb{R} \), we have

\[
| (\phi_n)_{a}(x) - (\phi_k)_{a}(x) | \leq \| M_{\phi_n} - M_{\phi_k} \| \tag{5.1}
\]

(see (2.1)). Since \( (M_{\phi_n})_{n \in \mathbb{N}} \) converges to \( M \) in \( \mathcal{A} \), we conclude that \( (\phi_n)_{a} \) converges uniformly on \( \mathbb{R} \) to a continuous function \( \mu_{(a)} \) and that there exists a constant \( C \) such that

\[
| (\phi_n)_{a}(x) | \leq C, \forall a \in [-\ln \rho(S^{-1}), \ln \rho(S)], \forall x \in \mathbb{R}, \forall n \in \mathbb{N}.
\]

Moreover, we have \( \lim_{x \to +\infty} \mu_{(a)}(x) = 0 \). We obtain that \( (\phi_n)_{a} \) converges to \( \mu_{(a)} \) with respect to the weak topology of \( L^2(\mathbb{R}) \). Thus we can identify the symbol of \( M \) with the function \( \mu \) defined by

\[
\mu(x + ia) = \mu_{(a)}(x), \forall a \in [-\ln \rho(S^{-1}), \ln \rho(S)], \forall x \in \mathbb{R}.
\]

Taking into account (5.1), it is clear that \( \mu \) is continuous on \( \Omega \).

Given \( \lambda \in \sigma(M) \setminus \{0\} \), there exists \( \gamma \in \mathcal{M} \) such that \( \lambda = \gamma(M) \). Then we have

\[
\lambda = \lim_{n \to \infty} \gamma(M_{\phi_n}) = \gamma(M_{\phi_n}) = \mu(\alpha).
\]

Notice that \( \alpha \) is independent of \( n \). Consequently, we have

\[
\lambda = \lim_{n \to \infty} \gamma(M_{\phi_n}) = \lim_{n \to \infty} \gamma(\phi_n) = \mu(\alpha).
\]

Since \( \mu \) is equal to the symbol \( \nu \) of \( M \), we conclude that

\[
\sigma(M) \setminus \{0\} \subset \nu(\Omega).
\]

Taking into account the result of the previous section, the proof is complete. \( \square \)

6. Spectrum of multipliers on \( L^2(\mathbb{R}^k) \)

Theorems 1.2-1.4 can be generalized for multipliers on \( L^2(\mathbb{R}^k) \), \( k > 1 \), where \( \omega \) is a weight on \( \mathbb{R}^k \) satisfying the following condition

\[
\omega = \omega_1 \times \ldots \times \omega_k,
\]
where $\omega_1, ..., \omega_k$ are weights on $\mathbb{R}$. From now on, $\omega$ denotes a weight on $\mathbb{R}^k$ having the above form.

Given $\phi \in C_c(\mathbb{R}^k)$, the Fourier transform $\widehat{\phi}$ is defined on $\mathbb{C}^k$. Set

$$e_m = (e_{m,1}, ..., e_{m,k}),$$

where $e_{m,i} = 0$, if $m \neq i$ and $e_{m,m} = 1$. For $m = 1, ..., k$, let $S_m$ be the translation by $e_m$ defined on $L_2^\omega(\mathbb{R}^k)$. Introduce

$$U = \{z = (z_1, ..., z_k) \in \mathbb{C}^k ; \exists z_i \in [-\ln \rho(S_i^{-1}), \ln \rho(S_i)], \text{ for } i = 1, ..., k\}.$$ 

We have (see [7] and [8]) the following representation theorem for multipliers on $L_2^\omega(\mathbb{R}^k)$.

**Theorem 6.1.** Let $M$ be a multiplier on $L_2^\omega(\mathbb{R}^k)$. Then there exists $\nu \in L^\infty(U)$ such that

$$\int_{\mathbb{R}^k} (Mf)(x)e^{-ix \cdot z} dx = \nu(z) \int_{\mathbb{R}^k} f(x)e^{-ix \cdot z} dx, \forall f \in C^\infty(\mathbb{R}^k),$$

for all $z \in U$ and for almost every $z \in \delta(U)$.

Given a multiplier $M$ on $L_2^\omega(\mathbb{R}^k)$, we call symbol of $M$ the function $\nu$ introduced in the previous theorem. Moreover, in [8] the following result was established.

**Proposition 6.2.** We have $z = (z_1, ..., z_k) \in U$ if and only if

$$e^{-iz_m} \in \sigma(S_m), \text{ for } m = 1, ..., k.$$ 

The set $U$ is related to the joint spectrum of $S_1, ..., S_k$. Let $A$ be a commutative Banach algebra with unit $I$. We recall the following

**Definition 6.3.** The joint spectrum $\sigma_s(A_1, ..., A_k)$ of the operators $A_1, ..., A_k \in A$ is the set

$$\{(\alpha_1, ..., \alpha_k) \in \mathbb{C}^k ; \sum_{m=1}^k (A_m - \alpha_m I)J_m \text{ is not invertible in } A, \forall (J_1, ..., J_k) \in \mathbb{A}^k\}.$$ 

In general $\sigma_s(A_1, ..., A_k) \neq \sigma(A_1) \times \cdots \times \sigma(A_k)$ and the determination of $\sigma_s(A_1, ..., A_k)$ is a quite difficult problem. However, in the spaces $L_2^\omega(\mathbb{R}^k)$, we have the equality

$$\sigma_s(S_1, ..., S_k) = \sigma(S_1) \times \cdots \times \sigma(S_k)$$

(see ([8])). Using Theorem 6.1, Proposition 6.2 and the arguments in Section 4, we obtain the following

**Theorem 6.4.** For $\phi \in C_c(\mathbb{R}^k)$, we have

$$\sigma(M\phi) = \overline{\phi(U)}.$$ 

Moreover, repeating the arguments in Sections 2-5, we obtain
**Theorem 6.5.** Let $M \in \mathcal{A}$ and let $\nu$ be the symbol of $M$. Then $\nu$ is a continuous function on $\mathcal{U}$ and we have

$$\sigma(M) = \overline{\nu(\mathcal{U})}.$$ 

Let $M \in \mathcal{M}$ and let $\nu$ be its symbol. Then

$$\nu(\mathcal{U}) \subset \sigma(M).$$

**References**


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Violeta Petkova
LMAM
Université de Metz UMR 7122
Ile du Saulcy
57045 Metz Cedex 1, France.
e-mail: petkova@univ-metz.fr