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HIGHER SPECTRAL FLOW AND AN ANALYTIC BIVARIANT JLO COCYCLE

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Abstract. Given a smooth fibration of closed manifolds and a family of generalised Dirac operators along the fibers, we define an associated bivariant JLO cocycle. We then prove that, for any $\ell \geq 0$, our bivariant JLO cocycle is analytic when we endow functions on the total manifold with the $C^{\ell+1}$ Banach topology and functions on the base manifold with the C^{ℓ} Banach topology. This means that our bivariant coycle is entire in the sense of Connes. As a by-product of our theorem, we deduce that the bivariant JLO cocycle is analytic for the Fréchet smooth topologies. As an application we prove that our JLO bivariant cocycle computes the Chern character of the Dai-Zhang higher spectral flow.

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Introduction

Our objective in this paper is to give a bivariant analytic Chern character sufficiently general to encompass the families index theorem for generalised Dirac operators. In other words we make explicit the long held view that the Bismut formalism may be incorporated into noncommutative geometry. From the authors' point of view this question arose from a discussion at Oberwolfach (we thank Masoud Khalkhali and Alain Connes for comments). A number of results in different algebraic and/or geometric situations have been obtained previously for instance in [25, 30, 19, 26]. (There is also the related question of the bivariant version of the Connes-Moscovici residue cocycle but we defer that to another place.)

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The first issue is to choose a bivariant framework for this problem. In Meyer's thesis [24] we have found an appropriate formalism. Using Meyer's ideas we define a bivariant JLO (Jaffe-Lesniewski-Osterwalder [20]) cocycle that encompasses the local families index theorem [8, 9]. Our first main result is to prove that the bivariant JLO cocycle is analytic in the sense of the formalism of [24], when one considers the C^ℓ topology on the algebra of functions on the base, and the $C^{\ell+1}$ topology on the algebra of functions on the total space of the fibration. We thank Ralf Meyer for his helpful comments on our approach to this result.

To describe our results we need some notation. Throughout we shall consider a locally trivial fibration $F \to M \xrightarrow{\pi} B$ of closed manifolds endowed with smooth metrics. Following [8], we fix an hermitian vector bundle $E \to M$ whose fibers are modules over the Clifford algebra of the fiberwise tangent bundle $T_vM = \mathrm{Ker}(\pi_*)$. While we formulate our initial results in a general way our main interest lies in the case of odd dimensional fibers for E as this is not as well understood as is the even dimensional situation. We follow [9, 8] and introduce on E a quasiconnection ∇. We choose a family of generalized Dirac operators parametrised smoothly by B denoted D, and different superconnections as in [9]. They will be for us given by

$$\mathbb{A}_\sigma(A) := \mathbb{B}_\sigma + A,$$

where A is a zero-th order fiberwise σ-pseudodifferential operator with coefficients in differential forms of positive degree (in applications ≥ 2) on B. Here, the superconnection $\mathbb{B}_\sigma$ is defined by

$$\mathbb{B}_\sigma = \nabla + \sigma D$$

with σ being a Clifford variable as in Quillen's work [27]. We focus on $\mathbb{A}_\sigma(A)$, $\mathbb{B}_\sigma$ or in superconnections obtained by metric rescalings. In order to simplify the exposition of the analyticity results, we restrict ourselves to the case $\mathbb{A}(0) = \mathbb{B}_\sigma$ and briefly explain later how the proofs extend to the general case.

Our main application is to connect our bivariant Chern character with the Dai-Zhang higher spectral flow. The method we use draws on some ideas in [18], [12] and [27] on superconnections and the JLO cocycle. We show using [16] that the bivariant JLO Chern character in the case of the Bismut superconnection gives the Bismut local formula that Dai-Zhang observe computes higher spectral flow. (The use of the Bismut superconnection avoids the complications that enter into our bivariant formula when $\nabla^2 \neq 0$.)

Thus our approach explains the Dai-Zhang results in terms of bivariant cyclic cohomology. In the single operator case these results are known from Connes [13] and Getzler [18]. We choose here to extend Getzler's method to deal with families. In [26], similar issues are discussed for interesting noncommutative bivariant situations, but only encompassing in the commutative case trivial fibrations and flat quasiconnections ∇. Our motivation for giving a detailed treatment of the general commutative case has to do with some applications in noncommutative settings, see for instance [4]. We are aware of a need to use our point of view and results in current work of colleagues A. Gorokhovsky, J.-M. Lescure and B.-L. Wang. Thus we have written the exposition so that it will adapt immediately to foliations and to spectral flow for twisted families.

The conceptual framework for our results is that we establish the following commutative square.

$$\begin{array}{ccc}
K^1(M) & \xrightarrow{\mathrm{SF}(D,\cdot)} & K^0(B) \\
\Big\downarrow \mathrm{Ch} & & \Big\downarrow \mathrm{ch} \\
HCA_1(C^\infty(M)) & \xrightarrow{\mathrm{JLO}(D)} & H^{even}(B,\mathbb{C})
\end{array}$$

We digress to explain the notation in this diagram. The left vertical arrow represents the analytic Chern character, while the right vertical arrow represents the usual Chern character with appropriate normalizations. Our family of generalised Dirac operators parametrised by B defines an element $[D]$ of $KK^1(M,B)$ and the top horizontal arrow, which is the higher spectral flow map defined in [16] when the K^1 class of D is trivial, is still well defined in general as the Kasparov product $\cap[D]$ by $[D]$. Finally $\mathrm{JLO}(D)$ is our analytic bivariant Chern character which takes values in even de Rham cohomology of B.

Statement of the main results. The bivariant JLO cochain is defined by the sequence $(\psi_n)_n$ of functionals which, for $(f_0, \cdots, f_n) \in C^\infty(M)^{n+1}$, are given by the formula

$$\psi_n(f_0, \cdots, f_n) := \langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle_{n,\mathbb{B}_\sigma} \in \Omega^*(B),$$

where the multilinear functional on the right hand side is a generalised JLO functional whose precise form is explained at the beginning of Section 2. Our main results are then as follows.

Theorem 3.3. We assume that the fibers of our fibration are odd dimensional manifolds. For any $\ell \geq 0$, $\psi = (\psi_{2n+1})_{n\geq 0}$ is an analytic bivariant cyclic cocycle on the Banach algebras $(C^{\ell+1}(M), C^\ell(B))$. More precisely, ψ is a bounded morphism from the analytic bornological completion of the universal differential algebra of $C^{\ell+1}(M)$ to the Banach algebra of ℓ regular differential forms on B.

Theorem 5.5. Assume that the index class of D in $K^1(B)$ is trivial so that the higher spectral flow $\mathrm{SF}(D, U)$ be well defined, for any $U \in GL_N(C^\infty(M))$. Then the following relation holds in the even de Rham cohomology of the base manifold B:

$$\frac{1}{\sqrt{\pi}} < \mathrm{JLO}(D), \mathrm{ch}(U) >= \mathrm{ch}(\mathrm{SF}(D, U)).$$

In fact, we prove more as we derive the previous theorem from:

Theorem 5.8. We have the following equality as differential forms:

$$\int_0^1 \tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})dt = -\sum_{k\geq 0}(-1)^k k!\,\langle\langle U^{-1}, [\mathbb{B}_\sigma, U], [\mathbb{B}_\sigma, U^{-1}], \cdots, [\mathbb{B}_\sigma, U]\rangle\rangle_{2k+1,\mathbb{B}_\sigma},$$

where $\mathbb{B}_t = \mathbb{B}_\sigma + tU^{-1}[\mathbb{B}_\sigma, U]$ is the usual linear path of superconnections.

We will explain in the appendix how our point of view relates to the Hochschild-Kostant-Rosenberg-Connes (HKRC) map. In addition we see as a corollary of our arguments, and of the main result of [16], that $\mathrm{JLO}(D)$ coincides up to the HKRC map with the topological map $H^{odd}(M) \to H^{even}(B)$ given by

$$\omega \longmapsto \int_{M/B} \omega \wedge \hat{A}(TM|B).$$

To understand the structure of our exposition we now expand on the List of Contents. Section 1 gives the differential geometric framework: families of generalised Dirac operators, connections and bivariant functionals. In Section 2 we introduce our bivariant JLO multilinear functionals and discuss the identities they satisfy essentially following [17] but adapted to families. Our first objective, to understand the entire property for bivariant JLO, begins in Section 3. We summarise Meyer's point of view at the beginning of this Section for the reader's convenience and then state our main theorem. Section 4 contains the proof: the argument is a series of estimates that establish that our JLO is analytic in the sense of Meyer (and hence entire). In Section 5 we prove the commutative diagram above. Finally in the Appendix we establish the compatibility of Section 5 with the HKRC map.

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1. Preliminary results

1.1. **Connections.** We begin by introducing further notation and a general framework for our discussion. As above we denote by T_vM the fiberwise tangent bundle $T_vM := \mathrm{Ker}(\pi_*) \subset TM$ then we are assuming there is a Clifford homomorphism of algebra bundles

$$c : Cl(T_vM \otimes \mathbb{C}) \longrightarrow \mathrm{End}(E) \text{ with } c(\xi)^2 = |\xi|^2 \text{ for } \xi \in T_vM,$$

where $Cl(T_vM\otimes\mathbb{C})$ denotes the Clifford algebra bundle associated with the hermitian bundle $T_vM\otimes\mathbb{C}$, and $\mathrm{End}(E)$ is the algebra bundle of endomorphism of E. We assume as usual that E is endowed with a Clifford connection ∇^E and consider the Dirac operator D associated with this connection. Then D is a fiberwise first order differential operator acting on smooth sections of E and can be regarded as a family of elliptic operators along the fibers smoothly parametrized by the elements of the base manifold B, i.e. $D=(D_b)_{b\in B}$ where $D_b : C^\infty(M_b, E|_{M_b})\to C^\infty(M_b, E|_{M_b})$ is an essentially self-adjoint generalized Dirac operator. Notice that we are using self-adjoint operators while the authors of [8] rather use skew-adjoint operators.

Using the metric, we fix the horizontal distribution $H=(T_vM)^\perp$ so that

$$TM = H\oplus T_vM \text{ and for any } b\in B, \pi_{*,m}: H_m\to T_{\pi(m)}B \text{ is a linear isomorphism.}$$

The dual vector bundle π^*T^*B can be identified with the subbundle of T^*M consisting of forms that vanish on vertical vectors. Using the splitting given by H, we deduce the existence of a restriction projection $\varrho: T^*M\to\pi^*T^*B$. This projection extends to exterior powers and we obtain, in the obvious notation,

$$\varrho: C^\infty(M, E\otimes\Lambda T^*M)\to C^\infty(M, E\otimes\Lambda\pi^*T^*B).$$

Notice that $C^\infty(M, E\otimes\Lambda\pi^*T^*B)$ is a module over the algebra of differential forms on the base manifold B. More precisely, this module structure is obtained using the pull-back map associated with the projection π. So if we denote by $\xi\omega$ the action of a differential form ω on B, on a smooth section ξ of $E\otimes\Lambda\pi^*T^*B$, then

$$(\xi\omega)(m) := \xi(m)\wedge\omega(\pi(m)), \text{ or } \xi\omega := \xi\wedge\pi^*\omega.$$

Here $\wedge$ is denoting the usual action of differential forms on the exterior algebra.

For any $m\in\mathbb{Z}$, we denote by $\Psi^m(M|B,E)$ the space of (1-step polyhomogeneous) classical pseudodifferential operators of order m, acting along the fibers of $\pi: M\to B$, see [2]. The local coefficients of such operators are thus smooth in the base variables and the space $\Psi^m(M|B;E)$ is a module over the algebra $C^\infty(B)$ of smooth functions on B. We shall also need the space of such operators with coefficients in differential forms on the base. More precisely, we set

$$\psi^m(M|B,E;\Lambda^*B) = \Psi^m(M|B,E)\widehat{\otimes}_{C^\infty(B)}\Omega^*(B).$$

According to [2], the space $\Psi^m(M|B,E;\Lambda^*B)$ can be endowed with a complete smooth topology as a projective tensor product of such topological spaces. An element of $\psi^m(M|B,E;\Lambda^*B)$ is thus equivariant for the action of the algebra $\Omega^*(B)$ of smooth differential forms on B. As a consequence we can compose an element P of $\psi^m(M|B,E;\Lambda^kB)$ with an element Q of $\psi^{m'}(M|B,E;\Lambda^hB)$ to get an element QP of $\psi^{m+m'}(M|B,E;\Lambda^{k+h}B)$. We shall denote by $\psi^\infty(M|B,E;\Lambda^*B)$ the algebra obtained in this way, i.e.

$$\psi^\infty(M|B,E;\Lambda^*B) := \bigcup_{m\in\mathbb{Z}}\psi^m(M|B,E;\Lambda^*B).$$

As we shall see, the order of the pseudodifferential operators will not be involved in the gradings used in the sequel. In particular, for us, $\psi^\infty(M|B,E;\Lambda^*B)$ is $\mathbb{Z}_2$-graded by the parity of the degree of the forms on B. We also denote by $\psi^{-\infty}(M|B,E;\Lambda^*B)$ the ideal of fiberwise smoothing operators with coefficients in differential forms on B, i.e.

$$\psi^{-\infty}(M|B,E;\Lambda^*B) := \bigcap_{m\in\mathbb{Z}}\psi^m(M|B,E;\Lambda^*B).$$

Introduce the fiber product $G=M\times_B M := \{(m,m')\in M\times M, \pi(m)=\pi(m')\}$, which is a smooth groupoid. By the fiberwise Schwartz theorem applied to fiberwise smoothing operators, $\psi^{-\infty}(M|B,E;\Lambda^*B)$ can and will be identified with the convolution algebra of smooth sections of the bundle $\mathrm{Hom}(E)\otimes\Lambda^*T^*B$ over G whose fiber is

$$(\mathrm{Hom}(E)\otimes\Lambda^*T^*B)_{m,m'} := \mathrm{Hom}(E_{m'},E_m)\otimes\Lambda^*T^*_{\pi(m)=\pi(m')}B.$$

Definition 1.1. *We define the operator* ∇ *by*

$$\nabla := \varrho\circ\nabla^E : C^\infty(M, E\otimes\Lambda\pi^*T^*B)\to C^\infty(M, E\otimes\Lambda T^*M)\to C^\infty(M, E\otimes\Lambda\pi^*T^*B),$$

Then ∇ *is called a quasi-connection.*

Remark 1.2. *In Bismut's viewpoint ∇ is a connection on an infinite dimensional Fréchet bundle on B.*

Lemma 1.3. *The quasi-connection ∇ increases the degree of the forms by one and satisfies the Leibniz rule*

$$\nabla(\xi\omega) = (-1)^{\partial\xi}\xi d_B\omega + (\nabla\xi)\omega,$$

where $\partial\xi$ is the form degree of the section ξ. Moreover, the curvature operator ∇^2 of ∇ is a fiberwise first order differential operator with coefficients in 2-forms on the base manifold B.

Proof. This argument is analogous to one introduced in [6]. As ∇^E is a connection and choosing the module action on the right, we have:

$$\begin{aligned}\nabla^E(\xi\omega) &= (\nabla^E\xi)\wedge\pi^*\omega + (-1)^{\partial\xi}\xi\wedge d\pi^*\omega\\ &= (\nabla^E\xi)\omega + (-1)^{\partial\xi}\xi\wedge\pi^* d_B\omega\\ &= (\nabla^E\xi)\omega + (-1)^{\partial\xi}\xi d_B\omega.\end{aligned}$$

Therefore, applying ϱ, we obtain

$$\nabla(\xi\omega) = \varrho(\nabla^E\xi)\omega + (-1)^{\partial\xi}\xi d_B\omega = (\nabla\xi)\omega + (-1)^{\partial\xi}\xi d_B\omega.$$

Therefore, we have by the classical computation $\nabla^2(\xi\omega) = (\nabla^2\xi)\omega$, i.e. ∇^2 is $\Omega^*(B)$ linear, and hence it is a fiberwise differential operator with coefficients in 2-forms on the base. By reducing to local cordinates on the base manifold B, one computes ∇^2 using the local expression of ∇^E. It is then easy to check that ∇^2 is indeed a (fiberwise) first order differential operator, see also [5] for more details. □

We shall say that an $\Omega^*(B)$-linear map on $C^\infty(M; E\otimes\pi^*\Lambda^*B)$ has degree $k\in\mathbb{Z}$, if it increases the form degree of the sections by k. Hence, such a map sends $C^\infty(M; E\otimes\pi^*\Lambda^h B)$ into $C^\infty(M; E\otimes\pi^*\Lambda^{h+k}B)$. We then set $\partial T = k$ and denote by $\mathcal{C}^k$ the filtration obtained in this way, that is $\mathcal{C}^k$ is the space of such maps with degree $\leq k$, which are $\Omega^*(B)$-linear operators.

Lemma 1.4. *Let T be an $\Omega^*(B)$-linear map of degree k, acting on $C^\infty(M; E\otimes\pi^*\Lambda^*B)$. Then the graded commutator*

$$\partial(T) = [\nabla, T] := \nabla\circ T - (-1)^k T\circ\nabla,$$

is an $\Omega^*(B)$-linear map which belongs to $\mathcal{C}^{k+1}$.

Proof. Fix $T\in\mathcal{C}^k$. For $\xi\in C^\infty(M; E\otimes\pi^*\Lambda^h B)$ and $\omega\in\Omega^*(B)$, we can write

$$\begin{aligned}\partial(T)(\xi\omega) &= \nabla((T\xi)\omega) - (-1)^k T(\nabla\xi)\omega - (1)^{k+h}(T\xi)d_B\omega\\ &= \nabla(T\xi)\omega + (-1)^{h+k}(T\xi)d_B\omega - (-1)^k T(\nabla\xi)\omega - (1)^{k+h}(T\xi)d_B\omega\\ &= (\partial(T)\xi)\omega.\end{aligned}$$

□

Proposition 1.5. *The derivation $\partial:\mathcal{C}^*\to\mathcal{C}^*$ preserves the subspace $\psi^{-\infty}(M|B, E;\Lambda^*B)$. More precisely, the derivation ∂ preserves each $\psi^h(M|B, E;\Lambda^*B)$ for $h\in\mathbb{Z}$.*

Proof. A classical computation shows that, in local coordinates over a small open set V of M, the quasi-connection ∇ has the expression

$$\nabla = d^\nu\otimes I_N + \omega, \qquad \text{with } \omega\in M_N(C^\infty(V, \pi^*(T^*B))),$$

where we have used a vector bundle isomorphism $E|_V\to V\times\mathbb{C}^N$ and d^ν denotes the transverse de Rham derivative (in the direction $\pi^*(\Lambda T^*B)$) given with obvious notations by $d^\nu = \varrho\circ d$. Taking commutators with the 0-th order term ω clearly preserves $\psi^h(V|B, E;\Lambda^*B)$, since ω belongs to $\psi^0(V|B, E;\Lambda^1 B)$. Using a trivialization

$$(x;b) = (x_1,\cdots,x_p;b_1,\cdots,b_q): V\longrightarrow\mathbb{R}^p\times\mathbb{R}^q.$$

of the fibration $V\to U$ and the open set $U\subset B$, one finds that there exists $A\in M_{q,p}(C_c^\infty(\mathbb{R}^n))$ such that

$$d^\nu(f) = \sum_{i=1}^q[\frac{\partial f}{\partial b_i} + \sum_{j=1}^p A_{ij}\frac{\partial f}{\partial x_j}]db_i.$$

The expression $\sum_{i=1}^q db_i \sum_{j=1}^p A_{ij} \frac{\partial}{\partial x_j}$ is an element of $\psi^1(V|B;\Lambda^1 B)$, a scalar operator. Therefore, when tensored with the identity of $\mathbb{C}^N$ it has a diagonal matrix as fiberwise principal symbol. Such a diagonal matrix graded commutes with the principal symbol of any element of $\psi^h(V|B,E;\Lambda^* B)$. So, the commutator with this operator preserves the order of the pseudodifferential operators. It thus remains to compute the commutator of $d_B = \sum_{i=1}^q db_i \frac{\partial}{\partial b_i}$ with an element of $\psi^h(V|B,E;\Lambda^* B)$. Since the elements P of $\psi^h(M|B,E;\Lambda^* B)$ are $\Omega^*(B)$-linear, we can restrict to $P \in \psi^h(M|B,E)$ and by pseudolocality, we can even assume that $P \in \psi^h(V|B, V\times\mathbb{C}^N)$ is given in the local coordinates (x,b) by

$$P(f)(x;b) = \frac{1}{(2\pi)^{p/2}} \int_{V_b\times\mathbb{R}^p} a(x;b;\xi) e^{i(x-x')\xi} f(x';b) dx' d\xi,$$

where $f \in C_c^\infty(V,\mathbb{C}^N)$ and a is the local total symbol of the classical fiberwise pseudodifferential operator P, an $N\times N$ matrix. A simple computation shows that the commutator $[d_B\otimes I_N, P]$ is given by the same local formula but replacing a by the matrix $d_B(a)$. Since this latter matrix is a classical symbol of order h, the proof is complete. □

1.2. **Bivariant cochains.** As before we are considering a smooth locally trivial fibration $\pi: M \to B$ of closed manifolds, together with the fiberwise generalized Dirac operator D acting on the smooth sections of the Clifford bundle E over M. The dimension of the fibers is denoted by p and the dimension of the base is p'. Recall that $\Omega^*(B,E)$ is the space of smooth sections over M of the bundle $E\otimes\pi^*\Lambda^* T^* B$. This is a graded module over the graded Grassmann algebra of differential forms on the base manifold B. We defined in Section 1.1 the quasi-connection ∇ associated with a connection on E and the choice of a horizontal distribution H.

Our bivariant (n,k)-cochains are linear maps $f: \mathcal{B}_n = \mathcal{A}^{\otimes n} \to L_k$ where $L = \oplus_k L_k$ is a graded vector space (or a graded module over some algebra) or a graded algebra that will often be endowed with a 'connection'. More precisely, we are interested in the graded spaces

$$L = \Omega^*(B;E), \quad L = \Omega^*(B), \quad L_{-\infty} = \psi^{-\infty}(M|B,E;\Lambda^* B) \text{ and } L = \mathcal{L}(\mathcal{E}).$$

Here $\psi^{-\infty}(M|B,E;\Lambda^* B)$ is the algebra of fiberwise smoothing operators with coefficients in forms. The space $\mathcal{E}$ is the Hilbert module over the C^*-algebra $C(\Lambda^* T^* B)$, of continuous sections of $\Lambda^* T^* B$ over B, which is the completion of the smooth sections over M of the bundle $E\otimes\pi^*\Lambda^* T^* B$. (For background on Hilbert C* modules see [22].) We choose connections on these graded spaces (except for the last one)

$$\nabla, d_B = \text{de Rham differential and } [\nabla,\cdot]$$

respectively. Note that we use the convention that all the commutators are graded ones. The space $\mathrm{Hom}(\mathcal{B},L)$ will then be bi-graded and we shall use the total grading for the commutators. For any $P\in\psi^h(M|B,E;\Lambda^* B)$ and any $b\in B$, the operator P_b belongs to $\psi^h(M_b, E|_{M_b})\otimes\Lambda^*(T_b^* B)$. Therefore, for $h\le 0$, P_b extends to a $\Lambda^*(T_b^* B)$-linear bounded operator of the Hilbert space $L^2(M_b, E|_{M_b})\otimes\Lambda^*(T_b^* B)$, hence an element of

$$B(L^2(M_b, E|_{M_b}))\otimes\Lambda^*(T_b^* B).$$

Next, using a basis of the finite dimensional vector space $\Lambda^*(T_b^* B)$), we define a $\Lambda^*(T_b^* B)$-valued graded trace

$$\tau : L^1(L^2(M_b, E|_{M_b}))\otimes\Lambda^*(T_b^* B). \longrightarrow \Lambda^*(T_b^* B),$$

where $L^1(L^2(M_b, E|_{M_b}))$ is the usual ideal of trace class operators on the Hilbert space $L^2(M_b, E|_{M_b})$. The expression "graded trace" means that τ vanishes on graded commutators, the grading being produced by the degree of the forms in $\Lambda^*(T_b^* B)$. The classical theory of pseudodifferential operators shows that $\psi^h(M_b, E|_{M_b}) \subset L^1(L^2(M_b, E|_{M_b}))$, for $h < -p$. Putting these traces together, we inherit a graded trace

$$\tau : L_h = \psi^h(M|B,E;\Lambda^* B) \longrightarrow \Omega^*(B) \text{ for any } h < -p.$$

We shall for simplicity restrict τ to fiberwise smoothing operators and only consider

$$\tau : L_{-\infty} = \psi^{-\infty}(M|B,E;\Lambda^* B) \longrightarrow \Omega^*(B).$$

Lemma 1.6. *The graded trace τ is closed, i.e. it satisfies the relation $\tau\circ\partial = d_B\circ\tau$.*

Proof. (*cf* [5]). Fix a $P \in \psi^{-\infty}(M, E \otimes \Lambda^*(T^*B))$. Using the module structure, we can assume that $P \in \psi^{-\infty}(M, E)$. Notice that each such P has a smooth Schwartz kernel k_P. In local coordinates, $\nabla = d^\nu + M_\omega$ with $d^\nu = \varrho \circ d$ the de Rham derivative in the direction $\pi^*(\Lambda T^*B)$ and M_ω a zero-th order differential operator with coefficients in 1-forms on B. Clearly, the trace of the commutator $[M_\omega, P]$ is the integral of the trace of a commutator and hence is trivial. Using compactly supported smooth cut-off functions, we can assume that the smooth kernel k_P is supported within a trivial open set diffeomorphic to $U \times U' \times W$ where U and U' are trivializing open sets in the typical fiber manifold F and W is a trivializing open set in the base B. The operator d^ν is given in the local coordinates $(x_1, \cdots, x_p; b_1, \cdots, b_q)$ of $U \times W$, by

$$d^\nu = d_B + \sum_{i=1}^q db_i \sum_{j=1}^p A_{ij} \frac{\partial}{\partial x_j} \text{ with } A \in M_{q,p}(C_c^\infty(U \times W)) \text{ and } d_B = \sum_{i=1}^q db_i \frac{\partial}{\partial b_i}.$$

We observe that

$$\tau([d_B, P])(b) = \int_{M_b} (d_B k_P)(b; x, x) dx = (d_B \circ \tau)(P)(b).$$

It thus remains to show that $\tau([\frac{\partial}{\partial x_j}, P]) = 0$. But since P is fiberwise smoothing and $\frac{\partial}{\partial x_j}$ is a fiberwise first order differential operator, the proof is complete. □

We follow [27] and introduce an extra Clifford variable σ of degree 1 and central in the graded sense (it graded commutes with all operators). Hence we replace the algebra $L_{-\infty}$ by $L_{-\infty}[\sigma]$. We assume that the fibers of our fibration are odd dimensional, so p is odd. Then we extend the graded closed trace τ so that

$$\partial : L_{-\infty}[\sigma] \to L_{-\infty}[\sigma] \text{ and } \tau_\sigma : L_{-\infty}[\sigma] \longrightarrow L = \Omega^*(B),$$

by setting $\tau_\sigma(T + \sigma S) := \tau(S)$. We may also consider the algebra $\psi^\infty(M, E; \Lambda^* B)[\sigma]$ in the sequel. The total degree of an element A in one of these extensions then takes into account σ and will be denoted $|A|$.

Lemma 1.7. *The map τ_σ is a graded trace on the Clifford extension $L_{-\infty}[\sigma]$, with values in the graded algebra $\Omega^*(B)$. Moreover, it satisfies the relation $\tau_\sigma \circ \partial + d_B \circ \tau_\sigma = 0$.*

Proof. Let $A = T + \sigma S$ and $A' = T' + \sigma S'$ be elements of $L_{-\infty}[\sigma]$ with degrees k and k' respectively. This means that T and T' have respectively degrees k and k' while S and S' have respectively degrees $(k-1)$ and $(k'-1)$. We then compute

$$\begin{aligned} \tau_\sigma(AA') &= (-1)^k \tau(TS') + \tau(ST') \\ &= (-1)^k (-1)^{k(k'-1)} \tau(S'T) + (-1)^{(k-1)k'} \tau(T'S) \\ &= (-1)^{kk'} \left[\tau(S'T) + (-1)^{k'} \tau(T'S)\right] \\ &= (-1)^{kk'} \tau_\sigma(A'A). \end{aligned}$$

In the same way we have

$$\tau_\sigma([\nabla, T] - \sigma[\nabla, S]) = -\tau([\nabla, S]) = -d_B \tau(S) = -d_B \tau_\sigma(T + \sigma S).$$

□

1.3. **The heat semigroup and Duhamel.** The operator ∇ is used to associate with the generalized Dirac operator D, different superconnections [9]. They will be for us given by

$$\mathbb{A}_\sigma(A) := \mathbb{B}_\sigma + A,$$

where A is a zero-th order fiberwise pseudodifferential operator with coefficients in differential forms of positive degree (in applications ≥ 2) on B. Here, the superconnection $\mathbb{B}_\sigma$ is defined by

$$\mathbb{B}_\sigma = \nabla + \sigma D.$$

We are mainly interested in the superconnection $\mathbb{B}_\sigma$ or in the Bismut superconnection together with its metric rescalings. Inorder to simplify the exposition of the analyticity results, we restrict ourselves to the case $\mathbb{A}(0) = \mathbb{B}_\sigma$ and shall briefly explain later how the proofs extend to the general case.

We have $\mathbb{B}_\sigma^2 = D^2 + X$ where $X = \nabla^2 - \sigma[\nabla, D]$. Note that the operator X is a fiberwise differential operator of order one with coefficients in differential forms of positive degree ≤ 2.

Definition 1.8. *Following* [8] *we will use the notation* $e^{-u\mathbb{B}_\sigma^2}$ *to denote the semigroup (that is, the solution to the heat equation) given by the following finite perturbative sum of strong integrals*

$$e^{-u\mathbb{B}_\sigma^2} = \sum_{m\geq 0} (-u)^m \int_{\Delta(m)} e^{-uv_0 D^2} X e^{-uv_1 D^2} \cdots X e^{-uv_m D^2} dv_1 \cdots dv_m.$$

where $\Delta(m) = \{(u_0, \cdots, u_m) \in \mathbb{R}^{m+1}, \sum u_j = 1\}$ *is the m-simplex.*

Since the base manifold B is finite dimensional, the above sum is finite. Note also that classical results show that the operator D is a self-adjoint regular operator on the Hilbert $C(B, \Lambda^* B)$-module $\mathcal{E}$, see for instance [29] or [7]. Hence the heat operator e^{-tD^2} can be viewed as an adjointable (bounded) operator on $\mathcal{E}$, say it belongs to $\mathcal{L}(\mathcal{E})$. Moreover, since the operator e^{-tD^2} is a smoothing operator, the heat operator $e^{-u\mathbb{B}_\sigma^2}$ associated with the superconnection $\mathbb{B}_\sigma$ defined above, is also a smoothing operator but with coefficients in differential forms of the base B. As a consequence, for any $u > 0$, the fiberwise graded trace $\tau_\sigma(e^{-u\mathbb{B}_\sigma^2})$ makes sense as a differential form on the base manifold B.

Lemma 1.9. *(Duhamel principle) For any element* A *of the algebra* $\psi^\infty(M|B, E; \Lambda^* B)[\sigma]$*, the following equality holds in* $\mathcal{L}(\mathcal{E})$

$$[A, e^{-\mathbb{B}_\sigma^2}] = -\int_0^1 e^{-s\mathbb{B}_\sigma^2}[A, \mathbb{B}_\sigma^2] e^{-(1-s)\mathbb{B}_\sigma^2} ds.$$

Proof. This lemma can be proved following [8]. We sketch the argument. Following [28] p. 263-264 the Duhamel formula is known to be satisfied by the family D^2 using just the fact that for ξ_0 in our Hilbert space of L^2 sections $\xi_t = e^{-tD^2}\xi_0$ solves the heat equation. If $\xi_0 \in \mathcal{E}$ then using the Schwartz kernel for e^{-tD^2} we see that $\xi_t \in \mathcal{E}$ from which Duhamel follows in $\mathcal{E}$. Now if we replace $e^{-s\mathbb{B}_\sigma^2}$ and $e^{-(1-s)\mathbb{B}_\sigma^2}$ by the finite expansion sums, we obtain

$$e^{-s\mathbb{B}_\sigma^2}[A, \mathbb{B}_\sigma^2]e^{-(1-s)\mathbb{B}_\sigma^2} = \sum_{m,m'\geq 0} (-1)^{m+m'} s^m (1-s)^{m'} \int_{\Delta(m)\times\Delta(m')}$$

$$e^{-v_0 sD^2} X e^{-v_1 sD^2} \cdots X e^{-v_m sD^2}[A, D^2 + X] e^{-w_0(1-s)D^2} X e^{-w_1(1-s)D^2} \cdots X e^{-w_{m'}(1-s)D^2} dv_1 \cdots dv_m dw_1 \cdots dw_{m'}.$$

Now if we integrate over $(0, 1)$, make a suitable change of variables using

$$\sum_{j=0}^{m} sv_j + \sum_{i=0}^{m'} (1-s)w_i = 1,$$

and apply the Duhamel principle for D^2 we obtain the result. □

It is worth pointing out that the operator $[A, e^{-\mathbb{B}_\sigma^2}]$ is a fiberwise smoothing operator with coefficients in $\Omega^*(B)$. Moreover, for any $s \in (0, 1)$ the operator $e^{-s\mathbb{B}_\sigma^2}[A, \mathbb{B}_\sigma^2]e^{-(1-s)\mathbb{B}_\sigma^2}$ is also fiberwise smoothing. It is then straightforward to check, using the dominated convergence theorem, that the following holds

$$\tau_\sigma([A, e^{-\mathbb{B}_\sigma^2}]) = -\int_0^1 \tau_\sigma(e^{-s\mathbb{B}_\sigma^2}[A, \mathbb{B}_\sigma^2]e^{-(1-s)\mathbb{B}_\sigma^2}) ds.$$

Lemma 1.10. *Consider for any* $u \in [0, 1]$*, a superconnection* $\mathbb{A}_u$*, given by* $\mathbb{A}_u := \mathbb{B}_\sigma + uA$*, where* A *is a (odd for the grading) zero-th order fiberwise pseudodifferential operator with coefficients in positive degree differential forms on* B*. Then we have*

$$\frac{d}{du} e^{-\mathbb{A}_u^2} = -\int_0^1 e^{-s\mathbb{A}_u^2}[A, \mathbb{A}_u] e^{-(1-s)\mathbb{A}_u^2} ds$$

in the strong operator topology of $\mathcal{L}(\mathcal{E})$.

Proof. We set for any small real number $h \neq 0$, $Y_u(h) := [A, \mathbb{A}_u] + hA^2$. Then from the definition of $e^{-\mathbb{A}_v^2}$ we deduce that

$$\frac{1}{h}\left[e^{-\mathbb{A}_{u+h}^2} - e^{-\mathbb{A}_u^2}\right] = \sum_{k\geq 1} h^{k-1} \int_{\Delta(k)} e^{-u_0\mathbb{A}_u^2} Y_u(h) e^{-u_1\mathbb{A}_u^2} \cdots Y_u(h) e^{-u_k\mathbb{A}_u^2} du_1 \cdots du_k.$$

Both sides are well defined as operators on $\mathcal{E}$ as they are fiberwise smoothing operators. Applying both sides to elements of $\mathcal{E}$ we end the proof by letting $h \to 0$. □

We shall also need the following lemma.

Lemma 1.11. *For any $A_0, \cdots, A_n$ in the algebra $\psi^\infty(M|B, E; \Lambda^* B)[\sigma]$ we have,*

$$\tau_\sigma\left([\sigma D, A_0 e^{-u_0\mathbb{B}_\sigma^2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2}]\right) = 0 \text{ for } u_j > 0.$$

Proof. By definition $e^{-u_j\mathbb{B}_\sigma^2}$ is given by the perturbative sum

$$e^{-u_j\mathbb{B}_\sigma^2} = \sum_{m\geq 0} (-u_j)^m \int_{\Delta(m)} e^{-u_j v_0 D^2} X e^{-u_j v_1 D^2} \cdots X e^{-u_j v_m D^2} dv_1 \cdots dv_m.$$

For $u_0 > 0$, we know that the operator $\sigma D A_0 e^{-u_0\mathbb{B}_\sigma/2}$ is fiberwise smoothing with coefficients in differential forms and that its degree is $|A_0| + 1$. Therefore, the graded tracial property of τ_σ shows that

$$\tau_\sigma\left[(DA_0 e^{-u_0\mathbb{B}_\sigma^2/2})(e^{-u_0\mathbb{B}_\sigma^2/2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2})\right] = (-1)^{(|A_0|+1)\sum_{j=1}^n |A_j|} \tau_\sigma\left[(e^{-u_0\mathbb{B}_\sigma^2/2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2} \sigma D)(A_0 e^{-u_0\mathbb{B}_\sigma^2/2})\right]$$

Now as before, the operator $A_0 e^{-u_0\mathbb{B}_\sigma^2/2}$, as well as

$$e^{-u_0\mathbb{B}_\sigma^2/2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2} \sigma D$$

are fiberwise smoothing operators with coefficients in differential forms. Therefore,

$$\tau_\sigma\left[(e^{-u_0\mathbb{B}_\sigma^2/2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2} \sigma D)(A_0 e^{-u_0\mathbb{B}_\sigma^2/2})\right] = (-1)^{|A_0|(1+\sum_{j=1}^n |A_j|)} \tau_\sigma\left[(A_0 e^{-u_0\mathbb{B}_\sigma^2/2})(e^{-u_0\mathbb{B}_\sigma^2/2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2} \sigma D)\right].$$

Hence we have

$$\tau_\sigma\left[\sigma D A_0 e^{-u_0\mathbb{B}_\sigma^2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2}\right] = (-1)^{\sum_{j=0}^n |A_j|} \tau_\sigma\left[A_0 e^{-u_0\mathbb{B}_\sigma^2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2} \sigma D\right].$$

□

2. Multilinear functionals and identities

In this Section we record some useful identities satisfied by the multilinear functionals that enter into the bivariant JLO cocycle. Let $A_i \in \psi^\infty(M|B, E; \Lambda^* B)[\sigma]$ and with $\Delta(n)$ being as before, the n-simplex, we define multilinear functionals [17, 30]

$$\langle\langle A_0, \cdots, A_n\rangle\rangle_{\mathbb{B}_\sigma} := \int_{\Delta(n)} \tau_\sigma(A_0 e^{-u_0\mathbb{B}_\sigma^2} A_1 e^{-u_1\mathbb{B}_\sigma^2} \cdots A_n e^{-u_n\mathbb{B}_\sigma^2}) du_1 \cdots du_n \quad \in \Omega^*(B).$$

and

$$< A_0, \cdots, A_n >:= \int_{\Delta(n)} \tau_\sigma(A_0 e^{-u_0 D^2} \cdots A_r e^{-u_n D^2}) du_1 \cdots du_n \quad \in \Omega^*(B).$$

The following lemma is stated in the case of flat connections in [30] and is a straightforward extension of [17][Lemma 2.2]. We give the proof for completeness and since it will be used in the sequel.

Lemma 2.1. *Let $A_0, \cdots, A_n \in \psi^\infty(M|B, E; \Lambda^* B)[\sigma]$ and let $\epsilon_i = (|A_0| + \cdots + |A_{i-1}|)(|A_i| + \cdots + |A_n|)$.*

- *For $1 \leq i \leq n$,* $\langle\langle A_0, \cdots, A_n\rangle\rangle = (-1)^{\epsilon_i} \langle\langle A_i, \cdots, A_n, A_0, \cdots, A_{i-1}\rangle\rangle$;
- $\sum_{i=0}^n \langle\langle A_0, \cdots, A_i, 1, A_{i+1}, \cdots, A_n\rangle\rangle = \langle\langle A_0, \cdots, A_n\rangle\rangle$;

- $\langle\langle [\mathbb{B}_\sigma, A_0], A_1, \cdots, A_n\rangle\rangle + \sum_{i=1}^n (-1)^{|A_0|+\cdots+|A_{i-1}|}\langle\langle A_0, \cdots, [\mathbb{B}_\sigma, A_i], \cdots, A_n\rangle\rangle + d_B\langle\langle A_0, \cdots, A_n\rangle\rangle = 0$;
- *For* $0 \leq i < n$, $\langle\langle A_0, \cdots, A_{i-1}A_i, A_n\rangle\rangle - \langle\langle A_0, \cdots, A_iA_{i+1}, A_n\rangle\rangle = \left\langle\left\langle A_0, \cdots, [\mathbb{B}_\sigma^2, A_i], A_n\right\rangle\right\rangle$; *and for* $i = n$,

$$\langle\langle A_0, \cdots, A_{n-1}A_n\rangle\rangle - (-1)^{(|A_0|+\cdots+|A_{n-1}|)|A_n|}\langle\langle A_nA_0, A_1, \cdots, A_{n-1}\rangle\rangle = \left\langle\left\langle A_0, \cdots, A_{n-1}, [\mathbb{B}_\sigma^2, A_n]\right\rangle\right\rangle.$$

- *Let* $(\mathbb{B}_{\sigma,s} = \mathbb{B}_\sigma + sA)_s$ *be a 1-parameter family of superconnections associated with* σD *as in Lemma 1.10, then we have*

$$\frac{d}{ds}\langle\langle A_0, \cdots, A_n\rangle\rangle_{\mathbb{B}_{\sigma,s}} + \sum_{i=0}^n \langle\langle A_0, \cdots, A_i, [\mathbb{B}_{\sigma,s}, A], A_{i+1}, \cdots, A_n\rangle\rangle_{\mathbb{B}_{\sigma,s}} = 0.$$

Proof. The first relation is a consequence of the fact that τ_σ is a graded trace. Indeed, $\mathbb{B}_\sigma^2$ is homogeneous of even total degree. The second relation is also clear, see [17] for more details. Let us check the third relation. From the perturbative finite sum which defines $e^{-u\mathbb{B}_\sigma^2}$ we have seen in Lemma 1.9 that the Duhamel principle holds, hence we obtain the following Bianchi identity

$$[\mathbb{B}_\sigma, e^{-u\mathbb{B}_\sigma^2}] = -\int_0^1 e^{-us\mathbb{B}_\sigma^2}[\mathbb{B}_\sigma, \mathbb{B}_\sigma^2]e^{-u(1-s)\mathbb{B}_\sigma^2}ds = 0.$$

Therefore, the left hand side of the third relation coincides with

$$\int_{\Delta(n)} \tau_\sigma\left([\mathbb{B}_\sigma, A_0e^{-u_0\mathbb{B}_\sigma^2}A_1e^{-u_1\mathbb{B}_\sigma^2}\cdots A_ne^{-u_n\mathbb{B}_\sigma^2}]\right)du_1\cdots du_n + d_B\langle\langle A_0, \cdots, A_n\rangle\rangle.$$

On the other hand, by Lemma 1.11

$$\tau_\sigma([\sigma D, A_0e^{-u_0\mathbb{B}_\sigma^2}A_1e^{-u_1\mathbb{B}_\sigma^2}\cdots A_ne^{-u_n\mathbb{B}_\sigma^2}]) = 0.$$

Hence, the left hand side of the third relation coincides with

$$\int_{\Delta(n)}\left[\tau_\sigma\left([\nabla, A_0e^{-u_0\mathbb{B}_\sigma^2}A_1e^{-u_1\mathbb{B}_\sigma^2}\cdots A_ne^{-u_n\mathbb{B}_\sigma^2}]\right) + d_B\left(\tau_\sigma(A_0e^{-u_0\mathbb{B}_\sigma^2}A_1e^{-u_1\mathbb{B}_\sigma^2}\cdots A_ne^{-u_n\mathbb{B}_\sigma^2})\right)\right]du_1\cdots du_n.$$

Lemma 1.7 then completes the proof of the third item.

The fourth relation is again a consequence of Lemma 1.9 and is a straightforward generalization of the similar relation proved for a single operator in [17].

Now, notice that

$$\frac{d}{ds}\langle\langle A_0, \cdots, A_n\rangle\rangle_{\mathbb{B}_{\sigma,s}} = \sum_{i=0}^n\int_{\Delta(n)}\tau_\sigma(A_0e^{-u_0\mathbb{B}_{\sigma,s}^2}\cdots A_i\frac{de^{-u_i\mathbb{B}_{\sigma,s}^2}}{ds}A_{i+1}e^{-u_{i+1}\mathbb{B}_{\sigma,s}^2}\cdots A_ne^{-u_n\mathbb{B}_{\sigma,s}^2})du_1\cdots du_n.$$

But, Duhamel's formula 1.10 shows that

$$\frac{de^{-\mathbb{B}_{\sigma,s}^2}}{ds} + \int_0^1 e^{-u\mathbb{B}_{\sigma,s}^2}[\mathbb{B}_{\sigma,s}, A]e^{-(1-u)\mathbb{B}_{\sigma,s}^2}du = 0.$$

The proof is thus complete. □

Definition 2.2. *The bivariant JLO cochain is defined by the sequence* $(\psi_n)_n$ *given for* $(f_0, \cdots, f_n) \in C^\infty(M)^{n+1}$ *by the formula*

$$\psi_n(f_0, \cdots, f_n) := \langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle_{\mathbb{B}_\sigma}.$$

One deduces from Lemma 4.3 that for any C^1 function f on the closed manifold M, the commutator $[\mathbb{B}_\sigma, f]$ is a bounded operator with values in horizontal 1-forms. For simplicity, we work with smooth functions and smooth forms, although the constructions work obviously with less regularity, and leave it to the interested reader to transpose the statements for more restrictive regularity conditions.

Lemma 2.3. *The sequence $\psi = (\psi_n)_{n\geq 0}$ for n odd (resp. for n even) is an odd (formal) cyclic cocycle for the algebras $(C^\infty(M), C^\infty(B))$, i.e. it maps the universal differential algebra of $C^\infty(M)$ to the graded Grassmann algebra $\Omega^*(B)$, its components of odd form degree (resp. of even form degree) are trivial, and it satisfies the cocycle relation*

$$\psi\circ(b+B)+d_B\circ\psi=0.$$

Remark 2.4. *The above lemma is stated under the assumption that the fibers are odd dimensional. When these fibers are even dimensional an analogous result holds.*

Proof. The third relation in Lemma 2.1 applied to $A_0 = f_0$ and $A_i = [\mathbb{B}_\sigma, f_i]$ for $i \geq 1$ gives

$$\langle\langle[\mathbb{B}_\sigma,f_0],\cdots,[\mathbb{B}_\sigma,f_n]\rangle\rangle - \sum_{i=1}^n(-1)^i\left\langle\left\langle f_0,[\mathbb{B}_\sigma,f_1],\cdots,[\mathbb{B}_\sigma^2,f_i],\cdots,[\mathbb{B}_\sigma,f_n]\right\rangle\right\rangle + d_B\left\langle\left\langle f_0,[\mathbb{B}_\sigma,f_1],\cdots,[\mathbb{B}_\sigma,f_n]\right\rangle\right\rangle=0.$$

On the other hand computing $(B\psi_{n+1})(f_0,\cdots,f_n)$, we find, using the second item of Lemma 2.1:

$$(B\psi_{n+1})(f_0,\cdots,f_n)=\langle\langle[\mathbb{B}_\sigma,f_0],\cdots,[\mathbb{B}_\sigma,f_n]\rangle\rangle\,.$$

Using the last item of Lemma 2.1, we finally deduce that

$$(b\psi_{n-1})(f_0,\cdots,f_n)+\sum_{i=1}^n(-1)^i\left\langle\left\langle f_0,[\mathbb{B}_\sigma,f_1],\cdots,[\mathbb{B}_\sigma^2,f_i],\cdots,[\mathbb{B}_\sigma,f_n]\right\rangle\right\rangle=0.$$

Therefore, we obtain the desired result $B\psi_{n+1} + b\psi_{n-1} + d_B\psi_n = 0$. □

3. Analyticity of the bivariant JLO cocycle

We are using cyclic homology for bornological algebras due to R. Meyer [24] and will need some preliminaries which we now describe.

3.1. Review of bivariant analytic cyclic homology.

For the convenience of the reader, we summarise in this subsection what we need about bivariant analytic homology. The reader is encouraged to consult [24] for more details, especially for the definitions and properties of bornologies. See also [11] for the basic (non-trivial) concepts of bornological functional analysis. Our task here is to adapt this formalism to the families situation.

The idea of using a bornology in the study of entire cyclic cohomology is due to Connes [15] (see pages 370-371). Given a locally convex topological algebra A, it is proposed there to use the bounded subsets on A to define entire cyclic cohomology. Meyer develops this idea using bornological functional analysis in a form that is appropriate for this paper in [24].

Here A, A_1, A_2 etc will denote complete convex topological algebras. We will use a family of different bornologies on A_j denoted generically by $\mathfrak{S}(A_j)$. An algebra A equipped with a particular bornology will be denoted $(A, \mathfrak{S}(A))$. Denote by ΩA the universal differential graded algebra of A. Following [24], Section 3, we introduce the following notions.

Definition 3.1. *(i) For $S \in \mathfrak{S}(A)$ $(dS)^\infty$ denotes the union over n of elements $ds_1ds_2\ldots ds_n \in \Omega^n(A)$ where $s_1,\ldots,s_n$ are from S. Let $\langle S\rangle = S\cup\{1\}$ where 1 denotes an additional unit and not the identity of A (we use A^+ to denote the adjunction of this unit to A) and then define*

$$\langle S\rangle(dS)^\infty = S(dS)^\infty\cup(dS)^\infty\cup S\subset\Omega A,$$

$$S(dS)^{ev}=\langle S\rangle(dS)^\infty\cap\Omega^{ev}A,$$

$$\langle S\rangle(dS)^{odd}=\langle S\rangle(dS)^\infty\cap\Omega^{odd}A.$$

(ii) The notation $\langle a_0\rangle da_1\dots da_n \in \Omega^n A$ *means either* $a_0da_1\dots da_n$ *or* $da_1\dots da_n$ *depending on context.*
(iii) $\mathfrak{S}_{an}$ *is the bornology on* ΩA *generated by* $\langle S\rangle (dS)^\infty$ *for all* $S\in\mathfrak{S}(A)$ *and* $\Omega_{an}A$ *denotes the completion of* ΩA *in the bornology* $\mathfrak{S}_{an}$*. Equivalently* $\mathfrak{S}_{an}$ *is generated by the union over* n *of the sets*

$$\{\langle s_0\rangle ds_1\ ds_2\dots ds_n | s_j\in S, S\in\mathfrak{S}(A)\}.$$

Remark 3.2. *If* A *is Fréchet, then* A *is already complete in the bornology given by taking the bounded sets in the Fréchet topology, see*[26] *for instance.*

In this paper A will always be one of the Frèchet algebras $C^\infty(M)$ or $C^\infty(B)$ where $F\to M\to B$ is a fibration of compact smooth manifolds. However they will be equipped with bornologies defined by the subsets bounded in certain families of norms.

If $(V,\mathfrak{S}(V))$ and $(W,\mathfrak{S}(W))$ are complete convex bornological spaces then bounded linear maps $\ell: V\to W$ are linear maps with the property that $\ell(S)\in\mathfrak{S}(W)$ whenever $S\in\mathfrak{S}(V)$. This notion extends to multilinear maps as well. Moreover bounded linear maps $\ell:\Omega_{an}A\to W$ are in bijection with bounded linear maps on ΩA equipped with the bornology $\mathfrak{S}_{an}$. These in turn are in bijection with linear maps $\ell:\Omega A\to W$ satisfying $\ell(\langle S\rangle(dS)^\infty)\in\mathfrak{S}(W)$, for any $S\in\mathfrak{S}(V)$.

We now explain some key results. We denote by $n!\mathfrak{S}_{an}$ the bornology on ΩA generated by the union over n and $S\in\mathfrak{S}(A)$ of the sets $n!\langle S\rangle\langle dS\rangle(dS)^{2n}$ which are defined to be

$$\{n!\langle s_0\rangle ds_1\ ds_2\dots ds_{2n}, | s_j\in S\}\cup\{n!\langle s_0\rangle ds_1\ ds_2\dots ds_{2n+1}|s_j\in S\}.$$

Let $\mathcal{C}(A)$ be the algebra ΩA completed in the bornology $n!\mathfrak{S}_{an}$. Meyer equips $\Omega_{an}(A)$ with the Hochschild boundary b and then with Connes' operator B satisfying the usual relations $b^2=0=B^2=Bb+bB$ and thus defining a bicomplex and the cyclic homology of $\Omega_{an}A$. The pair (b,B) extend also to bounded maps on $C(A)$. Then $(C(A),b+B)$ is a $\mathbb{Z}_2$-graded complex of complete bornological vector spaces.

For us the key result is that Meyer shows that his analytic cyclic cohomology of A (being the homology of $((\Omega_{an}(A))',b+B)$, where $(\Omega_{an}(A))'$ denotes bounded maps to the scalars) is the same as Connes' entire cyclic cohomology of A. The idea of the proof is to consider the dual complex $C(A)'$ of bounded linear maps $C(A)\to\mathbb{C}$. These are just bounded linear maps $(\Omega A, n!\mathfrak{S}_{an})\to\mathbb{C}$. The bounded linear functionals on $(\Omega A, n!\mathfrak{S}_{an})$ are those linear maps $\Omega A\to\mathbb{C}$ that remain bounded on all sets of the form $n!\langle S\rangle\langle dS\rangle(dS)^{2n}$. Identifying $\Omega A\cong\sum_{n=0}^\infty \Omega^n A$ and $\Omega^n A\cong A^+\hat\otimes A^{\hat\otimes^n}$, $C(A)'$ becomes the space of families $(\phi_n)_{n\in\mathbb{Z}_+}$ of $n+1$-linear maps $\phi_n: A^+\times A^n\to\mathbb{C}$ satisfying the entire growth condition

$$|\phi_n(\langle a_0\rangle, a_1,\dots,a_n)|\le const(S)/[n/2]!$$

for all $\langle a_0\rangle\in\langle S\rangle, a_1\dots,a_n\in S$ and for all $S\in\mathfrak{S}(A)$. Here $[n/2]:=k$ if $n=2k$ or $n=2k+1$ and $const(S)$ is a constant depending on S but not on n. The boundary on $C(A)$ is composition with $B+b$.

We may also use the results of [24] to define the bivariant cyclic cohomology of a pair A_1,A_2 to be the homology of the complex of bounded linear maps from $\Omega_{an}(A_1)$ to $\Omega_{an}(A_2)$. In this paper we also consider the smooth exterior algebras $\Omega^*(M),\Omega^*(B)$ (that is smooth sections of the exterior bundle) associated with the smooth manifolds M and B. Regarding these as differential graded algebras we may use the same notation and definitions as we introduced above for the universal differential graded algebras $\Omega C^\infty(M)$ and $\Omega C^\infty(B)$ for these smooth differential algebras $\Omega^*(M),\Omega^*(B)$. We will equip these smooth algebras with various bornologies which we give in the next subsection.

3.2. **The analyticity theorem.** We denote as usual for any $\ell\ge 0$, by $C^\ell(M)$ the space of complex valued functions on the smooth manifold M which are of class C^ℓ. Then, the semi-norms

$$p_q(f):=\sup_{\|X_j\|\le 1}\|X_1\circ\cdots\circ X_q(f)\|_\infty,\quad 0\le q\le\ell \text{ and } X_j \text{ are vector fields on } M,$$

induce a structure of a Banach algebra on $C^\ell(M)$. We shall denote by Σ_ℓ the bornology on $C^\ell(M)$, and also its restriction to $C^\infty(M)$, which is given by the bounded sets of the norm $\max_{0\le q\le\ell} p_q$. We also introduce, d_{X_j} to denote the operator $i_{X_j}\circ d_B$ and the complete bornological algebra $(\Omega^*_\ell(B),\Sigma_\ell)$ of ℓ regular differential forms on B with the bornology given by the bounded sets of the Banach structure associated with the semi-norms given on $\Omega^k_\ell(B)$ by

$$p_q(\omega) := \sup_{\|Y_j\|\leq 1} \frac{1}{2^{kq}} \|d_{X_1} \circ \cdots d_{X_q}(i_{Z_1} \circ \cdots \circ i_{Z_k}\omega)\|, \quad 0 \leq q \leq \ell \text{ and } Y_j, Z_i \text{ vector fields on } B.$$

Our goal is to prove that the bivariant JLO cochain constructed in the formal spirit of Quillen's seminal paper [27], is a bounded cyclic cocycle from the analytic completion of the universal differential algebra associated with the underlying complete bornological algebra $(C^{\ell+1}(M), \Sigma_{\ell+1})$ on the one hand and the complete bornological algebra $(\Omega^*_\ell(B), \Sigma_\ell)$ of ℓ regular differential forms on B on the other hand. Again, we shall only consider smooth forms and the restriction of $\Sigma_{\ell+1}$ to them. As a corollary we shall obtain an entire bivariant cyclic cocycle, following Connes [15]. Moreover, we obtain as a corollary an analytic bivariant cocycle for the Fréchet topology on the algebra $C^\infty(M)$ and the Fréchet topology on the graded differential algebra $\Omega^*(B)$.

We are now ready to state our first theorem. Recall that the fibers of our fibration are odd dimensional. There is a similar statement in the even case.

Theorem 3.3. *For any $\ell \geq 0$, $\psi = (\psi_{2n+1})_{n\geq 0}$ is an analytic bivariant cyclic cocycle on the Banach algebras $(C^{\ell+1}(M), C^\ell(B))$. More precisely, ψ is a bounded morphism from the analytic bornological completion of the universal differential algebra of $C^{\ell+1}(M)$ to the Banach algebra of ℓ regular differential forms on B.*

Corollary 3.4. *The bivariant JLO cocycle is analytic for the Fréchet C^∞-topologies.*

Proof. If S is a bounded set for the Fréchet C^∞ topology of $C^\infty(M)$, then for any $\ell \geq 0$ S is bounded in the $C^{\ell+1}$ topology. Therefore, if A is a subset of $\langle S\rangle(dS)^\infty$ then applying Theorem 3.3, its image under the morphim defined by ψ will be contained in some set $\langle S'\rangle(dS')^\infty$ for a bounded set S' in the C^ℓ topology of $C^\infty(B)$. Since this is true for any $\ell \geq 0$, we deduce that $\psi(A)$ is bounded for the C^∞ topology. □

Corollary 3.5. *For any $U \in GL_N(C^\infty(M))$ and for any $\ell \geq 0$, the following series of differential forms on B converges in the C^ℓ-topology to a closed differential form whose cohomology class is denoted $< \mathrm{JLO}(D), U >$:*

$$\sum_{k\geq 0} (-1)^k k! \left\langle\left\langle U^{-1}, [\mathbb{B}_\sigma, U], \cdots, [\mathbb{B}_\sigma, U^{-1}], [\mathbb{B}_\sigma, U]\right\rangle\right\rangle_{2k+1}.$$

Proof. This corollary is the precise rephrasing of the following fact. The analytic bivariant cocycles over $(C^\infty(M), C^\infty(B))$ pair with analytic cyclic homology classes of $C^\infty(M)$ to yield an analytic cyclic homology class of $C^\infty(B)$ (we also need here the fact that the HKRC map is bounded). See [24] for more details. □

Recall that a sequence $(\phi_n)_{n\geq 0}$ of cochains $\phi_n : C^\infty(M)^{\otimes n+1} \to \mathbb{C}$ is entire in the sense of Connes' definition [15] for the C^s norm $\|\cdot\|_s$ if and only if for any bounded set S in $(C^\infty(M), \|\cdot\|_s)$ there exists a constant $C(S)$ such that

$$|\phi_n(f, \cdots, f_n)| \leq C(S), \quad \text{for any } f_i \in S \text{ and any } n \geq 0.$$

This allows us to define in the same way entire cocycles for the Fréchet topology. Another consequence of Theorem 3.3 is the following:

Corollary 3.6. *Let C be a closed current on the base manifold B of degree $N \in \{0, \cdots, \dim(B)\}$. Then the following sequence*

$$\psi^C = (< C, \psi_n >)_{n-N\in 2\mathbb{Z}+1},$$

is an entire cyclic cocycle on the algebra $C^\infty(M)$. Moreover, the following series converges in $\mathbb{C}$ to the pairing of the Chern character of U with the composition of $\mathrm{JLO}(D)$ with C:

$$\sum_{k\geq 0} (-1)^k k! \left(C, \left\langle\left\langle U^{-1}, [\mathbb{B}_\sigma, U], \cdots, [\mathbb{B}_\sigma, U^{-1}], [\mathbb{B}_\sigma, U]\right\rangle\right\rangle_{2k+1}\right).$$

Proof. Computing $(b+B)\psi^C$ we find

$$b\psi^C_{n-1} + B\psi^C_{n+1} = < C, b\psi_{n-1} + B\psi_{n+1} > = - < C, d_B\psi_n > = 0.$$

The last equality is true since C is closed. It remains to check the entire property. Notice that the closed current C defines a periodic cohomology class on $C^\infty(B)$ and hence an analytic class in particular. Composing this class with the JLO analytic bivariant cocycle, we obtain an analytic bivariant class over $C^\infty(M)$ and the algebra $\mathbb{C}$ of scalars. Now, translating the analytic property into the entire property, see [24], we conclude. □

4. Proof of Theorem 3.3

The proof is long and will be split into many subparts.

4.1. **Estimates.** The proof of Theorem 3.3 rests on establishing some estimates on our bivariant JLO functional. We collect the preliminary facts in this subsection. We denote by $d_v f$ the differential of f in the fiberwise direction and by $d_H f$ the differential of f in the horizontal direction defined by the horizontal distribution H. We choose the metric on M so that H and the fiberwise bundle T_vM are orthogonal. Recall that if c is the fiberwise Clifford representation then for $f \in C^\infty(M)$ we have $[D, f] = c(d_v f)$. Note also that $[\nabla, f] = d_H f \wedge \cdot$

For a vector field Y on B, we denote by $\tilde{Y}$ the horizontal vector field on M satisfying $\pi_*\tilde{Y} = Y$. Recall that ∂ denotes the graded (with respect to the degree of the forms) commutator associated with the quasi-connection ∇, a (exterior) graded derivation of the algebra $\psi^\infty(M|B, E; \Lambda^*B)$ of fiberwise pseudodifferential operators with coefficients in horizontal differential forms. We denote, for any horizontal (i.e. H valued) vector field Z on M, by ∇_Z the composition $i_Z \circ \nabla$ where i_Z is contraction by Z. As usual, for $P \in \psi^h(M|B, E; \Lambda^k B)$ we also denote by $\partial_Z(P)$ the element $[\nabla_Z, P]$ of $\psi^h(M|B, E; \Lambda^k B)$.

If $P \in \psi^h(M|B, E)$ with $h \le 0$, then we set

$$\|P\| := \sup_{b\in B} \|P_b\|,$$

where the norm $\|P_b\|$ is the operator norm on the L^2 sections. In general, if $P \in \psi^h(M|B, E; \Lambda^k B)$ for $h \le 0$ and $k \ge 0$ we define the uniform norm of P by the same expression, except that now $\|P_b\|$ is obtained by taking the supremum over k-multivectors $Z \in \Lambda^k(T_bB)$ of norm ≤ 1, of the operator norms $\|i_Z P_b\|$.

Lemma 4.1. *For any $q \ge 0$,*

$$\sup_{\|Y_1\|\le 1,\cdots,\|Y_q\|\le 1} \|(I+D^2)^{-1/2}[\partial_{\tilde{Y}_1}\cdots\partial_{\tilde{Y}_q}](D^2)(I+D^2)^{-1/2}\| = \alpha_q(D^2) < +\infty,$$

$$\sup_{\|Y_1\|\le 1,\cdots,\|Y_q\|\le 1} \|(I+D^2)^{-1/2}[\partial_{\tilde{Y}_1}\cdots\partial_{\tilde{Y}_q}](D)\| < +\infty$$

and

$$\sup_{\|Y_1\|\le 1,\cdots,\|Y_q\|\le 1} \|[\partial_{\tilde{Y}_1}\cdots\partial_{\tilde{Y}_q}](D)(I+D^2)^{-1/2}\| < +\infty.$$

where the Y_j's are vector fields on B. The maximum of the two last supremums for D will be denoted $\alpha_q(D)$.

Proof. The same proof works for both operators and we only give the proof for D^2. We first point out that the operator $[\partial_{Y_1}\cdots\partial_{Y_q}](D^2)$ is a second order fiberwise differential operator, with smooth coefficients. Therefore, the operator

$$(I+D^2)^{-1/2}[\partial_{\tilde{Y}_1}\cdots\partial_{\tilde{Y}_q}](D^2)(I+D^2)^{-1/2},$$

is a zero-th order fiberwise pseudodifferential operator whose norm is finite. Moreover, as B is compact, by a partition of unity argument we may assume that we are given a local orthonormal basis $(\partial_1, \cdots, \partial_b)$ of the tangent bundle to B over an open set $U \subset B$. Then, we can replace the operators ∂_{Y_j} by operators of the form $\tilde{\partial}_j := \partial_j + \omega(\partial_j)$, where ω is a matrix of differential 1-forms. Now, the finite family of operators $(I+D^2)^{-1/2}[\tilde{\partial}_{j_1}\cdots\tilde{\partial}_{j_q}](D^2)(I+D^2)^{-1/2}$, for $1 \le j_1, \cdots, j_q \le b$, is uniformly bounded over U. Since the vector fields $Y_1, \cdots Y_q$ all have norm ≤ 1, the proof is thus complete. □

Lemma 4.2. *For any $q \geq 0$,*

$$\sup_{\|Z_1\|\leq 1, \|Z_2\|\leq 1, \|Y_1\|\leq 1, \cdots, \|Y_q\|\leq 1} \|(I+D^2)^{-1/2}[\partial_{\tilde{Y}_1}\cdots\partial_{\tilde{Y}_q}](i_{\tilde{Z}_1\wedge\tilde{Z}_2}\nabla^2)\| = \beta_q < +\infty,$$

where $Z_1, Z_2, Y_1, \cdots, Y_q$ are vector fields on B that we view through their unique horizontal lifts.

Proof. The proof follows the same lines as the previous lemma. More precisely, using the compactness of B we can reduce to local coordinates. But as can be checked in these local coordinates, since the operator $i_{Z_1\wedge Z_2}\nabla^2$ is a smooth family of differential operators of order 1, the smooth family of zero-th order operators $(I+D^2)^{-1/2}[\partial_{Y_1}\cdots\partial_{Y_q}](i_{Z_1\wedge Z_2}\nabla^2)$ is uniformly bounded with bound comparable with the product $\|Z_1\| \times \|Z_2\| \times \Pi_{j=1}^q \|Y_j\|$. □

Recall that for $f \in C^\infty(M)$, $\|f\|_s := \max_{0\leq j\leq s} p_j(f)$.

Lemma 4.3. *For any $s \geq 0$, there exists a constant $C_s \geq 0$ such that*

$$\|[\partial_{\tilde{Y}_s}\circ\cdots\circ\partial_{\tilde{Y}_1}](f)\| \leq C_s\|f\|_s \text{ and } \|[\partial_{\tilde{Y}_s}\circ\cdots\circ\partial_{\tilde{Y}_1}]([D,f])\| \leq C_s\|f\|_{s+1},$$

for any $f \in C^\infty(M)$ and any vector fields Y_j of norm ≤ 1.

Proof. Since M is compact, there exists a constant $C > 0$ only depending on the distribution H such that $\|\tilde{Y}\| \leq C\|Y\|$, for any Y. On the other hand, for any $j \leq s$, we have:

$$[\partial_{\tilde{Y}_j}\circ\cdots\circ\partial_{\tilde{Y}_1}](f) = [\tilde{Y}_j\circ\cdots\circ\tilde{Y}_1](f).$$

The RHS means the multiplication operator by the function $[\tilde{Y}_j\circ\cdots\circ\tilde{Y}_1](f)$. Therefore, by definition of the semi-norm p_j, we deduce that

$$\|[\tilde{Y}_j\circ\cdots\circ\tilde{Y}_1](f)\| \leq p_j(f)\|\tilde{Y}_j\|\times\cdots\times\|\tilde{Y}_1\| \leq C^j p_j(f).$$

□

Lemma 4.4. *Fix any $\epsilon \in]0,1/2]$, then for fiberwise pseudodifferential operators $(A_j)_{0\leq j\leq N}$ and $(B_j)_{0\leq j\leq N}$ with $A_j \in \psi^0(M|B,E)$ and $B_j \in \psi^2(M|B,E)$ for any j, we have:*

$$\| < A_0, B_0, \cdots, A_N, B_N > \| \leq \left(\frac{\pi}{2\epsilon}\right)^{N+1} \|e^{-(1-\epsilon)D^2}\|_1 \times \Pi_{j=0}^N \|A_j\|\|(I+D^2)^{-1/2}B_j(I+D^2)^{-1/2}\|.$$

Remark 4.5. *It is straightforward to obtain similar estimates for pseudodifferential operators A_j, B_j under the assumption that the sum of the orders is at most $2N+1$, see Lemma 4.6 below. This extends estimates proved in* [17] *with a single operator D and with particular differential operators.*

Proof. By inspection we have

$$< A_0, B_0, \cdots, A_N, B_N > = < A_0(I+D^2)^{1/2}, (I+D^2)^{-1/2}B_0, \cdots, A_N(I+D^2)^{1/2}, (I+D^2)^{-1/2}B_N > .$$

Therefore, using Hölder's inequality fiberwise, we obtain (writing $d\underline{u} = du_1\cdots du_N, d\underline{v} = dv_0dv_1\cdots dv_N$):

$$\begin{aligned}
&\| < A_0, B_0, \cdots, A_N, B_N > \| \\
&\leq \int_{\Delta(2N+1)} \Pi_{j=0}^N \|A_j(I+D^2)^{1/2}e^{-u_jD^2}\|_{1/u_j}\|(I+D^2)^{-1/2}B_je^{-v_jD^2}\|_{1/v_j} d\underline{u}d\underline{v} \\
&\leq \Pi_{j=0}^N \|A_j\|\|(I+D^2)^{-1/2}B_j(I+D^2)^{-1/2}\| \\
&\qquad \int_{\Delta(2N+1)} \Pi_{j=0}^N \|(I+D^2)^{1/2}e^{-u_jD^2}\|_{1/u_j}\|(I+D^2)^{1/2}e^{-v_jD^2}\|_{1/v_j} d\underline{u}d\underline{v} \\
&\leq \Pi_{j=0}^N \|A_j\|\|(I+D^2)^{-1/2}B_j(I+D^2)^{-1/2}\| \\
&\qquad \int_{\Delta(2N+1)} \Pi_{j=0}^N \|(I+D^2)^{1/2}e^{-u_j\epsilon D^2}\|\|(I+D^2)^{1/2}e^{-v_j\epsilon D^2}\|\|\tau(e^{-(1-\epsilon)D^2})^{u_j}\tau(e^{-(1-\epsilon)D^2})^{v_j}\| d\underline{u}d\underline{v}
\end{aligned}$$

$$\leq \|e^{-(1-\epsilon)D^2}\|_1 \Pi_{j=0}^N \|A_j\| \|(I+D^2)^{-1/2} B_j (I+D^2)^{-1/2}\|$$
$$\int_{\Delta(2N+1)} \Pi_{j=0}^N \|(I+D^2)^{1/2} e^{-u_j \epsilon D^2}\| \|(I+D^2)^{1/2} e^{-v_j \epsilon D^2}\| d\underline{u} d\underline{v}.$$

But, for any $\alpha > 0$, we have by the spectral theorem in $\mathcal{L}(\mathcal{E})$ (see for instance [7]),

$$\|(I+D^2)^{1/2} e^{-\alpha D^2}\| \leq \frac{e^{\alpha - 1/2}}{\sqrt{2\alpha}}.$$

Therefore,

$$\Pi_{j=0}^N \|(I+D^2)^{1/2} e^{-u_j \epsilon D^2}\| \|(I+D^2)^{1/2} e^{-v_j \epsilon D^2}\| \leq \frac{e^{\epsilon - 1/2}}{(2\epsilon)^{N+1}} \times \Pi_{j=0}^N (u_j v_j)^{-1/2}.$$

Now we complete the proof by computing, using beta functions, the following integral:

$$\int_{\Delta(2N+1)} \Pi_{j=0}^N (u_j v_j)^{-1/2} d\underline{u} d\underline{v} = \pi^{N+1}.$$

□

We shall also need the intermediate estimate corresponding to $p+1$ entries of second order fiberwise pseudodifferential operators in $< \cdots >_{n+p+1}$. In fact a similar method of proof establishes our next result.

Lemma 4.6. *For any $\epsilon \in]0, 1/2]$, for any $A_0, \cdots, A_N \in \psi^0(M|B, E)$ and any $B_{j_0}, \cdots, B_{j_p} \in \psi^2(M|B, E)$ with $p < N$ and $0 \leq j_0 < \cdots < j_p \leq N$, the following estimate holds*

$$\left\| \langle A_0, \cdots, A_{j_0}, B_{j_0}, A_{j_0+1}, \cdots, A_{j_1}, B_{j_1}, \cdots, A_{j_p}, B_{j_p}, A_{j_p+1}, \cdots, A_N \rangle \right\| \leq$$
$$\frac{\|e^{-(1-\epsilon)D^2}\|_1 \pi^{p+1} p!}{\epsilon^{p+1} N!} \times \Pi_{i=0}^N \|A_i\| \Pi_{i=0}^p \|(I+D^2)^{-1/2} B_{j_i} (I+D^2)^{-1/2}\|.$$

Proof. We apply again the method of proof of Lemma 4.4 and use the equality

$$\int_{u_0+\cdots+u_N+v_{j_0}+\cdots+v_{j_p}=1} \frac{du_1 \cdots du_N dv_{j_0} \cdots dv_{j_p}}{\sqrt{u_{j_0} \cdots u_{j_p} v_{j_0} \cdots v_{j_p}}} = \frac{\pi^{p+1} p!}{N!}.$$

More precisely, we have

$$\|\tau(A_0 e^{-u_0 D^2} \cdots A_{j_0} e^{-u_{j_0} D^2} B_{j_0} e^{-v_{j_0} D^2}$$
$$A_{j_0+1} e^{-u_{j_0+1} D^2} \cdots A_{j_p} e^{-u_{j_p} D^2} B_{j_p} e^{-v_{j_p} D^2} A_{j_p+1} e^{-u_{j_p+1} D^2} \cdots A_N e^{-u_N D^2})\| \leq$$
$$\Pi_{i=0}^N \|A_i\| \Pi_{i=0}^p \|(I+D^2)^{-1/2} B_{j_i} (I+D^2)^{-1/2}\| \|e^{-(1-\epsilon)D^2}\|_1 \Pi_{i=0}^p \|(I+D^2)^{1/2} e^{-u_{j_i} D^2}\| \|(I+D^2)^{1/2} e^{-v_{j_i} D^2}\|.$$

Next we apply the spectral theorem in $\mathcal{E}$ to estimate

$$\|(I+D^2)^{1/2} e^{-u_{j_i} D^2}\| \|(I+D^2)^{1/2} e^{-v_{j_i} D^2}\| \leq \frac{e^{-1/2+\epsilon u_{j_i}}}{\sqrt{2u_{j_i}\epsilon}} \frac{e^{-1/2+\epsilon v_{j_i}}}{\sqrt{2v_{j_i}\epsilon}} \leq \frac{1}{\sqrt{2u_{j_i}\epsilon}\sqrt{2v_{j_i}\epsilon}}.$$

The rest of the proof is straightforward. □

4.2. **Completion of the proof of the theorem.** Recall from [24] that the universal differential graded algebra $\Omega C^\infty(M)$ is endowed with the analytic bornology $\Sigma_{\ell+1}$ generated by the sets $\langle S \rangle (dS)^\infty$ where S describes the bounded subsets of $C^{\ell+1}(M)$ for the Banach algebra topology recalled in the beginning of subsection 3.1. Recall that $\Omega^*(B)$ is similarly endowed with the bornology given by the bounded sets for the Banach algebra topology of C^ℓ forms.

In order to estimate the semi-norms of $\psi_N(f_0, \cdots, f_N)$, we need to expand into its homogeneous components. We denote by $\mathcal{J}$ the subset of $\{0,1\}^3$ given by

$$\mathcal{J} = \{(1,0,0); (0,1,0); (0,0,1)\}$$

For $\alpha \in \mathcal{J}$ we denote by $\alpha^{(j)}$ the j-th component of α, $j = 1, 2, 3$. So only one of the integers $\alpha^{(j)}$ is non trivial and equals 1. We shall set $b^{\alpha^{(j)}}$ in a given expression to mean that when $\alpha^{(j)} = 1$, we take into account b but when $\alpha^{(j)} = 0$ then we simply erase b from the expression. For instance

$$(a_0, \cdots, a_k, b^{\alpha^{(j)}}, a_{k+1}, \cdots, a_n),$$

equals the $(n+2)$-tuple $(a_0, \cdots, a_k, b, a_{k+1}, \cdots, a_n)$ when $\alpha^{(j)} = 1$ and the $(n+1)$-tuple $(a_0, \cdots, a_n)$ when $\alpha^{(j)} = 0$. For $\alpha \in \mathcal{J}$, we set

$$X^\alpha(b) := [\nabla, b]^{\alpha^{(1)}} (\sigma[\nabla, D])^{\alpha^{(2)}} \nabla^{2\alpha^{(3)}}.$$

For any $m \geq 0$, $n = (n_0, \cdots n_m) \in \mathbb{N}^{m+1}$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{J}^m$, we define an $\sum_{j=0}^m n_j + \sum_{i=1}^m \alpha_i^{(1)}$ cochain $\phi^m_{\alpha,n}$ with values in $m + \sum_{i=1}^m \alpha_i^{(3)}$ differential forms on B, by the formula

$$\begin{aligned}\phi^m_{\alpha,n}(f_0, \cdots, f_{n_0}, g_1^{\alpha_1^{(1)}}, f_{n_0+1}, \cdots, f_{n_0+n_1}, \cdots, g_m^{\alpha_m^{(1)}}, f_{n_0+n_1+\cdots+n_{m-1}+1}, \cdots, f_{n_0+\cdots+n_m}) :=\\ < f_0, \sigma[D, f_1], \cdots, \sigma[D, f_{n_0}], X^{\alpha_1}(g_1), \sigma[D, f_{n_0+1}], \cdots, \sigma[D, f_{n_0+n_1}],\\ \cdots, X^{\alpha_m}(g_m), \sigma[D, f_{n_0+\cdots n_{m-1}+1}], \cdots, \sigma[D, f_{n_0+\cdots n_m}] > .\end{aligned}$$

Lemma 4.7. *The cochains ψ_N of the JLO cocycle can be expanded as a finite algebraic sum over $m \geq 0$, $n = (n_0, \cdots n_m) \in \mathbb{N}^{m+1}$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{J}^m$ of the bihomogeneous cochains $\phi^m_{\alpha,n}$.*

Proof. We first replace in ψ_N, each factor $e^{-u_j \mathbb{B}_\sigma^2}$ by its definition, a finite perturbative sum, and by using a straightforward change of variables, we easily deduce that $\psi_N(f_0, \cdots, f_N)$ is a finite signed sum over $(m_0, \cdots, m_N) \in \mathbb{N}^{N+1}$ of the terms

$$\left\langle f_0; \overbrace{X, \cdots, X}^{m_0 \text{ times}}; [\mathbb{B}_\sigma, f_1]; \overbrace{X, \cdots, X}^{m_1 \text{ times}}; \cdots; [\mathbb{B}_\sigma, f_N]; \overbrace{X, \cdots, X}^{m_N \text{ times}} \right\rangle$$

Now, replacing X by its value $\nabla^2 - \sigma[\nabla, D]$ and $[\mathbb{B}_\sigma, f_j]$ by its value $[\nabla, f_j] - \sigma[D, f_j]$, it is easy to rewrite each such term as a finite signed sum of appropriate $\phi^m_{\alpha,n}$'s. □

We have chosen to expand the JLO cocycle as a finite combination of bihomogeneous cochains with respect to the cochain grading and the form grading. By doing so, our formulae are explicit enough to be paired with closed currents on the base.

Proposition 4.8. *The bihomogeneous family $(\phi^m_{\alpha,n})_{m,n,\alpha}$ is a bounded morphism from $(\Omega C^\infty(M), \Sigma_{\ell+1})$ to $(\Omega^*(B), \Sigma_\ell)$.*

Proof. For simplicity, we denote by i_Y either contraction by the vector field Y over B, or by its horizontal lift on M or on $\psi^\infty(M|B, E; \Lambda^*B)$ and $\psi^\infty(M|B, E; \Lambda^*B)[\sigma]$. Then for fiberwise smoothing operators $T \in \psi^{-\infty}(M|B, E; \Lambda^*B)[\sigma]$, we have

$$(i_Y \circ \tau_\sigma)(T) = (\tau_\sigma \circ i_Y)(T) \text{ and } (d_B \circ \tau_\sigma)(T) + (\tau_\sigma \circ \partial)(T) = 0.$$

Therefore, if $d_Y := i_Y \circ d_B$ is the derivative in the direction Y in B, then

$$(d_Y \circ \tau_\sigma)(T) + (\tau_\sigma \circ \partial_Y)(T) = 0 \text{ and } (d_Y \circ \tau)(T) = (\tau \circ \partial_Y)(T).$$

We need to estimate for $0 \leq s \leq \ell$, the semi-norms

$$p_s(\phi^m_{\alpha,n}(f_0, \cdots, f_N)) \text{ where } N = n_0 + \cdots + n_m + m \text{ odd},$$

for given $f_0, \cdots, f_N$ in a bounded set S for the $C^{\ell+1}$ topology. So, we assume that there exists a constant $C \geq 0$ such that $\|f_j\|_t \leq C$ for any $0 \leq t \leq \ell + 1$ and for $0 \leq j \leq N$.

For the convenience of the reader, we explain the proof for $m = 0$, giving a guide to the general case.

Step I: $m = 0$

We begin by estimating the p_s semi-norms of functions on B. They are given by

$$\phi^0_N(f_0, \cdots, f_N) = \langle f_0, \sigma[D, f_1], \cdots, \sigma[D, f_N] \rangle .$$

Let $\|\cdot\|_\alpha$ denote the supremum over B of the fiberwise α-Schatten norm of a compact operator on L^2 sections. Using Hölder's inequality for each $b \in B$ and taking the supremum over B, we have:

$$\begin{aligned} \|\phi_N^0(f_0,\cdots,f_N)\| &\leq \int_{\Delta(N)} \|\tau\left(f_0 e^{-u_0 D^2}[D,f_1]e^{-u_1 D^2}\cdots[D,f_N]e^{-u_N D^2}\right)\| du_1\cdots du_N \\ &\leq \int_{\Delta(N)} \|f_0\|\|e^{-u_0 D^2}\|_{1/u_0}\|[D,f_1]\|\|e^{-u_1 D^2}\|_{1/u_1}\cdots\|[D,f_N]\|\|e^{-u_N D^2}\|_{1/u_N} \\ &\leq \frac{\|e^{-D^2}\|_1}{N!}\times\|f_0\|\Pi_{j=1}^N\|[D_b,f_j]\|. \end{aligned}$$

Using Lemma 4.3, we deduce

$$\begin{aligned} \|\phi_N^0(f_0,\cdots,f_N)\| &\leq C_0\frac{\|e^{-D^2}\|_1}{N!}\times\|f_0\|\|f_1\|_1\cdots\|f_N\|_1 \\ &\leq \|e^{-D^2}\|_1\frac{C_0 C^{N+1}}{N!}. \end{aligned}$$

In the same way, let $Y_1,\cdots,Y_s$ be vector fields on B with norms ≤ 1. Using Lemma 1.7, we can write

$$d_{Y_1}\cdots d_{Y_s}\phi_N^0(f_0,\cdots,f_N) = \int_{\Delta(N)} \tau\left([\partial_{Y_1}\cdots\partial_{Y_s}](f_0 e^{-u_0 D^2}[D,f_1]e^{-u_1 D^2}\cdots[D,f_N]e^{-u_N D^2})\right) du_1\cdots du_N.$$

Note that the operators $\partial_{Y_{j_1}}\cdots\partial_{Y_{j_k}}(f)$ and $\partial_{Y_{j_1}}\cdots\partial_{Y_{j_k}}[D,f]$ are zero-th order differential operators on M and by Lemma 4.3, we can estimate

$$\|\partial_{Y_{j_1}}\cdots\partial_{Y_{j_k}}(f)\| \leq C_k\|f\|_k \text{ and } \|\partial_{Y_{j_1}}\cdots\partial_{Y_{j_k}}[D,f]\| \leq C_k\|f\|_{k+1}.$$

Therefore, one can apply the Hölder inequality exactly, as in the case $s = 0$ treated above, and deduce the required estimates for all the terms that involve no derivatives of the fiberwise smoothing operators $e^{-u_j D^2}$. Thus, we may concentrate on terms of the form

$$\tau\left(A_0[\partial_{Y_{k_1^0}}\cdots\partial_{Y_{k_{\beta_0}^0}}]e^{-u_0 D^2}A_1[\partial_{Y_{k_1^1}}\cdots\partial_{Y_{k_{\beta_1}^1}}]e^{-u_1 D^2}\cdots A_N[\partial_{Y_{k_1^N}}\cdots\partial_{Y_{k_{\beta_N}^N}}]e^{-u_N D^2}]\right),$$

where A_j is a zero-th order pseudodifferential operator, $0 \leq \beta_l \leq s$ and $\Sigma\beta_l$ is at most s and is prescribed by the number of derivatives applied to get the operators A_j out of the operators f_0 and $[D,f_j]$. Now, apply again Duhamel's formula:

$$\partial_Y e^{-uD^2} = -u\int_0^1 e^{-utD^2}\partial_Y(D^2)e^{-u(1-t)D^2}dt = -\int_0^u e^{-tD^2}\partial_Y(D^2)e^{-(u-t)D^2}dt.$$

For instance,

$$\begin{aligned} &\int_{\Delta(N)} \tau(A_0\partial_Y e^{-u_0 D^2}A_1 e^{-u_1 D^2}\cdots A_N e^{-u_N D^2})du_1\cdots du_N \\ &\qquad = \int_{\Delta(N+1)} \tau(A_0 e^{-v_0 D^2}\partial_Y(D^2)e^{-v_1 D^2}A_1 e^{-v_2 D^2}\cdots A_N e^{-v_{N+1}D^2})dv_1\cdots dv_{N+1} \\ &\qquad = \left\langle A_0,\partial_Y(D^2),\sigma A_1,\cdots,\sigma A_N\right\rangle_{N+1} \end{aligned}$$

So, the norm of this term can be estimated using Lemma 4.6 and Lemma 4.1. More pecisely, we get for any $\epsilon \in]0,1/2]$ (one can take here $\epsilon = 1/2$ for simplicity)

$$\begin{aligned} &\|\left\langle A_0,\partial_Y(D^2),A_1,\cdots,A_N\right\rangle_{N+1}\| \\ &\qquad \leq \frac{\pi\|e^{-(1-\epsilon)D^2}\|_1}{\epsilon N!}\|(I+D^2)^{-1/2}\partial_Y(D^2)(I+D^2)^{-1/2}\|\Pi_{i=0}^N\|A_i\| \\ &\qquad \leq \frac{\pi\alpha_1(D^2)\|\tau(e^{-(1-\epsilon)D^2})\|}{\epsilon N!}\Pi_{i=0}^N\|A_i\| \end{aligned}$$

It is clear now by an easy finite induction on the number of vector fields that there exists a constant $C_s(\epsilon)\geq 0$ such that for any zero-th order pseudodifferential operators A_j, we have

$$\|\tau_\sigma\left(A_0[\partial_{Y_{k^0_1}}\cdots\partial_{Y_{k^0_{\beta_0}}}]e^{-u_0D^2}A_1[\partial_{Y_{k^1_1}}\cdots\partial_{Y_{k^1_{\beta_1}}}]e^{-u_1D^2}\cdots A_N[\partial_{Y_{k^N_1}}\cdots\partial_{Y_{k^N_{\beta_N}}}]e^{-u_ND^2}]\right)\|\leq\frac{C_s(\epsilon)}{N!}\Pi_{i=0}^N\|A_i\|.$$

Again, the operators A_j are here derivatives of f_0 or $[D,f_j]$'s, so Lemma 4.3 finishes the proof in this case.

Step II: general m

The general case involves differential forms on the base manifold B obtained from commutators of functions with ∇, commutators of D with ∇ and also with the curvature ∇^2. The latter presents some difficulties. The commutators of functions with ∇ are easy to handle, as they give zero-th order differential operators and can be estimated using the Hölder inequality again and Lemma 4.3. On the other hand, commutators of D with ∇ and terms involving ∇^2 introduce additional derivatives in the fiberwise direction, as these operators are first order fiberwise differential operators with coefficients in differential forms of degree 1 for the first and degree 2 for the second. So, we cannot apply directly the argument of Lemma 4.6 and we need to give careful estimates for such terms. The worst situation arises when the entries $B_0,\cdots,B_N$ in the expression $<B_0,\cdots,B_N>$ are composed of 'too many' fiberwise pseudodifferential operators of positive orders. By this we mean that, in addition to the B_j's, we have the maximum number of commutators $[\nabla,D]$ or $[\nabla^2,D]$ and also the maximum number of directional derivatives of the heat kernel $e^{-u_jD^2}$. The latter introduce operators of order 2. Fortunately, and this seems to be a crucial point here, the base manifold is finite dimensional and we are taking at most ℓ directional derivatives. Hence the number of entries involving $[\nabla,D]$ or $[\nabla^2,D]$ is limited by the dimension and the number of derivatives is also bounded by ℓ.

Denote by k the degree of the differential form

$$\phi^m_{\alpha,n}(f_0,\cdots,f_{n_0},g_1^{\alpha_1^{(1)}},\cdots,g_m^{\alpha_m^{(1)}},\cdots,f_{\sum_{i=0}^m n_i}).$$

So, $k=m+\sum_{i=1}^m\alpha_i^{(3)}$. We fix vector fields $Z_1,\cdots,Z_k$ on the base manifold B with norms ≤ 1 and thus need to estimate, for $s\leq\ell$, the s seminorm of the function $i_{Z_1}\cdots i_{Z_k}\phi^m_{\alpha,n}(\cdots)$. This reduces to the computation of the s seminorm of terms $<A_0,\cdots,A_{N+q+q'}>$ where N entries A_j are zero-th order, q entries are first order and q' entries are second order, and where as explained above, we can assume that N is as large as allowed, while q and q' are bounded by $\sup(\ell,b)$, with $b=\dim(B)$. Indeed, for $N\geq q+q'$, we obtain the desired estimate as follows. Assume that the order is as follows:

$$<A_0,B_1,A_1,\cdots,B_q,A_q,C_1,A_{q+1},\cdots,C_{q'},A_{q+q'},A_{q+q'+1},\cdots,A_N>$$

where the A_j' are zero-th order, the B_j's are first order and the C_j' are second order. By using the Hölder inequality, we reduce to the issue of estimating the expression

$$\|e^{-uD^2}Ee^{-vD^2}A\|_{1/(u+v)}\leq\|e^{-uD^2}E(I+D^2)^{-1/2}\|_{1/u}\|(I+D^2)^{1/2}e^{-vD^2}A\|_{1/v}.$$

where E is at most second order. This gives for any $\epsilon\in]0,1/2]$

$$\|e^{-uD^2}Ee^{-vD^2}A\|_{1/(u+v)}$$
$$\leq\|e^{-u\epsilon D^2}(I+D^2)^{1/2}\|\|(I+D^2)^{-1/2}E(I+D^2)^{-1/2}\|\times\|(I+D^2)^{1/2}e^{-v\epsilon D^2}A\|\sup_b\tau(e^{-(1-\epsilon)D_b^2})^{u+v}.$$

Now, again we have by the spectral theorem

$$\|e^{-u\epsilon D^2}(I+D^2)^{1/2}\|\leq\frac{e^{\epsilon-1/2}}{\sqrt{2u\epsilon}},$$

and the estimate goes exactly as for the previous simpler cases. We thus obtain the existence of a constant $C(\epsilon)\geq 0$ such that

$$\|<A_0,B_1,A_1,\cdots,B_q,A_q,C_1,A_{q+1},\cdots,C_{q'},A_{q+q'},A_{q+q'+1},\cdots,A_N>\|$$
$$\leq\frac{C(\epsilon)}{N!}\Pi_{i=0}^N\|A_i\|\Pi_{i=1}^q\|(I+D^2)^{-1/2}B_i\|\Pi_{i=1}^{q'}\|(I+D^2)^{-1/2}C_i(I+D^2)^{-1/2}\|.$$

The rest of the proof is completed using Lemmas 4.3 and 1.7 and the Duhamel formula. □

4.3. **More general superconnections and transgression.** We show in this subsection how to extend Theorem 3.3 to more general superconnections associated with the odd operator σD and prove that the analytic bivariant cyclic homology class does not depend on such choice. Since the techniques are classical, we shall be brief. More precisely, we consider superconnections $\mathbb{A}$ given as

$$\mathbb{A} := \mathbb{B}_\sigma + A \text{ where } A \text{ is an odd element of } \Psi^0(M|B, E; \Lambda^* B)[\sigma],$$

whose differential forms degrees are positive. Recall that $\mathbb{B}_\sigma = \sigma D + \nabla$ so that $\mathbb{A} = \sigma D + \nabla + A$. Given such superconnection $\mathbb{A}$, we can write $\mathbb{A}^2 = D^2 + X'$ where $X' = \nabla^2 + A^2 + [\nabla, \sigma D + A]$ has only positive degree forms and is therefore nilpotent.

We define the heat kernel $e^{-\mathbb{A}^2}$ of the superconnection $\mathbb{A}$ by the usual finite Duhamel expansion where we simply replace the operator X by X'. More precisely, we have the following finite perturbative definition

$$e^{-\mathbb{A}^2} := \sum_{m\geq 0} \int_{\Delta(m)} e^{-v_0 D^2} X' e^{-v_1 D^2} \cdots X' e^{-v_m D^2} dv_1 \cdots dv_m.$$

where $\Delta(m) = \{(u_0, \cdots, u_m) \in \mathbb{R}^{m+1}, \sum u_j = 1\}$ is again the m-simplex. Notice that the following expression makes sense for any elements A_i of $\Psi^\infty(M|B, E; \Lambda^* B)[\sigma]$:

$$\langle\langle A_0, \cdots, A_n \rangle\rangle_{\mathbb{A}} := \int_{\Delta(n)} \tau_\sigma(A_0 e^{-u_0 \mathbb{A}^2} \cdots A_n e^{-u_n \mathbb{A}^2}) du_1 \cdots du_n.$$

As with the superconnection $\mathbb{B}_\sigma$ which corresponds to $A = 0$, we define for any odd integer n:

$$\psi_n(f_0, \cdots, f_n) := \langle\langle f_0, [\mathbb{A}, f_1], \cdots, [\mathbb{A}, f_n] \rangle\rangle_{\mathbb{A}}$$

Proposition 4.9. *Given a superconnection $\mathbb{A}$ associated with σD as above, the cochains $(\psi_n)_n$ form a morphism* JLO($\mathbb{A}$) *from the universal graded algebra of $C^\infty(M)$ to the graded algebra $\Omega^*(B)$ of differential forms on B, which is a bivariant cyclic cocycle. Moreover,* JLO($\mathbb{A}$) *is bounded from $\Sigma_{\ell+1}$ to Σ_ℓ and hence also for the Fréchet topologies.*

Proof. Notice first that the algebraic relations proved in Lemma 2.1 (all of them besides the last one) are still valid with $\mathbb{A}$ replacing $\mathbb{B}_\sigma$. Hence, the collection $(\psi_n)_n$ is again a bivariant cyclic cocycle by exactly the same proof as for Lemma 2.3. The proof of boundedness is a rephrasing of the proof of 3.3. The only difference is that we have to deal with new terms involving $A^2 + [\mathbb{B}_\sigma, A]$. The term $A^2 + [\nabla, A]$ is a zero-th order fiberwise pseudodifferential operator with coefficients in positive degree forms and causes no trouble. We only have to explain how to estimate terms involving $-\sigma[D, A] = -\sigma \sum_{k>0} [D, A_{[k]}]$, where $A_{[k]}$ is the component of A which increases the degree form by k. But this is done using the following modification of Lemma 4.2 and which is proved in the same way by reducing to local coordinates:

$$\sup_{\|Z_1\|\leq 1, \cdots, \|Z_k\|\leq 1, \|Y_1\|\leq 1, \cdots, \|Y_q\|\leq 1} \|(I + D^2)^{-1/2} [\partial_{Y_1} \cdots \partial_{Y_q}](i_{Z_1 \wedge \cdots Z_k}[D, A_{[k]}])\| = \beta'_q < +\infty,$$

We omit the proof here. Then the rest of the proof is tedious but is exactly a rephrasing of the proof given in the previous subsection. □

Remark 4.10. *We will show that the analytic cohomology class of* JLO($\mathbb{A}$) *coincides with the analytic cohomology class* JLO(D).

We now proceed to prove the main result of this subsection, namely the transgression formula for our JLO analytic bivariant cocycle. We follow the method adopted in [17]. Set, for any superconnection $\mathbb{B}_\sigma$ associated with σD, and with V a homogeneous fiberwise pseudodifferential operator with coefficients in differential forms on the base B:

$$\mathrm{Ch}(\mathbb{B}_\sigma, V)(f_0, \cdots, f_n) := \sum_{i=0}^{n} (-1)^{i|V|} \langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n] \rangle\rangle_{\mathbb{B}_\sigma},$$

and

$$\alpha^*(\mathbb{B}_\sigma, V)(f_0, \cdots, f_n) := \sum_{i=1}^n (-1)^{(i-1)(|V|+1)} \langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [V, f_i], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle_{\mathbb{B}_\sigma}.$$

Lemma 4.11. *Let V be a fiberwise pseudodifferential operator with coefficients in forms on the base manifold B.*

- *Assume that the pseudodifferential order of V is ≤ 1, then the cochain $\mathrm{Ch}(\mathbb{B}_\sigma, V)$ is analytic with respect to the bornologies $\Sigma_{\ell+1}$ of $C^\infty(M)$ and Σ_ℓ of $\Omega^*(B)$.*
- *Assume that the pseudodifferential order of V is ≤ 0, then the cochain $\alpha^*(\mathbb{B}_\sigma, V)$ is analytic with respect to the bornologies $\Sigma_{\ell+1}$ of $C^\infty(M)$ and Σ_ℓ of $\Omega^*(B)$.*

Proof. Let us prove for instance the first item, the second being easier since V is bounded. Applying the definition of $e^{-u\mathbb{B}_\sigma^2}$ in the expression

$$\langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle_{\mathbb{B}_\sigma},$$

we reduce to the estimates of terms of the form

$$< f_0, X, \cdots, X, [\mathbb{B}_\sigma, f_1], X, \cdots, X, \cdots, [\mathbb{B}_\sigma, f_i], X, \cdots, X; V; \\ X, \cdots, X, [\mathbb{B}_\sigma, f_{i+1}], X, \cdots, X, \cdots, [\mathbb{B}_\sigma, f_n], X, \cdots, X >$$

The point is to apply the argument of Step II in the proof of Theorem 3.3, by simply adding one operator of order 1 in the entries. Recall that this can be done as long as the number of operators of order 1 or 2 is not too big with respect to the number of operators of order 0. But, notice that V only appears once, the first order operator X has coefficients in differential forms of positive degree only and hence cannot appear more than the dimension of B times. Therefore, since we only need the estimates for n large, the same proof works and we obtain the required estimates exactly as in Step II of the proof of Theorem 3.3. Notice that we have to estimate the sum of $n + 1$ terms of the form

$$\langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle_{\mathbb{B}_\sigma},$$

but since the estimate involves $1/n!$, multiplication by n doesn't alter the proof. □

Proposition 4.12. *The following identity holds*

$$(d_B + (-1)^{|V|}(b + B)) \,\mathrm{Ch}(\mathbb{B}_\sigma, V) + \mathrm{Ch}(\mathbb{B}_\sigma, [\mathbb{B}_\sigma, V]) + (-1)^{|V|}\alpha^*(\mathbb{B}_\sigma, V) = 0.$$

Proof. For $0 \leq i \leq n$, we apply the third relation of Lemma 2.1 to the operators

$$A_0 = f_0, A_j = [\mathbb{B}_\sigma, f_j] \text{ for } 1 \leq j \leq i, A_{i+1} = V \text{ and } A_j = [\mathbb{B}_\sigma, f_{j-1}] \text{ for } j \geq i + 2.$$

So, for $i = 0$ for instance, this means that we apply that relation to $A_0 = f_0, A_1 = V$ and $A_j = [\mathbb{B}_\sigma, f_{j-1}]$ for $j \geq 2$. For any i this gives us a relation $\theta_i + d_B\theta'_i = 0$ where $(-1)^{i|V|}\theta_i = X_1^i + X_2^i + X_3^i = 0$ where

$$X_1^i = (-1)^{i|V|} \langle\langle [\mathbb{B}_\sigma, f_0], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle$$

$$X_2^i = \sum_{1\leq j\leq i} (-1)^{i|V|+j-1} \left\langle\left\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma^2, f_j], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\right\rangle\right\rangle + \\ \sum_{j=i+1}^n (-1)^{j-1+(i+1)|V|} \left\langle\left\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma^2, f_j], \cdots, [\mathbb{B}_\sigma, f_n]\right\rangle\right\rangle$$

and

$$X_3^i = (-1)^{i(|V|+1)} \langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_i], [\mathbb{B}_\sigma, V], [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle.$$

Finally, θ'_i is given by

$$\theta'_i = \langle\langle f_0, [\mathbb{B}_\sigma, f_1], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle.$$

Thus the expression $\sum_{i=0}^n (-1)^{i|V|}(\theta_i + d_B\theta'_i) = 0$ allows us to write

$$\sum_{i=0}^n X_1^i + \sum_{i=0}^n X_2^i + \sum_{i=0}^n X_3^i + d_B \operatorname{Ch}(\mathbb{B}_\sigma, V) = 0.$$

Now we have, by inspection, the relation

$$\sum_{i=0}^n X_3^i = \operatorname{Ch}(\mathbb{B}_\sigma, [\mathbb{B}_\sigma, V])(f_0, \cdots, f_n).$$

Similarly, we leave it to the reader to directly compute $\sum_{i=0}^n X_2^i$. One finds

$$\sum_{i=0}^n X_2^i = (-1)^{|V|}[b \operatorname{Ch}(\mathbb{B}_\sigma, V) + \alpha^*(\mathbb{B}_\sigma, V)](f_0, \cdots, f_n).$$

Next, using the second relation of Lemma 2.1, we see that

$$B \operatorname{Ch}(\mathbb{B}_\sigma, V)(f_0, \cdots, f_n) = \sum_{i=0}^n (-1)^{(i+1)|V|} \langle\langle [\mathbb{B}_\sigma, f_0], \cdots, [\mathbb{B}_\sigma, f_i], V, [\mathbb{B}_\sigma, f_{i+1}], \cdots, [\mathbb{B}_\sigma, f_n]\rangle\rangle .$$

The conclusion follows immediately. □

We are now in position to prove

Theorem 4.13. *Let $\mathbb{A}$ be the superconnection $\mathbb{A} := \mathbb{B}_\sigma + A$, where A is a fiberwise pseudodifferential operator of zero-th order with coefficients in differential forms of positive degree on the base manifold B. In particular, we are assuming here that A is odd for the total grading. Then the JLO analytic cocycle* JLO($\mathbb{A}$) *associated with the superconnection $\mathbb{B}_\sigma(A)$ is cohomologuous to the JLO analytic cocycle* JLO(D) *associated with the superconnection $\mathbb{B}_\sigma = \nabla + \sigma D$.*

Proof. Let $\mathbb{B}_{\sigma,s} := \mathbb{B}_\sigma + sA$ be the smooth linear path of superconnections associated with σD. Then we can write, using the fifth relation of Lemma 2.1:

$$\frac{d}{ds} \langle\langle f_0, [\mathbb{B}_{\sigma,s}, f_1], \cdots, [\mathbb{B}_{\sigma,s}, f_n]\rangle\rangle_{\mathbb{B}_{\sigma,s}} =$$
$$- \sum_{i=0}^n \langle\langle f_0, [\mathbb{B}_{\sigma,s}, f_1], \cdots, [\mathbb{B}_{\sigma,s}, f_i], [\mathbb{B}_{\sigma,s}, A], [\mathbb{B}_{\sigma,s}, f_{i+1}], \cdots, [\mathbb{B}_{\sigma,s}, f_n]\rangle\rangle_{\mathbb{B}_{\sigma,s}} +$$
$$\sum_{i=1}^n \langle\langle f_0, [\mathbb{B}_{\sigma,s}, f_1], \cdots [\mathbb{B}_{\sigma,s}, f_{i-1}], [A, f_i], \cdots, [\mathbb{B}_{\sigma,s}, f_n]\rangle\rangle_{\mathbb{B}_{\sigma,s}}$$

Notice that $|A| = 1$ while $|[\mathbb{B}_{\sigma,s}, A]| = 2$. Hence we obtain

$$\frac{d}{ds} \langle\langle f_0, [\mathbb{B}_{\sigma,s}, f_1], \cdots, [\mathbb{B}_{\sigma,s}, f_n]\rangle\rangle_{\mathbb{B}_{\sigma,s}} = -\operatorname{Ch}(\mathbb{B}_{\sigma,s}, [\mathbb{B}_{\sigma,s}, A])(f_0, \cdots, f_n) + \alpha^*(\mathbb{B}_{\sigma,s}, A)(f_0, \cdots, f_n).$$

But we know from Proposition 4.12 that

$$-\operatorname{Ch}(\mathbb{B}_{\sigma,s}, [\mathbb{B}_{\sigma,s}, A]) + \alpha^*(\mathbb{B}_{\sigma,s}, A) = [d_B - (b + B)] \operatorname{Ch}(\mathbb{B}_{\sigma,s}, A),$$

which completes the proof since $\operatorname{Ch}(\mathbb{B}_{\sigma,s}, A)$ is an even bivariant cochain. □

5. Compatibility with the higher spectral flow

5.1. **Higher spectral flow.** An application of our analytic JLO cocycle comes from its relation with the higher spectral flow. Using our analyticity result we explain in this Section how to prove the equality between the Chern character of the higher spectral flow and the corresponding JLO pairing, which is well defined in the Fréchet topologies. Higher spectral flow, introduced in [16] for a family of fiberwise self-adjoint elliptic operators $D = (D_b)_{b\in B}$, is only well defined under the assumption that the K^1 class defined by the family is trivial. We assume this from now on. Our proof that the Chern character of higher spectral flow coincides with the pairing with our analytic bivariant JLO cocycle is a generalization of Getzler's proof in the case of a single operator [18]. Recall that the fiberwise generalized Dirac operator D defines a class $[D]$ in the Kasparov group $KK^1(M,B)$ [21], and hence using the Kasparov product, a homomophism $K^1(M)\to K^0(B)$ which assigns to $U\in K^1(M)$ the class $U\cap[D]$. This Kasparov product is an index map which is described below using either families of Toeplitz operators or the notion of higher spectral flow. Denoting by HCA analytic cyclic cohomology we prove in the present Section commutativity of the following diagram:

$$\begin{array}{ccc} K^1(M) & \xrightarrow{\mathrm{SF}(D,\cdot)} & K^0(B) \\ \mathrm{Ch}\downarrow & & \downarrow\mathrm{ch} \\ HCA_1(C^\infty(M)) & \xrightarrow{\mathrm{JLO}(D)} & H^{even}(B,\mathbb{C}) \end{array}$$

Thus the analytic cyclic cohomology class $\mathrm{JLO}(D)$ is precisely the bivariant Chern-Connes character of $[D]$. Combining our result with the main result of [16], we deduce that $\mathrm{JLO}(D)$ coincides up to the (analytic) HKR-Connes map for M, with the topological map $H^{odd}(M)\to H^{even}(B)$ given up to constant by

$$\omega\longmapsto\int_{M/B}\omega\wedge\hat{A}(TM|B).$$

By using the results of [23], we know that the K^1 index of $D=(D_b)_{b\in B}$ is zero if and only if there exists a (smooth) spectral section P for D, that is, a smooth family of self-adjoint fiberwise pseudodifferential projections $P=(P_b)_{b\in B}$ acting on the L^2-sections such that for some smooth non-negative function ϱ on B,

$$P_b 1_{]\varrho(b),+\infty)}(D_b)=P_b \text{ and } P_b 1_{]-\infty,-\varrho(b)[}(D_b)=0,\quad \forall b\in B.$$

The following result is taken from [23].

Proposition 5.1. [23] *Let $D=(D_b)_{b\in B}$ be as before the fiberwise generalized Dirac operator along the smooth fibration $\pi:M\to B$ and assume that the index of D in $K^1(B)$ is trivial. Given a spectral section P for D, there exists a self-adjoint fiberwise zero-th order pseudodifferential operator $A\in\Psi^0(M|B;E)$ such that for any $b\in B$ the operator D_b+A_b is invertible and P_b coincides with $1_{[0,+\infty)}(D_b+A_b)$.*

We assume from now on that the operator A is chosen as in the previous proposition and thus associated with a fixed spectral section P for D. So $P=1_{[0,+\infty)}(D+A)$ and $D+A$ is a zero-th order perturbation of D and is a family of invertible operators.

Proposition 5.2. *For any $U\in GL_N(C^\infty(M))$, the operator*

$$PUP:=(P\otimes 1_N)\circ U\circ(P\otimes 1_N),$$

acting fiberwise on the image of $L^2(M_b,E)\otimes\mathbb{C}^N$ under the projection $P_b\otimes 1_N$, is a smooth family of Fredholm operators whose index class in $K^0(B)$ is denoted $\mathrm{Ind}(T_U)$. *Then,* $\mathrm{Ind}(T_U)$ *does not depend on the choice of the spectral section P and only depends on the K^1 class of U.*

Proof. Compare with [16]. For any fixed $b\in B$, the usual proof for a single Toeplitz operator on the odd dimensional closed manifold M_b shows that

$$P_bU_bP_b=(P_b\otimes 1_N)\circ U|_{M_b}\circ(P_b\otimes 1_N)$$

is a Fredholm operator in the Hilbert space $P_b(L^2(M_b,E|_{M_b}))^N$. Hence, PUP is a smooth family of Fredholm operators on the image of P. We then know that the homotopy class of this family defines a class in $K^0(B)$, see [1]. To explicitly define this class as the Atiyah-Singer index of a fiberwise elliptic operator, we follow [3] and define the zero-th order fiberwise elliptic pseudodifferential operator $T_{U,P}$ by

$$T_{U,P}:=I-P+PUP.$$

The index $\mathrm{Ind}(PUP)$ is, by definition, the Atiyah-Singer index class of $T_{U,P}$ in $K^0(B)$ [2]. If we choose another spectral section P' then the usual computation shows that the operator $T_{U,P'}$ is a perturbation of $T_{U,P}$ by a smooth family of finite rank operators, therefore the index class is unchanged. A homotopy class of invertibles U_t yields a homotopy class of principal symbols of the fiberwise operator $T_{U,P}$ and hence the index is unchanged. □

Definition 5.3. [16] *Assume that* $[0,1] \ni t \mapsto D_t := (D_{t,b})_{b\in B}$ *is a smooth path of fiberwise elliptic pseudo-differential operators such that the index class of the endpoints,* D_0 *and* D_1 *in* $K^1(B)$*, are trivial. Choose spectral sections* P_0, P_1 *for* D_0, D_1 *respectively and fix a spectral section* $Q = (Q_t)_{t\in[0,1]}$ *for the total family viewed as a fiberwise operator over* $B \times [0,1]$*. Then the spectral flow of the path* $(D_t)_{t\in[0,1]}$ *with respect to* P_0 *and* P_1 *is the class in* $K^0(B)$ *defined by:*

$$\mathrm{SF}(D; P_0, P_1) := [P_1 - Q_1] - [P_0 - Q_0].$$

It is easy to check that $\mathrm{SF}(D; P_0, P_1)$ does not depend on the choice of the global spectral section Q. In this paper we are mainly interested in the affine path $D_t := D + tU^{-1}[D, U]$ where D is a family of generalized Dirac operators over B whose index class in $K^1(B)$ is trivial, and U is a given element of $GL_N(C^\infty(M))$. In this case the endpoints are conjugate and we consider the spectral flow with respect to the spectral sections $P_0 = P$ and $P_1 = U^{-1}PU$, where P is a fixed spectral section for D. It turns out that the spectral flow does not depend on P either and is an invariant of the principal symbol of D and of the homotopy class of U. We denote it $\mathrm{SF}(D, U)$. Indeed, Dai and Zhang proved the following

Proposition 5.4. [16]. *We have in* $K^0(B)$, $\mathrm{Ind}(T_U) = -\,\mathrm{SF}(D, U)$.

5.2. **Second theorem and reduction to the third theorem.** Recall that the pairing of analytic bivariant cyclic homology with analytic cyclic homology reads in our case as follows:

$$< \mathrm{JLO}(D), U >:= \sum_{n\geq 0} (-1)^n n! \left\langle\left\langle U^{-1}, [\mathbb{B}_\sigma, U], \cdots, [\mathbb{B}_\sigma, U^{-1}], [\mathbb{B}_\sigma, U]\right\rangle\right\rangle_{\mathbb{B}_\sigma, 2n+1}$$

Theorem 5.5. *Assume that the index class of* D *in* $K^1(B)$ *is trivial. Then for any* $U \in GL_N(C^\infty(M))$*, the following relation holds in the even de Rham cohomology of the base manifold* B*:*

$$\frac{1}{\sqrt{\pi}} < \mathrm{JLO}(D), U >= \mathrm{ch}(\mathrm{SF}(D, U)) = -\,\mathrm{ch}(\mathrm{Ind}(T_U)).$$

where ch *is the usual Chern character on the manifold* B.

The last relation is clear from the previous proposition and the fact that the Chern character only depends on the K-theory class. The proof of this theorem is long and we split it into several lemmas and propositions.

Lemma 5.6. *Let* $\mathbb{B}$ *be a superconnection associated with the operator* σD *as in the previoous sections. Define the affine path of superconnections* $(\mathbb{B}_t)_{0\leq t\leq 1}$ *given by*

$$\mathbb{B}_t := \mathbb{B} + tU^{-1}[\mathbb{B}, U].$$

Then (i) the differential form $\int_0^1 \tau_\sigma(U^{-1}[\mathbb{B}, U]e^{-\mathbb{B}_t^2})dt$ *is a closed form on* B*,*
(ii) the cohomology class of $\int_0^1 \tau_\sigma(U^{-1}[\mathbb{B}, U]e^{-\mathbb{B}_t^2})dt$ *does not depend on the choice of superconnection* $\mathbb{B}$.

Proof. (1) Let $\mathbb{A} := dt\frac{\partial}{\partial t} + \mathbb{B}_t$ be the associated superconnection for the fibration $M \times [0,1] \to B \times [0,1]$. Therefore, the differential form $\tau_\sigma(e^{-\mathbb{A}^2})$ is closed in $B \times [0,1]$. A straightforward computation using the fact that $\tau_\sigma(e^{-\mathbb{B}_t^2})$ is itself closed in B, proves the following relation

$$d_B\left(\tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})\right) = \frac{d}{dt}\tau_\sigma(e^{-\mathbb{B}_t^2}),$$

Hence,

$$d_B \int_0^1 \tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})dt = \tau_\sigma(e^{-\mathbb{B}_1^2}) - \tau_\sigma(e^{-\mathbb{B}_0^2}) = \tau_\sigma(U^{-1}e^{-\mathbb{B}_0^2}U) - \tau_\sigma(e^{-\mathbb{B}_0^2}) = 0.$$

The last equality is deduced from the relation $\sigma U = U\sigma$ and the graded tracial property of the functional τ.

(2) Assume that we are given another superconnection $\mathbb{B}'$ associated with σD. Consider the corresponding affine path $\mathbb{B}'_t := \mathbb{B}' + U^{-1}[\mathbb{B}', U]$ as before, and the smooth family $\mathbb{A}_{t,s} = \mathbb{B}_t + s(\mathbb{B}'_t - \mathbb{B}_t)$ of superconnections associated with σD, where s also runs over $[0,1]$. We then set

$$\mathbb{D} := \mathbb{A}_{t,s} + dt\frac{\partial}{\partial t} + ds\frac{\partial}{\partial s}.$$

Clearly, $\mathbb{D}$ is a superconnection associated with σD but for the smooth fibration $M \times [0,1]^2 \to B \times [0,1]^2$. Therefore, the differential form $\tau_\sigma(e^{-\mathbb{D}^2})$ is closed in $B \times [0,1]^2$. Using this fact and that

$$d_B \tau_\sigma(e^{-\mathbb{A}_{t,s}^2}) = 0,$$

we obtain the relation

$$d_B \tau_\sigma(e^{-\mathbb{A}_{t,s}^2} \wedge K_{t,s}^2) = dt \wedge ds \left[\frac{\partial}{\partial s}\tau_\sigma(\frac{\partial \mathbb{A}_{t,s}}{\partial t} e^{-\mathbb{A}_{t,s}^2}) - \frac{\partial}{\partial t}\tau_\sigma(\frac{\partial \mathbb{A}_{t,s}}{\partial s} e^{-\mathbb{A}_{t,s}^2})\right],$$

where $K_{t,s} = dt \wedge \frac{\partial \mathbb{A}_{t,s}}{\partial t} + ds \wedge \frac{\partial \mathbb{A}_{t,s}}{\partial s}$. Now we can compute

$$\begin{aligned}
\int_0^1 \tau_\sigma(\frac{\partial \mathbb{A}_{t,1}}{\partial t} e^{-\mathbb{A}_{t,1}^2})dt - \int_0^1 \tau_\sigma(\frac{\partial \mathbb{A}_{t,0}}{\partial t} e^{-\mathbb{A}_{t,0}^2})dt &= \int_0^1 \frac{\partial}{\partial s}\left[\int_0^1 \tau_\sigma(\frac{\partial \mathbb{A}_{t,s}}{\partial t} e^{-\mathbb{A}_{t,s}^2})dt\right] ds \\
&= \int_{[0,1]^2} \frac{\partial}{\partial s}\tau_\sigma(\frac{\partial \mathbb{A}_{t,s}}{\partial t} e^{-\mathbb{A}_{t,s}^2})dt \wedge ds \\
&= \int_{[0,1]^2} \frac{\partial}{\partial t}\tau_\sigma(\frac{\partial \mathbb{A}_{t,s}}{\partial t} e^{-\mathbb{A}_{t,s}^2})dt \wedge ds \\
&\quad + d_B \int_{[0,1]^2} \tau_\sigma(e^{-\mathbb{A}_{t,s}^2} \wedge K_{t,s}^2) \\
&= \int_0^1 \left[\tau_\sigma(\frac{\partial \mathbb{A}_{1,s}}{\partial s} e^{-\mathbb{A}_{1,s}^2} - \tau_\sigma(\frac{\partial \mathbb{A}_{0,s}}{\partial s} e^{-\mathbb{A}_{0,s}^2}\right] ds \\
&\quad + d_B \int_{[0,1]^2} \tau_\sigma(e^{-\mathbb{A}_{t,s}^2} \wedge K_{t,s}^2).
\end{aligned}$$

Notice that

$$\mathbb{A}_{1,s} = U^{-1}\mathbb{A}_{0,s}U \text{ and } \frac{\partial \mathbb{A}_{1,s}}{\partial s} = \mathbb{B}'_1 - \mathbb{B}_1 = U^{-1}\frac{\partial \mathbb{A}_{0,s}}{\partial s}U.$$

Therefore, the proof is complete. □

The next proposition is an easy rephrasing of a result of Dai and Zhang:

Proposition 5.7. [16] *Let $\mathbb{B}$ be the Bismut superconnection associated with σD, then the cohomology class of the differential form $\frac{-1}{\pi^{1/2}}\int_0^1 \tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})dt$ coincides with the Chern character of the spectral flow, i.e.*

$$\mathrm{ch}(\mathrm{SF}(D,U)) = \frac{-1}{\pi^{1/2}}\left[\int_0^1 \tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2} dt\right].$$

The proof of this proposition relies on the good behaviour of the asymptotics of the rescaled Bismut superconnection. To sum up, in order to prove Theorem 5.5, we are reduced to proving the following auxiliary result.

Theorem 5.8. *When $\mathbb{B} = \mathbb{B}_\sigma$ and as differential forms on the base manifold B, we have the following equality:*

$$\int_0^1 \tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})dt = -\sum_{k\geq 0}(-1)^k k! \left\langle\left\langle U^{-1}, [\mathbb{B}_\sigma, U], [\mathbb{B}_\sigma, U^{-1}], \cdots, [\mathbb{B}_\sigma, U]\right\rangle\right\rangle_{2k+1,\mathbb{B}_\sigma}.$$

5.3. **Proof of the third theorem.** As explained before, the proof of this theorem follows the lines of [18] and we split the argument into a number of steps. First we double up our Hilbert space and replace U by

$$V := \begin{pmatrix} 0 & iU^{-1} \\ -iU & 0 \end{pmatrix}, \text{ so that } V^2 = I, \text{ and let } \tilde{\mathbb{B}} = \begin{pmatrix} \mathbb{B}_\sigma & 0 \\ 0 & -\mathbb{B}_\sigma \end{pmatrix}.$$

Consider the operator $\mathbb{A}$ associated with the fibration

$$M \times [0,1] \times [0,+\infty) \to B \times [0,1] \times [0,+\infty),$$

and given by

$$\mathbb{A} := \tilde{\mathbb{B}}_{t,x} + \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \text{ where } \tilde{\mathbb{B}}_{t,x} := \tilde{\mathbb{B}}_t + xV \text{ and } \tilde{\mathbb{B}}_t = \tilde{\mathbb{B}} - tV[\tilde{\mathbb{B}}, V].$$

We use graded commutators so that for instance $[\tilde{\mathbb{B}}, V] = \tilde{\mathbb{B}}V + V\tilde{\mathbb{B}}$. The differential d is again the de Rham differential on $[0,1] \times [0,+\infty)$. Moreover, we extend τ_σ to a supertrace τ_s given by

$$\tau_s(A) := \tau_\sigma(A_{11}) + \tau_\sigma(A_{22}).$$

It is then straightforward to check, using the Bianchi identity satisfied by $\mathbb{B}_\sigma$, that $d_B\tau_s(e^{-\mathbb{A}^2}) = 0$. Computing the square of $\mathbb{A}$ one finds using for instance the relation $V[\tilde{\mathbb{B}}, V]V = [\tilde{\mathbb{B}}, V]$:

$$\mathbb{A}^2 = Y_{t,x} + dxV + dtV[\tilde{\mathbb{B}}, V] \text{ where } Y_{t,x} = (\tilde{\mathbb{B}}_t)^2 + x(1-2t)[\tilde{\mathbb{B}}, V] + x^2.$$

Recall then that the differential form $\tau_s(e^{-\mathbb{A}^2})$ is automatically closed as a differential form on the manifold with boundary $B \times [0,1] \times [0,+\infty)$.

Lemma 5.9. *Let $R(x_0)$ denote the rectangle $[0,1] \times [0,x_0]$, then in $\Omega^*(B)$ we have: $\int_{\partial R(x_0)} \tau_s(e^{-\mathbb{A}^2}) = 0$.*

Proof. We have by a direct computation

$$\tau_s(e^{-\mathbb{A}^2}) = dx\tau_s(Ve^{-Y_{t,x}}) + dt\tau_s(V[\tilde{\mathbb{B}}, V]e^{-Y_{t,x}}).$$

Hence the closed differential form $\tau_s(e^{-\mathbb{A}^2})$ is given by

$$\tau_s(e^{-\mathbb{A}^2}) = dx\omega_{t,x} + dt\alpha_{t,x},$$

where $\omega_{t,x}$ and $\alpha_{t,x}$ are smooth families of differential forms on B. We thus have

$$\begin{aligned} \int_{R(x_0)} \tau_s(e^{-\mathbb{A}^2}) &= \int_0^1 [\alpha_{t,x_0} - \alpha_{t,0}]dt - \int_0^{x_0} [\omega_{1,x} - \omega_{0,x}]dx \\ &= \int_0^1 \int_0^{x_0} [\frac{\partial \alpha}{\partial x} - \frac{\partial \omega}{\partial t}]dtdx. \end{aligned}$$

The closedness of $\tau_s(e^{-\mathbb{A}^2})$ implies in particular that the component of $d_B\tau_s(e^{-\mathbb{A}^2})$ which contains a $dx \wedge dt$ is trivial. But this is precisely $\frac{\partial \alpha}{\partial x} - \frac{\partial \omega}{\partial t}$. □

We denote for $x > 0$ by γ_x the path $[0,1] \times \{x\}$ oriented in the direction of increasing $t \in [0,1]$. We also consider the path $\Gamma_t^x = \{t\} \times [0,x]$ for $t \in [0,1]$ and $x > 0$, oriented in the direction of increasing $y \in [0,x]$.

Lemma 5.10. *We have the following equality of the corresponding even forms*

$$\int_{\gamma_0} \tau_s(e^{-\mathbb{A}^2}) = 2 \times \int_0^1 \tau_\sigma(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})dt.$$

Proof. We have

$$Y_{t,0} = \tilde{\mathbb{B}}_t^2 = \begin{pmatrix} (\mathbb{B}_\sigma + tU^{-1}[\mathbb{B}_\sigma, U])^2 & 0 \\ 0 & (\mathbb{B}_\sigma - tU[\mathbb{B}_\sigma, U^{-1}])^2 \end{pmatrix}$$

Hence we obtain

$$\int_{\gamma_0} \tau_s(e^{-\mathbb{A}^2}) = -\int_0^1 \tau_s(V[\tilde{B}, V]e^{-Y_{t,0}})dt = \int_0^1 \tau_\sigma((U^{-1}[\mathbb{B}_\sigma, U])e^{-(\mathbb{B}_\sigma + tU^{-1}[\mathbb{B}_\sigma, U])^2}dt$$
$$-\int_0^1 \tau_\sigma(U[\mathbb{B}_\sigma, U^{-1}]e^{-(\mathbb{B}_\sigma + tU[\mathbb{B}_\sigma, U^{-1}])^2}dt = 2\int_0^1 \tau_\sigma(U^{-1}[\mathbb{B}_\sigma, U]e^{-(\mathbb{B}_\sigma + tU^{-1}[\mathbb{B}_\sigma, U])^2}dt.$$

Notice that only the even forms are relevent for us. □

Lemma 5.11. *We have*

$$\lim_{x\to+\infty} \left[\int_{\Gamma_1^x} \tau_s(e^{-\mathbb{A}^2}) + \int_{\Gamma_0^x} \tau_s(e^{-\mathbb{A}^2})\right] = 0.$$

Moreover,

$$\lim_{x\to+\infty} \left[\int_{\Gamma_0^x} \tau_s(e^{-\mathbb{A}^2})\right] =< \mathrm{JLO}(D), U > .$$

Proof. Only the term $-\tau_s(Ve^{-Y_{t,x}})$ contributes to the integrals $\int_{\Gamma_t^x}$. We thus need to compare $Y_{1,x}$ with $Y_{0,x}$. But notice that

$$Y_{1,x} = \tilde{\mathbb{B}}_1^2 - x[\tilde{\mathbb{B}}, V] + x^2 = V\tilde{\mathbb{B}}^2 V - x[\tilde{\mathbb{B}}, V] + x^2.$$

On the other hand, $V[\tilde{\mathbb{B}}, V]V = [\tilde{\mathbb{B}}, V]$ so that

$$Y_{1,x} = V\left[\tilde{\mathbb{B}}^2 - x[\tilde{\mathbb{B}}, V] + x^2\right]V = VY_{0,-x}V.$$

Hence,

$$\tau_s(Ve^{-Y_{1,x}}) = \tau_s(V^2e^{-Y_{0,-x}}V) = \tau_s(e^{-Y_{0,-x}}V) = \tau_s(Ve^{-Y_{0,-x}}).$$

Therefore we obtain

$$-\int_{\Gamma_1^x} \tau_s(e^{-\mathbb{A}^2}) = \int_{-\infty}^0 \tau_s(Ve^{-Y_{0,x}})dx.$$

Now, since $Y_{0,x} = \tilde{\mathbb{B}}^2 + x[\tilde{\mathbb{B}}, V] + x^2$ and using Duhamel we know that

$$-\int_{\mathbb{R}} \tau_s(Ve^{Y_{0,x}})dx = \sum_{k\geq 0} \left\langle\left\langle V, [\tilde{\mathbb{B}}, V], \cdots, [\tilde{\mathbb{B}}, V]\right\rangle\right\rangle_{\tilde{\mathbb{B}}} \int_{\mathbb{R}} x^k e^{-x^2}dx,$$

a series which converges in the Fréchet topology of $\Omega^*(B)$. Next, computing, $\langle\langle V, [\tilde{\mathbb{B}}, V], \cdots, [\tilde{\mathbb{B}}, V]\rangle\rangle_{\tilde{\mathbb{B}}}$ in terms of the multilinear functional corresponding to $\mathbb{B}_\sigma$, shows that it is trivial when the number of commutators $[\tilde{\mathbb{B}}, V]$ is even. Moreover, when k is odd clearly the integral $\int_{\mathbb{R}} x^k e^{-x^2}dx$ vanishes, and so

$$\int_{\mathbb{R}} \tau_s(Ve^{-Y_{0,x}})dx = 0 \text{ or equivalently } \lim_{x\to+\infty} \int_{\Gamma_1^x} \tau_s(e^{-\mathbb{A}^2}) = -\lim_{x\to+\infty} \int_{\Gamma_0^x} \tau_s(e^{-\mathbb{A}^2}).$$

If we integrate over $(0, +\infty)$ rather than $\mathbb{R}$ in the previous computation, then we obtain

$$\int_0^{+\infty} \tau_s(Ve^{-Y_{0,x}})dx = -\sum_{k\geq 0} \langle\langle V, [\tilde{\mathbb{B}}, V], \cdots, [\tilde{\mathbb{B}}, V]\rangle\rangle_{\tilde{\mathbb{B}}} \int_0^{\infty} x^{2k+1}e^{-x^2}dx$$
$$= -1/2 \sum_{k\geq 0} k!\langle\langle V, [\tilde{\mathbb{B}}, V], \cdots, [\tilde{\mathbb{B}}, V]\rangle\rangle_{\tilde{\mathbb{B}}}.$$

Hence

$$Ve^{-u_0\tilde{\mathbb{B}}^2}[\tilde{\mathbb{B}}, V]e^{-u_1\tilde{\mathbb{B}}^2} \cdots [\tilde{\mathbb{B}}, V]e^{-u_{2k+1}\tilde{\mathbb{B}}^2} =$$
$$(-1)^{k+1} \begin{pmatrix} U^{-1}e^{-u_0\mathbb{B}_\sigma^2}[\mathbb{B}_\sigma, U]e^{-u_1\mathbb{B}_\sigma^2} \cdots [\mathbb{B}_\sigma, U]e^{-u_{2k+1}\mathbb{B}_\sigma^2} & 0 \\ 0 & -Ue^{-u_0\mathbb{B}_\sigma^2}[\mathbb{B}_\sigma, U^{-1}]e^{-u_1\mathbb{B}_\sigma^2} \cdots [\mathbb{B}_\sigma, U^{-1}]e^{-u_{2k+1}\mathbb{B}_\sigma^2} \end{pmatrix}$$

So that, using the fact that the differential forms involved are even,

$$(-1)^{k+1}\langle\langle V,[\tilde{\mathbb{B}},V],\cdots,[\tilde{\mathbb{B}},V]\rangle\rangle_{2k+1}=\langle\langle U^{-1},[\mathbb{B}_\sigma,U],[\mathbb{B}_\sigma,U^{-1}],\cdots,[\mathbb{B}_\sigma,U^{-1}],[\mathbb{B}_\sigma,U]\rangle\rangle_{2k+1}$$
$$-\langle\langle U,[\mathbb{B}_\sigma,U^{-1}],[\mathbb{B}_\sigma,U],\cdots,[\mathbb{B}_\sigma,U],[\mathbb{B}_\sigma,U^{-1}]\rangle\rangle_{2k+1}.$$

We finally obtain

$$\lim_{x\to+\infty}\int_{\Gamma_0^x}\tau_s(e^{-\mathbb{A}^2})=\sum_{k\geq 0}(-1)^{k+1}k!\langle\langle U^{-1},[\mathbb{B}_\sigma,U],\cdots,[\mathbb{B}_\sigma,U^{-1}],[\mathbb{B}_\sigma,U]\rangle\rangle_{2k+1,\mathbb{B}_\sigma},$$

since JLO(D) is a cocycle. □

To end the proof of Theorem 5.8, we are reduced to the following

Lemma 5.12. *In the Fréchet topology of* $\Omega^*(B)$*, we have:* $\lim_{x_0\to+\infty}\int_{\gamma_{x_0}}\tau_s(e^{-\mathbb{A}^2})=0$.

Proof. Recall that

$$\int_{\gamma_{x_0}}\tau_s(e^{-\mathbb{A}^2})=\int_0^1\tau_s(V[\tilde{\mathbb{B}},V]e^{-Y_{t,x_0}})dt.$$

Moreover, an application of our analyticity theorem 3.3 shows that the following Duhamel expansion is convergent in the Fréchet topology of $\Omega^*(B)$, with sum precisely $\tau_s(V[\tilde{\mathbb{B}},V]e^{-Y_{t,x_0}})$

$$e^{-x_0^2}\sum_{k\geq 0}x_0^k(1-2t)^k\int_{\Delta(k)}\tau_s(V[\tilde{\mathbb{B}},V]e^{-u_0\tilde{\mathbb{B}}_t^2}[\tilde{\mathbb{B}},V]e^{-u_1\tilde{\mathbb{B}}_t^2}\cdots[\tilde{\mathbb{B}},V]e^{-u_k\tilde{\mathbb{B}}_t^2})du_1\cdots du_k.$$

Reproducing the estimates of the semi-norms $(p_q)_{q\geq 0}$ of the expression

$$\int_{\Delta(k)}\tau_s(V[\tilde{\mathbb{B}},V]e^{-u_0\tilde{\mathbb{B}}_t^2}[\tilde{\mathbb{B}},V]e^{-u_1\tilde{\mathbb{B}}_t^2}\cdots[\tilde{\mathbb{B}},V]e^{-u_k\tilde{\mathbb{B}}_t^2})du_1\cdots du_k$$

we see that we can find constants C_q depending on the unitary U such that

$$p_q(\int_{\Delta(k)}\tau_s(V[\tilde{\mathbb{B}},V]e^{-u_0\tilde{\mathbb{B}}_t^2}[\tilde{\mathbb{B}},V]e^{-u_1\tilde{\mathbb{B}}_t^2}\cdots[\tilde{\mathbb{B}},V]e^{-u_k\tilde{\mathbb{B}}_t^2})du_1\cdots du_k)\leq C_q^k/k!.$$

Finally notice that $\int_0^1|1-2t|^kdt=2/(k+1)$. As a result we deduce

$$p_q\left(\int_{\gamma_{x_0}}\tau_s(e^{-\mathbb{A}^2})\right)\leq e^{-x_0^2}\sum_{k\geq 0}(C_qx_0)^k/(k+1)!,$$

which converges to zero as $x_0\to+\infty$. □

Appendix A. The HKRC map

Let χ denote the usual HKR-Connes map from the (b,B) bicomplex $\Omega(C^\infty(B)$ to $\Lambda^*(B)$, see [14]. Recall that

$$\chi(f_0\otimes\cdots\otimes f_k):=\frac{1}{k!}f_0df_1\cdots df_k.$$

We prove that χ is bounded from $(\Omega(C^\infty(B)),\Sigma_{\ell+1})$ to $(\Omega^*(B),\Sigma_\ell)$.

Lemma A.1. *There is a constant* $C>0$ *such that for* $k\geq 1$ *and for any smooth functions* $f_0,f_1,\ldots,f)k$ *and for any* $\ell\geq 0$*, we have*

$$||f_0df_1\wedge\ldots\wedge df_k||_\ell\leq C||f_o||_\ell\prod_{j=1}^k||f_j||_{\ell+1}$$

Proof. Proof is by induction on k. Consider the case $k = 1$ so that we need to compute $||f_0 df_1||_\ell$ and hence we need $||L_{X_1} L_{X_2} \dots L_{X_\ell}(f_0 df_1)||_\infty$. For differential forms α, β we have

$$L_X(\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta.$$

Thus induction on ℓ proves that

$$||L_{X_1} L_{X_2} \dots L_{X_\ell}(f_0 df_1)||_\infty \leq 2^\ell ||f_0||_\ell ||f_1||_{\ell+1}.$$

Then because

$$p_\ell(f_0 df_1) = 2^{-\ell} \sup_{\{X_j : ||X_j|| \leq 1, j=1,\dots,\ell\}} 2^{-\ell k} \|L_{X_1} L_{X_2} \dots L_{X_\ell}(f_0 df_1)\|$$

we have $p_\ell(f_0 df_1) \leq ||f_0||_\ell ||f_1||_{\ell+1}$ and hence $||f_0 df_1||_\ell \leq ||f_0||_\ell ||f_1||_{\ell+1}$. Indeed, notice that

$$p_\ell(df) = 2^{-\ell} \sup_{\{X_j : ||X_j|| \leq 1, j=1,\dots,\ell\}} 2^{-\ell k} ||L_{X_1} L_{X_2} \dots L_{X_\ell}(df)$$

and that

$$\|L_{X_1} L_{X_2} \dots L_{X_\ell}(df)\| = \sup_{X, ||X|| \leq 1} \|L_{X_1} L_{X_2} \dots L_{X_\ell}(df)(X)\|.$$

So we have $p_\ell(df) = 2^{-\ell} p_{\ell+1}(f)$ and $p_0(df) = p_1(f)$. We also note that

$$||L_X f_0 df_1||_\infty \leq ||f_0||_1 ||df_1||_1 + ||f_0||_1 ||L_X(df_1)||_\infty \leq 2||f_0||_1 ||f_1||_2$$

which starts the induction over ℓ.

Next assume the result for k and let us deduce it for $k+1$. Thus we start with $\alpha = f_0 df_1 \wedge \dots \wedge df_k$, $\beta = df_{k+1}$. Then we introduce the notation, for ordered subsets of $\{1, \dots, \ell\}$, $J = \{j_1 \leq j_2 \leq \dots \leq j_m\}, 1 \leq m \leq \ell$ then

$$L_{X_1} L_{X_2} \dots L_{X_\ell}(\alpha \wedge \beta) = \sum_{J \subset \{1,\dots,\ell\}} L_J \alpha \wedge L_{J'} \beta$$

where J' is the ordered subset complementary to J. So

$$\|L_{X_1} L_{X_2} \dots L_{X_\ell}(\alpha \wedge \beta)\| \leq \sum_{J \subset \{1,\dots,\ell\}} \|L_J \alpha \wedge L_{J'} \beta\| \leq C \sum_{J \subset \{1,\dots,\ell\}} ||L_J \alpha||_\infty ||L_{J'} \beta||_\infty.$$

Next we have the estimate:

$$||L_J \alpha||_\infty \leq 2^{|J|k} p_{|J|}(\alpha).$$

So

$$C \sum_{J \subset \{1,\dots,\ell\}} ||L_J \alpha||_\infty ||L_{J'} \beta||_\infty \leq C \sum_{J \subset \{1,\dots,\ell\}} 2^{|J|k} p_{|J|}(\alpha) 2^{\ell - |J|} p_{\ell - |J|}(\beta).$$

Now

$$p_{\ell - |J|}(df_{k+1}) = 2^{-\ell} p_{\ell - |J| + 1}(f_{k+1})$$

and as $|J| \leq \ell, \ell - |J| \leq \ell$ we have

$$\begin{aligned} C \sum_{J \subset \{1,\dots,\ell\}} 2^{|J|k} p_{|J|}(\alpha) 2^{\ell - |J|} p_{\ell - |J|}(\beta) &\leq C \sum_{J \subset \{1,\dots,\ell\}} 2^{|J|k} 2^\ell ||\alpha||_\ell 2^{-\ell} ||f_{k+1}||_{\ell+1} \\ &= 2^{|J|k} ||f_0 df_1 \wedge \dots \wedge df_k||_\ell ||f_{k+1}||_{\ell+1} \\ &\leq 2^{\ell k} C' ||f_0||_\ell \prod_{j=1}^{k} ||f_j||_{\ell+1} ||f_{k+1}||_{\ell+1} \end{aligned}$$

Now using the definition

$$p_\ell(\alpha \wedge \beta) = 2^{-\ell k} \sup_{\{X_j : ||X_j|| \leq 1, j=1,\dots,\ell\}} ||L_{X_1} L_{X_2} \dots L_{X_\ell}(\alpha \wedge \beta)||$$

we obtain

$$p_\ell(\alpha \wedge \beta) \leq C' ||f_0||_\ell \prod_{j=1}^{k+1} ||f_j||_{\ell+1}.$$

So we have the desired result:

$$||\alpha \wedge \beta||_\ell \leq C' ||f_0||_\ell \prod_{j=1}^{k+1} ||f_j||_{\ell+1}.$$

□

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