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Index theory for foliations

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Trimester on Groupoids and Stacks

Slides and notes will be posted on
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by the end of the month

- Lecture 1: Index techniques for coverings.
- Lecture 2: Cheeger-Gromov invariant and Keswani's work.
- Lecture 3: Index theory for suspensions and the foliated ρ invariant.
- Lecture 4: Application of the foliated index maps to the gap-label problem for quasi-crystals.
- Lecture 5: Index theory for general foliations. Focus on homotopy invariance of Haefliger signature.

Lecture 1: Survey of index theory for coverings

1. Overview of some classical constructions of Hilbert modules associated with Galois coverings.
2. Dirac operators, the K-theory index (Miscenko-Fomenko/Connes-Skandalis) and relation with the type II index and the type I index.
3. Atiyah's index theorem, comparing traces on K-theory. Relation with BC-conjecture.
4. The signature operator and some words on the Novikov conjecture.
5. The odd case.

Some definitions

In these notes, Γ is a countable discrete group.

Definition

$\mathbb{C}\Gamma$ (resp. $\ell^1\Gamma$)) denotes the convolution $*$-algebra (resp. Banach $*$-algebra) of finitely supported complex valued functions (resp. summable complex valued functions) on Γ. δ_γ is the Dirac function at $\gamma \in \Gamma$. Rules

$$\delta_g \delta_h = \delta_{gh} \text{ and } (\delta_g)^* = \delta_{g^{-1}}.$$

Lemma

The left regular $*$-representation λ of $\ell^1\Gamma$ in $\ell^2\Gamma$

$$\lambda(f)(\xi) = f\xi, \quad f \in \ell^1\Gamma \textit{ and } \xi \in \ell^2\Gamma,$$

is injective.

Definition

- The norm closure of $\ell^1(\Gamma)$ in $B(\ell^2(\Gamma))$ is the reduced C^* algebra of the group Γ and is denoted $C_r^*\Gamma$.
- The completion of $\ell^1(\Gamma)$ with respect to $\sup_{\pi\in\hat{\Gamma}}\|\pi(f)\|$ is the maximal C^* algebra of Γ and is denoted $C_m^*\Gamma$.

Note: We have a $*$-homomorphism $\lambda : C_m^*\Gamma \longrightarrow C_r^*\Gamma$.

Example

If $\Gamma = \mathbb{Z}^n$, then

- $\mathbb{C}\Gamma \simeq \mathbb{C}[z, \bar{z}]$, $z \in T^n$ and $C_r^*\mathbb{Z}^n \simeq C(\mathbb{T}^n)$.
- For $z \in \mathbb{T}^n$, $\pi_z : \mathbb{Z}^n \to U(1)$ with $\pi_z(p_1, \cdots, p_n) := z_1^{p_1}\cdots z_n^{p_n}$ is a general element of $\widehat{\mathbb{Z}^n}$. So, $C_m^*Z^n \simeq C(\mathbb{T}^n) \simeq \mathbb{C}_r^*\mathbb{Z}^n$.

Data and notations...

- M is a closed Riemannian smooth manifold.
- $\pi : \tilde{M} \to M$ is a Galois Γ-cover with $\tilde{M}$ a smooth Riemannian manifold.
- F is a fundamental domain in $\tilde{M}$, i.e. a Borel subset F of $\tilde{M}$ such that $F\gamma \cap F \neq \emptyset \Rightarrow \gamma = 1$ and $\bigcup_{\gamma\in\Gamma} F\gamma = \tilde{M}$.
- A Lebesgue class measure μ on $M \rightsquigarrow$ a Γ-invariant measure $\tilde{\mu}$ on $\tilde{M}$.
- $p : E \to M$ is a hermitian vector bundle over M and $\tilde{E} = \pi^* E$.
- $L^2(M, E)$ (resp. $L^2(\tilde{M}, \tilde{E})$) is the space of (classes of) sections of E over M which are square μ-integrable (resp. of $\tilde{E}$ over $\tilde{M}$ which are square $\tilde{\mu}$-integrable).

MF and CS Hilbert modules

Lemma

The space $C_c(\tilde{M}, \tilde{E})$ is a right module over the $*$-algebra $\mathbb{C}\Gamma$ endowed with a $\mathbb{C}\Gamma$-valued non-negative hermitian product, for the rules:

$$(\xi f)(\tilde{m}) := \sum_{g\in\Gamma} (g\xi)(\tilde{m}) f(g^{-1})$$

$$\text{and } <\xi, \xi'>(g) := <\xi, g^{-1}\xi'>_{L^2(\tilde{M},\tilde{E})},$$

for $\xi, \xi' \in C_c(\tilde{M}, \tilde{E})$ and $f \in \mathbb{C}\Gamma$.

Definition

We set for any $\xi \in C_c(\tilde{M}, \tilde{E})$: $\phi(\xi) := \sum_{g\in\Gamma} R_g^* \xi \otimes \delta_g$.

Lemma

The section $s = \phi(\xi)$ is Γ-invariant: $\gamma s = s$ where for $u \in C^\infty(\tilde{E})$, and $v \in \mathbb{C}\Gamma$ we define $\gamma(u \otimes v) := R_\gamma^* u \otimes \delta_\gamma v$.

Lemma

1. *The space $C(\tilde{M}, \tilde{E} \otimes \mathbb{C}\Gamma)^\Gamma$, of Γ-invariant sections in the above sense, is a right $\mathbb{C}\Gamma$-module, endowed with a $\mathbb{C}\Gamma$-valued non-negative hermitian scalar product, for the rules:*

$$< \eta \otimes \delta_g, \eta' \otimes \delta_{g'} > := \int_F < \eta(\tilde{m}), \eta'(\tilde{m}) > d\tilde{\mu}(\tilde{m}) \delta_{g'^{-1}g}$$

$$\text{and } (s\delta_h)(\tilde{m}) := s(\tilde{m})\delta_h.$$

2. *The map ϕ is an isomorphism of $\mathbb{C}\Gamma$-modules which agrees with the hermitian products.*

Definition

We denote by ϑ_{alg}, ϑ_r and ϑ_m the bundles over M with total spaces the following quotient spaces:

$$\vartheta_{alg} := \tilde{M} \times_\Gamma \mathbb{C}\Gamma, \vartheta_r := \tilde{M} \times_\Gamma C^*_r\Gamma \text{ and } \vartheta_m := \tilde{M} \times_\Gamma C^*_m\Gamma).$$

Corollary

The space $C(M, E \otimes \vartheta_{alg})$ is a right $\mathbb{C}\Gamma$-module which is isomorphic to the right $\mathbb{C}\Gamma$-module $C_c(\tilde{M}, \tilde{E})$. Moreover, ϕ agrees with the hermitian products.

Definition

1. The completion of $C_c(\tilde{M}, \tilde{E})$ with respect to a unitary (injective) representation π of Γ is a Hilbert $C^*_\pi\Gamma$-module, denoted $\mathcal{E}_\pi$.
2. The completion of $C_c(\tilde{M}, \tilde{E})$ with respect to the norm $\|\xi\|_m := \sup_{\pi\in\hat{\Gamma}} \| < \xi, \xi > \|_\pi^{1/2}$ is denoted $\mathcal{E}_m$ and it is a Hilbert module over $C^*_m\Gamma$ called the maximal CS Hilbert module.
3. We shall denote by $\mathcal{E}_r$ the Hilbert $C^*_r\Gamma$-module $\mathcal{E}_\lambda$.
4. In the same way, we can complete $C(M, E \otimes \vartheta_{alg})$ with respect to the norm $\| < \eta, \eta > \|_\pi^{1/2}$ or to the maximal norm to get corresponding Hilbert modules called the MF Hilbert modules and denoted $\mathcal{E}_\pi^{MF}$, $\mathcal{E}_m^{MF}$ or $\mathcal{E}_r^{MF}$ when $\pi = \lambda$.

Proposition

The Hilbert modules $\mathcal{E}_r$ and $\mathcal{E}_r^{MF}$ over $C_r^*\Gamma$ are isomorphic. Same statement holds for the maximal completions.

Corollary

The natural morphism $\lambda_{\mathcal{E}} : C_c(\tilde{M}, \tilde{E}) \to \mathcal{E}_r$ extends into $\lambda_{\mathcal{E}} : \mathcal{E}_m \longrightarrow \mathcal{E}_r$, such that for any $\xi \in \mathcal{E}_m$ and any $a \in C_m^*\Gamma$,

$$\lambda_{\mathcal{E}}(\xi a) = \lambda_{\mathcal{E}}(\xi)\lambda(a).$$

Dirac operators

Notations...

- Fix a generalized Dirac operator $D : C^\infty(M; E) \to C^\infty(M, E)$.
- Denote by $\tilde{D} : C^\infty(\tilde{M}, \tilde{E}) \to C^\infty(\tilde{M}, \tilde{E})$ the (lifted) operator.
- $\tilde{D}$ is then a Γ invariant generalized Dirac operator on $\tilde{M}$.
- The linear operator $\tilde{D}$ is then $\mathbb{C}\Gamma$ linear

Lemma

The operator $\tilde{D}$ is a densely defined closable operator on $\mathcal{E}_m$ (resp. on $\mathcal{E}_r$). Its closure $\mathcal{D}_m$ (resp. $\mathcal{D}_r$) is a self-adjoint operator on $\mathcal{E}_m$ (resp. $\mathcal{E}_r$).

Definition

We set

$$\mathcal{D}_m^{MF} = \phi_m \circ \mathcal{D}_m \circ \phi_m^{-1} \text{ and } \mathcal{D}_r^{MF} = \phi_r \circ \mathcal{D}_r \circ \phi_r^{-1}$$

Proposition

1. *The MF-hilbert module $\mathcal{E}_m^{MF}$ (resp. $\mathcal{E}_r^{MF}$) is equal to $L^2(M, E \otimes \vartheta_m)$ (resp. $L^2(M, E \otimes \vartheta_r)$.*
2. *The operators $\mathcal{D}_m^{MF}$ (resp. $\mathcal{D}_r^{MF}$) and $D \otimes \vartheta_m$ (resp. $D \otimes \vartheta_r$) coincide.*

Proposition

The operator $\mathcal{D}_m$ *is regular. Same statement for* $\mathcal{D}_r$, $\mathcal{D}_m^{MF}$ *and* $\mathcal{D}_r^{MF}$.

Proposition

For any function f *whose restriction to* $\sigma(\mathcal{D}_m) \subset \mathbb{R}$ *is continuous, the operator* $f(\mathcal{D}_m)$ *is a normal regular operator which is self-adjoint when* f *is real. The same statement holds for* $\mathcal{D}_r$.

Proposition

We have

$$\mathcal{E}_m \otimes_\lambda \ell^2\Gamma \simeq L^2(\tilde{M}, \tilde{E}) \text{ and } f(\mathcal{D}_m) \otimes_\lambda I \simeq f(\tilde{D}).$$

In the same way, we have

$$\mathcal{E}_m \otimes_\epsilon \mathbb{C} \simeq L^2(M, E) \text{ and } f(\mathcal{D}_m) \otimes_\epsilon I \simeq f(D).$$

Remark

The above results on $\mathcal{D}_m$ *and* $\mathcal{D}_r$ *hold for the operators* $\mathcal{D}_m^{MF}$ *and* $\mathcal{D}_r^{MF}$ *by using conjugation with the isomorphism* ϕ*.*

The groupoid $G = \tilde{M} \times_\Gamma \tilde{M}$

We introduce

- The Lie groupoid whose space of units is M and whose space of arrows is $G = \tilde{M} \times_\Gamma \tilde{M}$.
- $C_c(G; \mathrm{End}(E))$, with $\mathrm{End}(E) = \mathrm{Hom}(s^*E, r^*E)$, is a $*$-algebra.
- There is a $*$-representation π of $C_c(G; \mathrm{End}(E))$ in $\mathcal{L}_{\mathbb{C}\Gamma}(C_c(\tilde{M}, \tilde{E}))$: $\pi(f)(\xi)(\tilde{m}) := \int_{\tilde{M}} f[\tilde{m}, \tilde{m}']\xi(\tilde{m}')d\tilde{\mu}(\tilde{m}')$.

Definition

1. $L^1(G, \mathrm{End}(E))$ is the Banach algebra for the norm
$$\|f\|_1 := \sup_{\tilde{m} \in \tilde{M}} \int \|f[\tilde{m}, \tilde{m}']\| d\tilde{\mu}(\tilde{m}').$$
2. $C^*_m(G; \mathrm{End}(E))$ is the completion of $L^1(G, \mathrm{End}(E))$ with respect to the collection of all continuous $*$-representations.
3. $C^*_r(G; \mathrm{End}(E))$ is the completion of $L^1(G, \mathrm{End}(E))$ with respect to $\|f\|_r := \|\lambda(f)\|$.

Proposition

- *π extends into $*$-representations π_r^E and π_m^E of $C_r^*(G;E)$ and $C_m^*(G;E)$ in $\mathcal{E}_r$ and $\mathcal{E}_m$ respectively.*
- *π_r and π_m are C^* algebras isomorphisms onto the compacts.*

Proposition

For any $f \in C_0(\mathbb{R})$ the $C_m^*\Gamma$-linear bounded operator $f(\mathcal{D}_m)$ is a compact operator on the Hilbert module $\mathcal{E}_m$.
Similar results hold for the operators $\mathcal{D}_r$, $\mathcal{D}_m^{MF}$ and $\mathcal{D}_r^{MF}$. Hence we get in this way elements of C^*-algebras of the groupoid G.

Proposition

The C^* algebras $C^*_r(G;\mathrm{End}(E))$, $C^*_r(G)$ and $C^*_r\Gamma$ are Morita equivalent. The same property is true for the C^* algebras $C^*_m(G;\mathrm{End}(E))$, $C^*_m(G)$ and $C^*_m\Gamma$.

Remark

Fix a smooth compactly supported continuous non-negative function f on $\tilde{M}$ with support in F and with $\int_{\tilde{M}}(f(\tilde{m}))^2 d\tilde{\mu}(\tilde{m}) = 1$, and define

$$\psi(\delta_h) := \sum_{g\in\Gamma} gf \otimes (gh)f \text{ that we view in } C_c(G).$$

Then, ψ is a $*$-homomorphism from $\mathbb{C}\Gamma$ to $C_c(G)$ which implements by extension the Morita isomorphism in K-theory.

Various indices and traces

Proposition

If $\tilde{Q}$ is a Γ-invariant parametrix for $\tilde{D}$ then the operators $\mathcal{R}_m = I - \mathcal{Q}_m \mathcal{D}_m$ and $\mathcal{S}_m = I - \mathcal{D}_m \mathcal{Q}_m$ are compact operators of $\mathcal{E}_m$. Same statement for $\mathcal{D}_r$.

Theorem

Let $\tilde{Q}$ be a Γ-invariant $\mathbb{Z}_2$-graded compactly supported parametrix for $\tilde{D}$. Set $\tilde{S}_m^+ = I - \tilde{Q}_m^- \tilde{D}_m^+$ and $\tilde{S}_m^- = I - \tilde{D}_m^+ \tilde{Q}_m^-$.
Then

$$\tilde{E} := \begin{pmatrix} (\tilde{S}^+)^2 & -\tilde{Q}^-(\tilde{S}^- + (\tilde{S}^-)^2) \\ -\tilde{S}^- \tilde{D}^+ & I_- - (\tilde{S}^-)^2 \end{pmatrix}$$

is an idempotent
and $[\tilde{E}] - [\pi_-] =: \mathrm{Ind}_\Gamma(\tilde{D}^+)$ belongs to $K_0(C_c^\infty(G; \mathrm{End}(E))$ and does not depend on the choice of $\tilde{Q}$.

Corollary

If we close the operators in $\mathcal{E}_m$ or in $\mathcal{E}_r$ then we get the index classes $\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)$ and $\mathrm{Ind}_\Gamma(\mathcal{D}_r^+)$ respectively.

- $\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)$ denotes also the corresponding element in $K_0(C_m^*(G))$ or even in $K_0(C_m^*\Gamma)$, and similarily for the reduced closure.
- The operator D^+ is elliptic on M and hence admits a Fredholm index

$$\mathrm{Ind}(D^+) = \dim(\mathrm{Ker}(D^+)) - \dim(\mathrm{Ker}(D^-)) \quad \in \mathbb{Z}.$$

Definition

- For $T \in C^*_m\Gamma$, we set
$$\tau_r(T) :=< \lambda(T)\delta_e, \delta_e > \text{ and } \tau_{av}(T) := \epsilon(T).$$
Then τ_r and τ_{av} are positive finite traces on $C^*_m\Gamma$.
- For $f \in C_c(G; \mathrm{End}(E))$ we set
$$\tau_r(f) := \int_F \mathrm{tr}(f[\tilde{m}, \tilde{m}])d\tilde{\mu}(\tilde{m}) \text{ and } \tau_{av}(f) := \sum_{g\in\Gamma} \int_F \mathrm{tr}(f[\tilde{m}, \tilde{m}g])d\tilde{\mu}(\tilde{m}).$$
Then τ_r and τ_{av} extend to positive traces on $C^*_m(G; \mathrm{End}(E))$.

Proposition

The above traces induces group morphisms from the K-theory of the corresponding C^*-algebras to the reals.

Proposition

For any $t > 0$, consider the Wassermann idempotent for the regular or the maximal completions

$$P(t\mathcal{D}) = \begin{bmatrix} e^{-t\mathcal{D}^-\mathcal{D}^+} & (-e^{-t\mathcal{D}^-\mathcal{D}^+/2})\frac{I-e^{-t\mathcal{D}^-\mathcal{D}^+}}{t\mathcal{D}^-\mathcal{D}^+}\sqrt{t}\mathcal{D}^- \\ -e^{-t\mathcal{D}^+\mathcal{D}^-/2}\sqrt{t}\mathcal{D}^+ & I - e^{-t\mathcal{D}^+\mathcal{D}^-} \end{bmatrix}.$$

Then $P(tD)$ is an idempotent and $[P(tD)] - [\pi_-] = \mathrm{Ind}_\Gamma(\mathcal{D}^+)$ in $K_0(\mathcal{K}_{C^*\Gamma}(\mathcal{E}))$.

Corollary

(McKean-Singer) (Also immediate directly!) We have for any $t > 0$,

$$\tau_r(\mathrm{Ind}_\Gamma(\mathcal{D}_r^+)) = \tau_r(e^{-t\mathcal{D}^-\mathcal{D}^+}) - \tau_r(e^{-t\mathcal{D}^-\mathcal{D}^+})$$

$$\text{and } \tau_{av}(\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)) = \tau_{av}(e^{-t\mathcal{D}^-\mathcal{D}^+}) - \tau_{av}(e^{-t\mathcal{D}^-\mathcal{D}^+}).$$

Lemma

The projection $p = p_+ \oplus p_-$ *onto the kernel of* $\tilde{D}$ *has finite* τ_r*-trace. So* $\tau_r(p_+)$ *and* $\tau_r(p_-)$ *exist in* $\mathbb{R}$*.*

Definition

(Atiyah) The ℓ^2 index of $\tilde{D}^+$ is $\mathrm{Ind}_{(2)}(\tilde{D}^+) = \tau_r(p_+) - \tau_r(p_-)$

Theorem

(Atiyah) Set as before

$$\tilde{S}^+ = I - \tilde{Q}^- \tilde{D}^+ \text{ and } \tilde{S}^- = I - \tilde{D}^+ \tilde{Q}^- \text{ so } \tilde{S}^\pm \in C_c^\infty(G; \mathrm{End}(E^\pm)).$$

Then for any $n \geq 1$, $\mathrm{Ind}_{(2)}(\tilde{D}^+) = \tau_r((\tilde{S}^+)^n) - \tau_r((\tilde{S}_-)^n)$.

Theorem

(Atiyah's ℓ^2-index theorem) We have $\mathrm{Ind}_{(2)}(\tilde{D}^+) = \mathrm{Ind}(D^+)$. *In particular,* $\mathrm{Ind}_{(2)}(\tilde{D}^+) \in \mathbb{Z}$.

Corollary

Denote by $\beta^i_{(2)}$ *the* ℓ^2 *dimension of the space of* L^2 *harmonic* i*-forms on* $\tilde{M}$*, then* $\sum_{i=0}^n (-1)^i \beta^i_{(2)} \in \mathbb{Z}$.

Proposition

$$\mathrm{Ind}(D^+) = (\tau_{av,*} \circ \mathrm{Ind}_\Gamma)(\mathcal{D}^+_m) \text{ and}$$

$$\mathrm{Ind}_{(2)}(\tilde{D}^+) = (\tau_{r,*} \circ \mathrm{Ind}_\Gamma)(\mathcal{D}^+_m) = (\tau_{r,*} \circ \mathrm{Ind}_\Gamma)(\mathcal{D}^+_r).$$

Corollary

1. *Denote by $\mu_{r,m}(\Gamma)$ the subgroup of $K_0(C^*_{r,m}\Gamma)$ built up from Γ-indices. Then the additive maps $\tau_{r,*}$ and $\tau_{av,*}$ are integral on $\mu_r(\Gamma)$ and $\mu_m(\Gamma)$ respectively.*
2. *The restrictions of $\tau_{r,*}$ and $\tau_{m,*}$ to $\mu_m(\Gamma)$ do coincide.*

Remark

For groups satisfying BC, the image of $\tau_{r,*}$ is integral.
If Γ satisfies the maximal BC, then the regular and averaged traces coincide on K-theory.
If Γ is for instance amenable then the two traces coincide on K-theory and are integral.

Corollary

(McKean-Singer) We have

$$\mathrm{Ind}(D^+) = \mathrm{Tr}(e^{-tD^-D^+}) - \mathrm{Tr}(e^{-tD^+D^-}) \text{ and}$$

$$\mathrm{Ind}_{(2)}(\tilde{D}^+) = \tau(e^{-t\tilde{D}^-\tilde{D}^+}) - \tau(e^{-t\tilde{D}^+\tilde{D}^-})$$

The signature case

- Again M has even dimension $n = 2m$.
- We have a well defined bilinear form B on $\mathrm{H}^m(M, \mathbb{Z})$ given by cup product.
- If $m = 2k + 1$ then B is antisymmetric, we set $\sigma(M) = 0$.
- If $m = 2k$, the signature of the quadratic form associated with B is $\sigma(M)$.

Definition

Let M and Y be two oriented smooth closed manifolds, then M and Y are (strongly) homotopy equivalent if there exist two orientation preserving smooth maps $f : M \to Y$ and $g : Y \to M$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_M$.

Lemma

If M and Y are homotopy equivalent even oriented closed smooth manifolds, then $\sigma(M) = \sigma(Y)$.

- The Hodge operator $*$ allows to write:

$$< \alpha, \beta > := \int_M \alpha \wedge *\beta.$$

- We have $* \circ * = (-1)^j$ on $\Omega^j(M)$.
- Set $\tau = \sqrt{-1}^{j(j-1)+m} *$ on $\Omega^j(M)$. Then $\tau \circ \tau = id$.

Lemma

We have $d^* = - * \circ d \circ *$ *and* $(d + d^*) \circ \tau + \tau \circ (d + d^*) = 0$.

We thus write $D = d + d^*$ as $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$

Proposition

We have $\mathrm{Ind}(D_+) = \sigma(M)$.

Definition

We define the space $\mathcal{H}^*(\tilde{M})$ of L^2 harmonic forms as the kernel of $\tilde{\Delta}$ on $\tilde{M}$.

Definition

Assume that m is even, then the ℓ^2 signature of $\tilde{M}$ is given by

$$\sigma_{(2)}(M) := \dim_r(\mathcal{H}^m_+(\tilde{M})) - \dim_r(\mathcal{H}^m_-(\tilde{M})).$$

Remark

The ℓ^2 signature can also be defined using ℓ^2 simplicial cohomology.

Corollary

(Atiyah) We have $\sigma(M) = \sigma_{(2)}(M)$.

Theorem

Let $\sigma_\Gamma(M) \in K_0(C^*_m\Gamma)$ be the Γ-index of the operator $\mathcal{D}^+_m$. Then $\sigma_\Gamma(M)$ is invariant under oriented homotopy equivalence.

Theorem

(Hirzebruch) There is a polynomial $\mathcal{L}(M) = \mathcal{L}(p_1, \cdots, p_\ell)$ *such that* $\sigma(M) =< \mathcal{L}(M), [M] >$.

Corollary

The characteristic number $< \mathcal{L}(M), [M] >$ *is an oriented homotopy invariant.*

Definition

Let $f : M \to B\Gamma$ classify $\tilde{M} \to M$. Then for any $x \in H^*(B\Gamma)$, we define higher signature $\sigma_x(M) :=< \mathcal{L}(M) \sqcup f^*(x), [M] >$.

Conjecture

(Novikov) The higher signatures are oriented homotopy invariants.

Remark

If $\sigma_x(M)$ *can be interpreted as the image of the* K*-theory signature* $\sigma_\Gamma(M)$ *under some additive map associated with* x*, then Novikov is proved.*

Assume now that M^{2m-1} is odd dimensional.

Lemma

If χ is a chopping function, then the operator $-e^{i\pi\chi(\mathcal{D}_m)}$ is a unitary operator on $\mathcal{E}_m$ and represents a class $\mathrm{Ind}_\Gamma(\mathcal{D}_m)$ in $K_1(C^*_m\Gamma)$ which does not depend on the choice of χ.
In particular, we have

$$\mathrm{Ind}_\Gamma(\mathcal{D}_m) = (\mathcal{D}_m - iI)(\mathcal{D}_m + iI)^{-1} \in K_1(C^*_m\Gamma).$$

Assume for simplicity that $\dim(M) = 2m-1$ with $m = 2\ell$.

Definition

The operator D on even forms, defined on forms of degree $2k$ by

$$D = (-1)^{\ell+k+1}(*d - d*) : \Omega^{2k}(M) \to \Omega^{2m-2k}(M) \oplus \Omega^{2m-2k-2}(M),$$

is the signature operator on M.

Proposition

The signature operator D is a first order self-adjoint elliptic operator.

Definition

The signature class $\sigma_\Gamma(M)$ in $K_1(C^*_m\Gamma)$ is the Γ-index of the operator $\mathcal{D}_m$ constructed out of the signature operator D.

Theorem

If M and M' are (strongly) homotopy equivalent then $\sigma_\Gamma(M) = \sigma_\Gamma(M')$.