Course at IHP, February 07

Lecture 2: Homotopy invariance of the Cheeger-Gromov invariant Keswani's K-theory proof

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Trimester on Groupoids and Stacks

1. Overview of integration theory in VN algebras.
2. De La Harpe-Skandalis determinants in VN algebras.
3. Eta invariants in VN algebras as determinants.
4. Application to Galois coverings and the Cheeger-Gromov invariant.
5. The Cheeger-Gromov invariant for measured foliations.
6. Homotopy invariance theorem and Keswani's proof.
7. Some words about the bordism proof of Piazza and Schick (see Piazza's talk).

Overview of integration theory in von Neumann algebras

$\mathcal{M}$ is a von Neumann algebra, identified with a weakly closed C^*-subalgebra of $B(H)$. Let τ be a positive trace on $\mathcal{M}$.

Definition

- τ is faithful if $[T \geq 0, \tau(T) = 0] \Longrightarrow T = 0$.
- τ is normal if for any increasing inductive system (T_i) of positive operators, we have $\tau(\sup_i T_t) = \sup_i \tau(T_i)$.
- τ is semi-finite if for any $T \geq 0$ with $\tau(T) = +\infty$, there exists $0 \leq S \leq T$, $S \neq 0$ with $\tau(S) < +\infty$
- $\mathcal{M}$ is semi-finite if for any non zero $T \geq 0$, $\exists$ a normal semi-finite trace τ with $\tau(T) \neq 0$. Then there always exists a normal faithful semi-finite trace on $\mathcal{M}$.

Remark

(Dixmier) If τ is normal, then τ is semi-finite iff $\forall T \geq 0$,

$$\tau(T) = \sup\{\tau(S), 0 \leq S \leq T, \tau(S) < +\infty\}.$$

Example

- $B(H)$ is semi-finite.
- $L^\infty(X,\mu)$ is semi-finite, when μ is σ-finite.
- Atiyah's VN algebra $\mathcal{M}_\Gamma$ is finite with trace τ_r: Here $\mathcal{M}_\Gamma = B(\ell^2\Gamma)^\Gamma$ is the algebra of operators on $\ell^2\Gamma$ which are invariant under the right action R of Γ. The trace is $\tau_r(T) =< T\delta_e, \delta_e >$.
- The VN algebra $\mathcal{M}_{G,E}$ is semi-finite with trace τ_r. Here the trace is given by $\tau_r := \tau_r \otimes \mathrm{Tr}$ when we identify $\mathcal{M}_{G,E}$ with the tensor product of the right VN algebra of Γ with $B(L^2(F,\tilde{E}))$.
- If (M,F) is a smooth foliated (closed) manifold with σ-finite holonomy invariant measure Λ, then $W^*_\Lambda(M,F)$ is semi-finite.

Fix from now a faithful normal semi-finite trace τ on $\mathcal{M}$.

Definition

The densely defined closed operator $T = U|T|$ is affiliated with $\mathcal{M}$ if U and the spectral projections $E_\lambda(|T|)$ belong to $\mathcal{M}$.

Definition

The $\mathcal{M}$-affiliated operator T is τ-measurable if for any $\epsilon > 0$, there exists a projection E in $\mathcal{M}$ such that $E(H) \subset Dom(T)$ and $\tau(I - E) \leq \epsilon$.

Remark

If $(\mathcal{M}, \tau)$ is finite, every $\mathcal{M}$-affiliated operator is τ-measurable.

Definition

(Nelson, Fack-Kosaki) For any $t > 0$, the t^{th} singular number $\mu_t(T)$ of T is

$$\mu_t(T) = \inf\{\|TE\|, E = E^2 = E^* \in \mathcal{M} \text{ and } \tau(I - E) \leq t\}.$$

Definition

Let $p \geq 1$ be fixed. A $\tau-$measurable operator T is p-summable if $t \mapsto \mu_t(T)$ belongs to the space $L^p(\mathbb{R}_+^*)$. The space of such operators is denoted $L^p(\mathcal{M}, \tau)$.

Theorem

(I. Segal) Define the spectral measure m of $|T|$ by

$$m(B) = \tau(1_B(|T|)), (B \text{ Borel subset of } \sigma(|T|).$$

and define $L^p_0(\mathcal{M}, \tau)$ as the space of $\mathcal{M}$-affiliated operators T such that the identity belongs to $L^1(\mathbb{R}_+, m)$, then this space is contained in the set of τ-measurable operators and coincides with the space $L^p(\mathcal{M}, \tau)$.

Lemma

The set $\mathcal{R}(\mathcal{M},\tau)$ of all finite τ-rank operators is a two-sided $*$-ideal in $\mathcal{M}$ which is contained in $\bigcap_{p\geq 1} L^p(\mathcal{M},\tau)$.

Proposition

We have for $S, T, S' \in \mathcal{M}$

1. $\mu_t(T) = d(T, \mathcal{R}_t(\mathcal{M},\tau))$.
2. $\mu_t(STS') \leq \|S\|\mu_t(T)\|S'\|$ *and* $|\mu_t(T) - \mu_t(S)| \leq \|T - S\|$.
3. $\mu_{t+s}(T + S) \leq \mu_t(T) + \mu_s(S)$.
4. $\mu_{t+s}(TS) \leq \mu_t(T)\mu_s(S)$ *and* $\mu_t(T^*) = \mu_t(T)$.
5. *If f is a non decreasing continuous function on $\mathbb{R}_+$ with $f(0) = 0$, then $\mu_t(f(T)) = f(\mu_t(T))$.*

Definition

An operator T is $\tau-$compact if it belongs to $\mathcal{M}$ and if the function $t \mapsto \mu_t(T)$ is zero at infinity. The space of τ-compact operators will be denoted $\mathcal{K}(\mathcal{M},\tau)$.

Corollary

$L^p(\mathcal{M},\tau) \cap \mathcal{M}$ are two-sided $*$-ideals in $\mathcal{M}$, called the Schatten ideals of $\mathcal{M}$.

Proposition

The space $\mathcal{K}(\mathcal{M},\tau)$ is a two-sided closed $*-$ideal in $\mathcal{M}$ which coincides with the closure of $L^p(\mathcal{M},\tau) \cap \mathcal{M}$ for any $p \geq 1$.

Proposition

The trace τ induces a group homomorphism

$$\tau_* : K_0(\mathcal{K}(\mathcal{M},\tau)) \longrightarrow \mathbb{R}.$$

De La Harpe-Skandalis Determinants

Proposition

Let $\gamma : [0,1] \to I + \mathcal{K}$ be a continuous path. For any $\epsilon > 0$, there exists a piecewise affine path $\gamma_\epsilon : [0,1] \to I + L^1$ such that $\|\gamma(t) - \gamma_\epsilon(t)\| \leq \epsilon$ for any $t \in [0,1]$.
Moreover, if $\gamma(0)$ and $\gamma(1)$ belong to $I + L^1$, then we can insure that $\gamma_\epsilon(i) = \gamma(i)$ for $i = 0, 1$.

Definition

Given a uniformly continuous L^1-piecewise C^1 path $\gamma : [0,1] \to \mathrm{GL}(I + L^1)$, we define the determinant $w^\tau(\gamma)$ by the formula

$$w^\tau(\gamma) := \frac{1}{2i\pi} \int_0^1 \tau(\gamma(t)^{-1}\gamma'(t))dt.$$

Proposition

(De La Harpe-Skandalis)

1. *Let* $\gamma : [0,1] \to \mathrm{GL}(I + L^1)$ *be a continuous* L^1*-piecewise* C^1 *path.*
   - *Assume that* $\|\gamma(t) - I\|_1 < 1$, *then for any* $t \in [0,1]$, *the operator* $\mathrm{Log}(\gamma(t))$ *is well defined in* $\mathcal{M}$ *and we have*

     $$w^\tau(\gamma) = \frac{1}{2\pi\sqrt{-1}} \left[\tau(\mathrm{Log}(\gamma(1)) - \tau(\mathrm{Log}(\gamma(0))\right].$$

   - *There exists* $\delta_\gamma > 0$ *such that for any continuous* L^1*-piecewise* C^1 *path* $\alpha : [0,1] \to I + L^1$ *with* $\|\alpha(t) - \gamma(t)\|_1 \leq \delta_\gamma$ *and* $\alpha(i) = \gamma(i), i = 0, 1$, *we have* $w^\tau(\alpha) = w^\tau(\gamma)$.
2. *If* γ *is continuous and uniformly piecewise* C^1, *then* $w^\tau(\gamma)$ *is well defined.*

Definition

Let $\gamma : [0,1] \to \mathrm{GL}(I + \mathcal{K})$ be a continous path. We define the determinant $w^\tau(\gamma)$ by $w(\gamma) := w(\alpha)$, for any continuous L^1-piecewise C^1 path $\alpha : [0,1] \to L^1(\mathcal{M}, \tau)$ satisfying

$$\|\alpha(t) - \gamma(t)\|_1 \leq \delta_\gamma \text{ and } \alpha(i) = \gamma(i), i = 0, 1.$$

Moreover, $w(\gamma)$ only depends on the homotopy class of γ with fixed endpoints.

Theorem

The determinant w^τ induces a homomorphism

$$w_*^\tau : K_1(S\mathcal{K}(\mathcal{M}, \tau)) \longrightarrow \mathbb{R}.$$

Proposition

Let β be the Bott isomorphism for the C^*-algebra $\mathcal{K}(\mathcal{M}, \tau)$. Then we have

$$w_*^\tau \circ \beta = \tau_*.$$

Type II eta invariants

Definition

A finitely summable operator D is a τ-measurable self-adjoint closed operator on H, such that there exists $k \geq 1$ with $(D+i)^{-k} \in L^1(\mathcal{M}, \tau)$.

Remark

If D is a finitely summable operator, then $e^{-sD^2} \in L^1(\mathcal{M}, \tau)$.

Lemma

1. *If f is bounded Borel and compactly supported, then $f(D)$ is τ-trace-class.*
2. *If $f \in \mathcal{S}(\mathbb{R})$, then $f(D)$ is τ-trace class.*
3. *For $f \in C_0(\mathbb{R})$ then $f(D)$ is τ-compact.*

Remark

Interesting examples arise with Connes-von Neumann spectral triples. e.g. Generalized almost periodic Dirac operators, measured foliations.

Let E_λ be the spectral projection on $(-\infty, \lambda)$ associated with D^2.

Lemma

The function $N(\lambda) = \tau(E_\lambda)$ is well defined, non-decreasing, non-negative and left continuous. It thus yields a Borel-Stieljes σ-finite measure ν on $\mathbb{R}$. So for any Borel function $f : \mathbb{R} \to [0, +\infty]$:

$$\tau(f(D^2)) = \int_{\mathbb{R}} f(x) d\nu(x),$$

Lemma

For any $\epsilon > 0$, the function $t \mapsto \tau(De^{-t^2 D^2})$ is integrable on $(\epsilon, +\infty)$.

Definition

We define a spectral number $\eta_\epsilon(D)$ for any finitely summable D by

$$\eta_\epsilon^\tau(D) = \eta_\epsilon(D) := \frac{1}{\sqrt{\pi}} \int_\epsilon^{+\infty} \tau(De^{-t^2D^2})dt.$$

Definition

A chopping function χ is a continuous odd function from $\mathbb{R}$ to $[-1, +1]$ such that

$$\lim_{x \to +\infty} \chi(x) = 1.$$

Remark

Two important examples: $\frac{2}{\pi}\arctan(x)$ *gives the Cayley transform and* $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ *plays a central role in Keswani's proof.*

Proposition

Set $\varphi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ *and* $U_t := -e^{i\pi\varphi(tD)}$. *For any* $(a, b) \in]0, +\infty[^2$ *the path* $(U_t)_{a \leq t \leq b}$ *admits a well defined* τ*-determinant in* $\mathbb{R}$.
Moreover, the limit of this τ*-determinant as* $b \to +\infty$ *exists and coincides with* $\eta_a(D)$.

Assume that the operator D satisfies the extra property

$$t \mapsto \tau(De^{-t^2 D^2}) \text{ is integrable at } 0,$$

then $\eta^\tau(D) := \eta_0(D)$ is the eta invariant of D with respect to the trace τ,

$$\eta(D) := \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \tau(De^{-t^2 D^2}) dt \quad \in \mathbb{R}.$$

Proposition

Under the above assumptions, we have

$$\eta(D) = \lim_{\epsilon \to 0} w^\tau((U_t)_{\epsilon \leq t \leq 1/\epsilon}).$$

Closed manifolds

- Let M be a closed odd dimensional manifold.
- $D : C^\infty(M, E) \to C^\infty(M, E)$ is a generalized Dirac operator on M.
- Then D is finitely summable with respect to $B(L^2(M, E))$. Moreover, we have the Bismut-Freed estimate

$$|\operatorname{Tr}(De^{-t^2 D^2})| = O(t) \text{ as } t \to 0.$$

Therefore, the spectral invariant $\eta(D)$ is well defined.

Lemma

Assume for simplicity that D has no kernel. For $\mathrm{Re}(s)$ large, $D|D|^{-(s+1)}$ is trace class and the function $\eta(s;D) := \tau(D|D|^{-(s+1)})$ has (a meromorphic extension to $\mathbb{C}$ whose) regular value at 0, given by:

$$\eta(0;D) = \eta(D).$$

Remark

When $\mathrm{Ker}(D)$ is not trivial, the function $\eta(s;D)$ is still regular at 0 with value $\eta(D)$.

Back to coverings

Recall that $\tilde{M} \to M$ is a Galois Γ-covering.

- The von Neumann algebra $\mathcal{M}_\Gamma$ has a normal faithful finite trace given by
$$\tau_r(T) :=< T\delta_e, \delta_e > .$$
- If $E \to M$ then $\mathcal{M}_{G,E}$ has a normal faithful semi-finite trace given by $\tau_r = \tau_r \otimes \mathrm{Tr}$.
- We have the $*$-representation
$$\lambda : C_r^*(G, \mathrm{End}(E)) \longrightarrow \mathcal{M}_{G,E}.$$
- Recall also from the first lecture that we have defined a representation π_r in the Hilbert module $\mathcal{E}_r$ which identifies $C_r^*(G, \mathrm{End}(E))$ with the C^*-algebra of compact operators of the $C_r^*\Gamma$ Hilbert module $\mathcal{E}_r$.

Proposition

The C^*-algebra $\mathcal{K}_{C_r^*\Gamma}(\mathcal{E}_r)$ of compact operators in the Hilbert module $\mathcal{E}_r$ embeds isometrically in the C^*-algebra $\mathcal{K}(\mathcal{M}_{G,E}, \tau_r)$ of τ_r-compact elements of the von Neumann algebra $\mathcal{M}_{G,E}$.

Considering now the maximal modules, we recall that π_m is an isomorphism from $C_m^*(G, \mathrm{End}(E))$ to $\mathcal{K}_{C_m^*\Gamma}(\mathcal{E}_m)$ and that we have defined in the first lecture a $*$-representation

$$p_{av} : C_m^*(G, \mathrm{End}(E)) \longrightarrow B(L^2(M, E)).$$

Proposition

The C^*-algebra morphism $p_{av} \circ \pi_m^{-1}$ sends $\mathcal{K}_{C_m^*\Gamma}(\mathcal{E}_m)$ inside the C^*-algebra $\mathcal{K}(L^2(M, E))$ of compact operators in the Hilbert space $L^2(M, E)$.

Proposition

The operator $\tilde{D}$ is finitely summable with respect to the von Neumann algebra $\mathcal{M}_{G,E}$ with tace τ_r.

Proposition

For any $f \in C_b(\mathbb{R})$, $f(\tilde{D})$ belongs to $\mathcal{M}_{G,E}$.
When $f \in C_0(\mathbb{R})$, $f(\tilde{D})$ is τ_r-compact and $f(D)$ is compact in $L^2(M, E)$.
Moreover, if $f \in \mathcal{S}(\mathbb{R})$ is Schwartz, then $f(\tilde{D})$ and $f(D)$ have finite traces in corresponding von Neumann algebras.

Proposition

For any chopping function χ, we define a continuous path of unitaries in $\mathcal{E}_m$ by setting

$$U_t := -\exp(i\pi\chi(t\mathcal{D}_m)).$$

Moreover, $I - U_t \in \mathcal{K}_{C^*_m\Gamma}(\mathcal{E}_m)$.

Corollary

For any chopping function χ, $-e^{i\pi\chi(t\tilde{D})}$ is a unitary operator in the von Neumann algebra $\mathcal{M}_{G,E}$. Moreover, this operator differs from the identity by a τ_r-compact operator. The corresponding statement is true for the operator $-e^{i\pi\chi(tD)}$ in the algebra $B(L^2(M, E))$.

Eta invariants for coverings

Proposition

(Cheeger-Gromov, Ramachandran) The operators $\tilde{D}$ satisfies the estimate

$$\tau_r(\tilde{D}e^{-t^2\tilde{D}^2}) = O(t) \text{ as } t \to 0.$$

Corollary

The eta invariant of the operator $\tilde{D}$ with respect to the trace τ_r on the von Neumann algebra $\mathcal{M}_{G,E}$ is well defined and is given by

$$\eta_{(2)}(\tilde{D}) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \tau_r(\tilde{D}e^{-t^2\tilde{D}^2}) dt.$$

Definition

The Cheeger-Gromov rho invariant of D is the real number

$$\rho(D) = \eta_{(2)}(\tilde{D}) - \eta(D).$$

Proposition

We have

$$\rho(D) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} ((\tau_r \circ \lambda) - (\mathrm{Tr} \circ \epsilon))(\mathcal{D}_m e^{-t^2 \mathcal{D}_m^2}) dt.$$

Definition

The Riemannian eta invariant $\eta(M)$ of M is the eta invariant of the signature operator. In the same way, the Riemannian ℓ^2 eta invariant $\eta_{(2)}(M)$ is the ℓ^2 eta invariant of the signature operator.

Keswani's proof of homotopy invariance

Theorem

Assume that $\Gamma = \pi_1(M)$ is torsion free and satisfies the maximal Baum-Connes conjecture. If the closed oriented odd dimensional manifolds M and M' are (strongly) Homotopy equivalent. Then $\rho(M) = \rho(M')$.

Proof.

- We denote equaly by Γ the isomorphic groups and consider the universal covers $\tilde{M} \to M$ and $\tilde{M}' \to M'$.
- We set $X = M \amalg (-M')$ with its Γ universal cover . Then we reduce the proof to showing that the signature ρ invariant of the manifold X is zero.

Using the homotopy equivalence, one constructs a continous piecewise smooth path $(LT_t^\epsilon)_{1/\epsilon \leq t \leq 2/\epsilon}$ of unitaries in $I + \mathcal{K}_{C_m^*\Gamma}(\mathcal{E}_m \oplus \mathcal{E}'_m)$ such that

1. $LT^\epsilon_{1/\epsilon} = \mathcal{U}_{1/\epsilon} \oplus (\mathcal{U}'_{1/\epsilon})^*$, $LT^\epsilon_{2/\epsilon} = I$.
2. LT^ϵ admits well defined determinants $w(LT^\epsilon)$ and $w_r(LT^\epsilon)$ with respect to the two von Neumann algebras.
3. $\lim_{\epsilon \to 0}(w(LT^\epsilon) - w_r(LT^\epsilon)) = 0$.

We now explain in more detail this construction.

Construction of the LT path

Set $S\omega = i^{p(p-1)+m} \star \omega$ for $\omega \in \Omega^p$ and same operator S' on M'. Set $B = d + d^*$, $B' = d' + d'^*$ the de Rham operators

Lemma

$S^2 = I$, $S^* = S$ and the signature operator D is $-iBS$. Hence $\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)$ is equal to the class of $(B_m + S)(B_m - S)^{-1}$.

Lemma

(Higson-Roe) There is a continuous path of gradings connecting $\begin{pmatrix} S' & 0 \\ 0 & -S \end{pmatrix}$ to $\begin{pmatrix} 0 & R \\ R^* & 0 \end{pmatrix}$ where R as well as the path are defined using the homotopy equivalence between M and M'.

Set $\mathcal{C}_m = \begin{pmatrix} \mathcal{B}'_m & 0 \\ 0 & \mathcal{B}_m \end{pmatrix}$ and $\mathcal{S} = \begin{pmatrix} S' & 0 \\ 0 & -S \end{pmatrix}$.

Proposition

There is a path $\alpha(t)$ such that $\mathcal{C}_m(t) = (\mathcal{C}_m + \alpha(t))(\mathcal{C}_m - \alpha(t))^{-1}$ is a well defined path connecting $(\mathcal{C}_m + \mathcal{S})(\mathcal{C}_m - \mathcal{S})^{-1}$ to I.

Proposition

Let $\varphi(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt$, then there is a smooth path connecting $(\mathcal{C}_m + \mathcal{S})(\mathcal{C}_m - \mathcal{S})^{-1}$ to

$$-e^{i\pi\varphi(\epsilon\mathcal{D}'_m)} \bigoplus -e^{-i\pi\varphi(\epsilon\mathcal{D}_m)}.$$

Corollary

There is a continuous piecewise smooth path denoted after normalization $(LT_t)_{\epsilon/2\leq t\leq\epsilon}$ and called the large time path which connects I to $-e^{i\pi\varphi(\epsilon\mathcal{D}'_m)} \bigoplus -e^{-i\pi\varphi(\epsilon\mathcal{D}_m)}$.

Small time path

Theorem

(Keswani) Let Y be compact with boundary and let (M, E, f) and (M', E', f') be two equivalent BD K-cycles over X (with f and f' Lipschitz). Let χ be smooth chopping with support of $\hat{\chi}$ a subset of $[-1, +1]$ and $\chi', \chi^2 - 1$ and $\chi(\chi^2 - 1)$ Schwartz with Fourier transforms in $[-1, +1]$. Then $\exists \mathcal{H}$, Hilbert space with an action of $C(Y)$, isomorphic to $L^2(M, E) \oplus H$ and to $H' \oplus L^2(M', E')$ and $\exists A, A'$ Kasparov-degenerate operators such that $\chi(D_E) \oplus A$ and $A' \oplus \chi(D'_{E'})$ are connected by a controlled path*.*

Using injectivity of the maximal Baum-Connes map, one then constructs a continuous piecewise smooth path $(ST_t^\epsilon)_{\epsilon/2 \leq t \leq \epsilon}$ of unitaries in $I + \mathcal{K}_{C_m^*\Gamma}(\mathcal{E}_m \oplus \mathcal{E}'_m)$ such that

- $ST_\epsilon^\epsilon = \mathcal{U}_\epsilon \oplus (\mathcal{U}'_\epsilon)^*$, $ST_{\epsilon/2}^\epsilon = I$.
- ST^ϵ admits well defined determinants $w(ST^\epsilon)$ and $w_r(ST^\epsilon)$ with respect to the two von Neumann algebras.
- $\lim_{\epsilon \to 0}(w(ST^\epsilon) - w_r(ST^\epsilon)) = 0$.

Corollary

There is a continuous piecewise smooth (controlled) path $(ST_t)_{\epsilon/2\leq t\leq\epsilon}$ such that

$$ST_\epsilon = -e^{i\pi\chi(\epsilon\mathcal{D}_m^E)} \bigoplus -e^{-i\pi\chi(\epsilon{\mathcal{D}'_m}^{E'})} \text{ and } ST_{\epsilon/2} = I.$$

This is a consequence of

Lemma

Let Y be a compact manifold with boundary as before. Let F be a Kasaprov-degenerate self-adjoint K-cycle on Y and let $\mathcal{F}_m$ be the induced operator on the Hilbert module $\mathcal{E}_m$. Then $-e^{i\pi\mathcal{F}_m} = I$.

End of the proof

- Concatenating these paths together with the path $(\mathcal{U}_t \oplus (\mathcal{U}'_t)^*)_{\epsilon \leq t \leq 1/\epsilon}$, we get a continuous piecewise smooth path $(\mathcal{V}_t)_{\epsilon/2 \leq t \leq 2/\epsilon}$ of unitaries in $I + \mathcal{K}_{C^*_m\Gamma}(\mathcal{E}_m \oplus \mathcal{E}'_m)$ such that
  1. $\mathcal{V}_{\epsilon/2} = \mathcal{V}_{2/\epsilon} = I$ and hence $\mathcal{V}$ is a loop.
  2. The determinants $w(\mathcal{V})$ and $w_r(\mathcal{V})$ with respect to the average representation and the regular representation respectively are well defined numbers.
  3. $\lim_{\epsilon \to 0}(w_r(\mathcal{V}) - w(\mathcal{V})) = \rho(X)$
- The loop $\mathcal{V}$ defines an element of $K_1(S\mathcal{K}_{C^*_m\Gamma}(\mathcal{E}_m \oplus \mathcal{E}'_m))$ and hence by Bott periodicity and Morita equivalence an element of $\beta^{-1}\mathcal{V} \in K_0(C^*_m\Gamma)$ such that

$$w_r(\mathcal{V}) - w(\mathcal{V}) = (\tau_r - \tau_{av})(\beta^{-1}\mathcal{V}).$$

- Using the assumption that the Baum-Connes map is surjective, the class $\beta^{-1}\mathcal{V}$ is the Γ-index of some operator $\mathcal{P}$.
- We have proved in the first lecture that $\tau_r(\operatorname{Ind}_\Gamma \mathcal{P})$ is the ℓ^2 index of and that $\tau_{av}(\operatorname{Ind}_\Gamma \mathcal{P})$ is the index of the operator P downstairs.
- Apply Atiyah's theorem to deduce that $w_r(\mathcal{V}) - w(\mathcal{V}) = 0$.
- Finally, when $\epsilon \to 0$ we deduce that $\rho(X) = 0$.