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Lecture 3: Index techniques for foliations

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Trimester on Groupoids and Stacks

1. More on index theory in von Neumann algebras.
2. C^*-algebras and W^* algebras of foliations given by suspension.
3. Invariant measures and traces. The Atiyah theorem for foliations.
4. Connes' measured index theorem and the foliated Cheeger-Gromov invariant.
5. Hilbert modules associated with foliations given by suspension.
6. The K-theory index and relation with the measured index.
7. Some words on higher index theorems for suspensions.
8. Application of the foliated index techniques to the gap-labelling conjecture.

More on index theory in VN algebras

As in the previous lecture, $\mathcal{M}$ is a von Neumann algebra in a Hilbert space H, equipped with a semi-finite normal faithful trace τ.

Lemma

For any τ-compact projection $e \in \mathcal{M}$, we have $\tau(e) < +\infty$.

Definition

An operator $T \in \mathcal{M}$ is called τ-Fredholm if there exists $S \in \mathcal{M}$ such that $1 - ST$ and $1 - TS$ are τ-compact.

Proposition

If $T \in \mathcal{M}$ is τ-Fredholm, the projections p_T and p_{T^*} onto $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$ are τ-finite.

Definition

The index $\mathrm{Ind}_\tau(T)$ of a τ-Fredholm operator T is defined by:

$$\mathrm{Ind}_\tau(T) := \tau(p_T) - \tau(p_{T^*}),$$

where p_T and p_{T^*} are the projections onto the kernel of T and T^* respectively.

Definition

A p-summable spectral triple is $(\mathcal{A}, \mathcal{M}, D)$ where $\mathcal{M} \subset B(\mathcal{H})$ is as before with a normal semi-finite faithful trace τ, $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{M}$, and D is a τ-measurable self-adjoint operator such that

- $\forall a \in \mathcal{A}$, $a(D+i)^{-1}$ belongs to $L^n(\mathcal{M}, \tau)$, for $n > p$.
- Every $a \in \mathcal{A}$ preserves $Dom(D)$ and $[D, a] \in \mathcal{M}$.
- $\forall a \in \mathcal{A}$, $a, [D, a] \in \cap_{n \geq 0} Dom(\delta^n)$ where $\delta(b) := [|D|, b]$.

The last condition is technical in view of an index theorem, it is satisfied in the examples. **We assume for simplicity that $\mathcal{A}$ is unital.**

Remark

When H is $\mathbb{Z}_2$-graded by an involution $\gamma \in \mathcal{M}$, with $\mathcal{A}$ even and D odd, $(\mathcal{A}, \mathcal{M}, D)$ is said to be even. Otherwise it is odd.

Example

Reorganizing the results of the previous lecture, we see that

- Any generalized Dirac operator D on a closed M yields a n-summable spectral triple $(C^\infty(M), B(L^2(M,E)), D)$.
- For a Galois covering $\tilde{M} \to M$ and a generalized Dirac operator D, the triple $(C^\infty(M), \mathcal{M}_{G,E}, \tilde{D})$ is a n-summable spectral triple.

- Fix a p-summable spectral triple and asume for simplicity that $\mathrm{Ker}(D) = 0$.
- Set $D = F|D|$.

Lemma

- $F = F^*$, $F^2 = 1$.
- $\forall n > p$, *and* $a \in \mathcal{A}$, $[F, a] \in L^n(\tilde{\mathcal{M}}, \tilde{\tau})$.

Assume that the spectral triple is even.

Proposition

For any self-adjoint idempotent $e \in M_n(\mathcal{A})$, *the operator*

$$F_e := e \circ (F \otimes 1_n) \circ e$$

is a τ*-Fredholm operator which anticommutes with the grading.*

Definition

We denote by $\mathrm{Ind}_{D,\tau}(e)$ the $(\tilde{\tau} \otimes \mathrm{Tr})$-index of F_e^+ acting from $e(H_+^n)$ to $e(H_-^n)$. The index map associated with the even spectral triple is $\mathrm{Ind}_{D,\tau} : K_0(\mathcal{A}) \to \mathbb{R}$, given by $\mathrm{Ind}_{D,\tau}(e) := \mathrm{Ind}_\tau(F_e^+)$.

Theorem

Let $(\mathcal{A}, \mathcal{M}, D)$ be an even p-summable VN spectral triple. Let F be the symmetry associated with D as above so that $F^2 = 1$. Then the formula:

$$\phi_{2k}(a^0, ..., a^{2k}) = (-1)^k \tau(\gamma a^0 [F, a^1] \cdots [F, a^{2k}]);$$

defines, for $k > p/2$, a $2k$-cyclic cocycle on the algebra $\mathcal{A}$ and we have for any projection e in $M_N(\mathcal{A})$:

$$\mathrm{Ind}_{D,\tau}(e) := \mathrm{Ind}_\tau((eFe)_+) = (\phi_{2k} \sharp \mathrm{Tr})(e, \cdots, e),$$

independently of the choice of k.

Index problem:

1. Give an abstract local formula for the τ index map $K_0(\mathcal{A}) \longrightarrow \mathbb{R}$ using cyclic cohomology $\rightsquigarrow$ Connes-Moscovici + Carey-Phillips.
2. Deduce *explicit* formulae in interesting examples.

C^* and W^*-algebras for $T \rtimes \Gamma$

- We introduce an extra compact space T with an action of Γ by homeomorphisms.
- The action of Γ is not locally free in general, as it will correspond to the monodromy action on a foliation.
- We consider the crossed product groupoid $\mathcal{G} := T \rtimes \Gamma$ with

$$s(\theta, \gamma) = \gamma^{-1}\theta \quad \text{and} \quad r(\theta, \gamma) = \theta.$$

- We denote by $\mathcal{A}_c$ the convolution $\star$-algebra of compactly supported continuous functions on $\mathcal{G}$, and by $L^1(\mathcal{G})$ its completion with respect to the Banach norm

$$\|f\|_1 := \sup_{\theta \in T} \sum_{\gamma \in \Gamma} |f(\theta, \gamma)|.$$

- The operations are then given by

$$(f * g)(\theta, \gamma) = \sum_{\gamma_1 \in \Gamma} f(\theta, \gamma_1) g(\gamma_1^{-1}\theta, \gamma_1^{-1}\gamma) \text{ and } f^*(\theta, \gamma) = \overline{f(\gamma^{-1}\theta, \gamma^{-1})}$$

Definition

For $\theta \in T$, the regular $*$-representation $\pi_\theta^{\mathrm{reg}}$ of $\mathcal{A}_c$ in $\ell^2(\Gamma)$ is defined by

$$\pi_\theta^{\mathrm{reg}}(f)(\xi)(\gamma) := \sum_{\gamma' \in \Gamma} f(\gamma\theta, \gamma{\gamma'}^{-1})\xi(\gamma').$$

Lemma

The $*$ representation $\pi_\theta^{\mathrm{reg}}$ is L^1-continuous. The family $(\pi_\theta^{\mathrm{reg}})_{\theta \in T}$ is injective and we complete $L^1(\mathcal{G})$ with respect to $\sup_{\theta \in T} \|\pi_\theta^{\mathrm{reg}}\|$ to get the reduced C^*-algebra $\mathcal{A}_r$ of the groupoid $\mathcal{G}$. We shall also denote it $C(T) \rtimes_r \Gamma$.

Definition

The completion of $L^1(\mathcal{G})$ w.r.t. all continuous $*$-representations is the maximal C^*-algebra of the groupoid $\mathcal{G}$, denoted $\mathcal{A}_m$ or $C(T) \rtimes_m \Gamma$.

Remark

Any continuous $*$-homomorphism from $L^1(\mathcal{G})$ in a C^*-algebra extends to a C^*-algebra morphism on $\mathcal{A}_m$. In particular, the homomorphism π^{reg} yields a C^*-algebra morphism $\pi^{\mathrm{reg}} : \mathcal{A}_m \longrightarrow \mathcal{A}_r$.

We fix a Γ-invariant Borel probability measure ν on the compact space T.

Definition

We define $W^*_{\mathrm{reg}}(\mathcal{G})$ as the algebra $L^\infty(T, B(\ell^2\Gamma); \nu)^\Gamma$ of classes modulo equality ν-a.e. of random families $T = (T_\theta)_{\theta \in T}$ of operators $T_\theta \in B(\ell^2(\Gamma))$, such that:

- For any Borel measurable ν square integrable functions ξ on $T \to \ell^2\Gamma$, the map $\theta \mapsto < T_\theta \xi_\theta, \xi_\theta >$ is Borel measurable.
- $\theta \mapsto \| T_\theta \|$ is ν-essentially bounded on T.
- For any $\gamma \in \Gamma$, we have $T_{\gamma\theta} = \gamma T_\theta$.

Remark

Here $\gamma T := R^*_\gamma \circ T \circ R^*_{\gamma^{-1}}$ *and* $(R^*_\alpha \xi)(\beta) := \xi(\beta\alpha)$.

Lemma

$W^*_{\mathrm{reg}}(\mathcal{G})$ *is a von Neumann algebra called the regular von Neumann algebra of* $\mathcal{G}$.

Lemma

The $*$*-representation* π^{reg} *takes values in* $W^*_{\mathrm{reg}}(\mathcal{G})$*, hence,*

$$\pi^{\mathrm{reg}} : \mathcal{A}_{\mathrm{r}} \longrightarrow W^*_{\mathrm{reg}}(\mathcal{G})$$

is a $*$*-representation which then yields a* $*$*-representation of* $\mathcal{A}_m$.

We set $\mathcal{G}_0 := (T \times \Gamma)/\sim$ where we identify (θ, γ) with $(\theta, \gamma\alpha)$ whenever $\alpha\theta = \theta$.

Definition

An element T of $W^*_{\rm triv}(\mathcal{G})$ is a class modulo equality ν-a.e. of random families $(T_\theta)_{\theta \in T}$ of operators $T_\theta \in B(\ell^2(\Gamma/\Gamma(\theta)))$, such that:

- For any Borel measurable ν square integrable section ξ of the Borel field $\ell^2(\Gamma/\Gamma(\theta))$, the map $\theta \mapsto < T_\theta \xi_\theta, \xi_\theta >$ is Borel measurable.
- $\theta \mapsto \| T_\theta \|$ is ν-essentially bounded on T.
- For any $\gamma \in \Gamma$, we have $T_{\gamma\theta} = \gamma T_\theta$.

Remark

Here $\gamma T := R^*_\gamma \circ T \circ R^*_{\gamma^{-1}}$ *and* $(R^*_\alpha \xi)([\beta]) := \xi([\beta\alpha])$. *Then we get a unitary*

$$R^*_\gamma : \ell^2(\Gamma/\Gamma(\theta)) \to \ell^2(\Gamma/\Gamma(\gamma\theta)).$$

Lemma

$W^*_{\mathrm{triv}}(\mathcal{G})$ is a von Neumann algebra called the trivial von Neumann algebra of $\mathcal{G}$.

Definition

For $f \in C_c(\mathcal{G})$ and $\theta \in T$, we set

$$\pi^{av}_\theta(f)(\xi)(x = [\alpha]) := \sum_{y \in \Gamma/\Gamma(\theta)} \sum_{[\beta]=y} f(\alpha\theta, \alpha\beta^{-1})\xi(y), \quad \xi \in \ell^2(\Gamma/\Gamma(\theta)).$$

Proposition

The family $\pi^{av}(f) = (\pi^{av}_\theta(f))_{\theta \in T}$ belongs to $W^*_{\mathrm{triv}}(\mathcal{G})$ and π^{av} extends to a $*$-representations of $\mathcal{A}_m$ in $W^*_{\mathrm{triv}}(\mathcal{G})$.

Definition

For $T = (T_\theta)_{\theta \in T} \geq 0$ in $W^*_{\rm reg}(\mathcal{G})$ (resp. in $W^*_{\rm triv}(\mathcal{G})$), we set

$$\tau^\nu(T) := \int_T < T_\theta(\delta_e), \delta_e > d\nu(\theta),$$

where in the regular case, δ_e is the δ function at the unit e, while in the second case it is the δ function of $[e] \in \Gamma/\Gamma(\theta)$.

Proposition

The functional τ^ν is a faithful normal positive finite trace.

Corollary

1. *The functional $\tau^\nu_{\rm reg} := \tau^\nu \circ \pi^{\rm reg}$ is a positive finite trace on $\mathcal{A}_r$.*
2. *The functional $\tau^\nu_{\rm av} := \tau^\nu \circ \pi^{\rm av}$ is a positive finite trace on $\mathcal{A}_m$.*

Proposition

We have

$$\tau_{\mathrm{reg}}^{\nu}(f) := \int_T f(\theta, e) d\nu(\theta), \quad f \in \mathcal{A}_c,$$

and

$$\tau_{\mathrm{av}}^{\nu}(f) := \int_T \left[\sum_{g \in \Gamma(\theta)} f(\theta, g) \right] d\nu(\theta), \quad f \in \mathcal{A}_c.$$

Remark

If we identify $\Gamma/\Gamma(\theta)$ with the orbit $\Gamma\theta$, then the representation π_θ^{av} is given by

$$\pi_\theta^{av}(f)(\xi)(\theta') = \sum_{\theta'' \in \Gamma\theta} \sum_{\alpha\theta'' = \theta'} f(\theta', \alpha)\xi(\theta'').$$

C^* and W^*-algebras of the foliation groupoid

- Γ acts freely and properly, on $\tilde{M} \times T$ by

$$(\tilde{m}, \theta)\gamma := (\tilde{m}\gamma, \gamma^{-1}\theta), \quad (\tilde{m}, \theta) \in \tilde{M} \times T \text{ and } \gamma \in \Gamma.$$

- The quotient space V is a foliated space $(V, \mathcal{F})$.
- **Note:** Paradigm for such situation is when $\Gamma_1 \hookrightarrow N \to M$ is a Galois cover, Γ_1 acts locally freely on T and $\Gamma = \pi_1(M)$ acts by composition.
- The monodromy cover of a leaf L_θ is

$$\tilde{M} \simeq \tilde{M} \times \{\theta\} \to (\tilde{M} \times \{\theta\})/\Gamma(\theta) \simeq L_\theta.$$

- The (monodromy) groupoid is $G = (\tilde{M} \times \tilde{M} \times T)/\Gamma$, with $G^{(0)} = V$,

$$s[\tilde{m}, \tilde{m}', \theta] = [\tilde{m}', \theta] \text{ and } r[\tilde{m}, \tilde{m}', \theta] = [\tilde{m}, \theta].$$

- Fix a (Lebesgue) measure dm on M and the induced Γ-invariant measure $d\tilde{m}$ on $\tilde{M}$.
- Let $E \to V$ be a vector bundle and $\tilde{E} \to \tilde{M} \times T$ its lift.
- Denote $\mathcal{B}_c^E := C_c^\infty(G, \mathrm{END}(E))$.
- Set $H_\theta = L^2(\widetilde{M} \times \{\theta\}; \tilde{E}_{\tilde{M}\times\{\theta\}})$.

Proposition

$\mathcal{B}_c^E$ is a $*$-algebra for the rules

$$(f_1 * f_2)[\tilde{m}, \tilde{m}', \theta] = \int_{\tilde{M}} f_1[\tilde{m}, \tilde{m}'', \theta] \circ f_2[\tilde{m}'', \tilde{m}', \theta]\, d\tilde{m}''$$

and $f^*[\tilde{m}, \tilde{m}', \theta] = (f[\tilde{m}', \tilde{m}, \theta])^*$.

Definition

For $f \in \mathcal{B}_c^E$ set

$$\pi_\theta^{reg}(f)(\xi)(\tilde{m}) := \int_{\tilde{M}} f[\tilde{m}, \tilde{m}', \theta](\xi(\tilde{m}'))d\tilde{m}', \quad \xi \in H_\theta.$$

Proposition

π_θ^{reg} is a $*$-representation in $\mathcal{H}_\theta$, which is continuous for the L^1 norm

$$\|f\|_1 := \sup_{(\tilde{m},\theta)\in\tilde{M}\times T} \int_{\tilde{M}} \|f[\tilde{m}, \tilde{m}', \theta]\|_E d\tilde{m}'.$$

Definition

The completion of $\mathcal{B}_c$ with respect to $\sup_{\theta\in T} \|\pi_\theta^{reg}(f)\|$ is the regular C^*-algebra of G with coefficients in E, denoted $\mathcal{B}_r^E$ and $\mathcal{B}_r$ when $E = V \times \mathbb{C}$.

Definition

The completion of $\mathcal{B}_c$ with respect to all L^1 continuous $*$-representations is the maximal C^*-algebra of G with coefficients in E, denoted $\mathcal{B}_m^E$ and $\mathcal{B}_m$ when $E = V \times \mathbb{C}$.

Definition

Let $W^*_{\text{reg}}(G; E) = L^\infty(T, L^2(\tilde{M}, \tilde{E}), \nu)^\Gamma$ be the space of classes modulo equality ν-a.e. of families $(T_\theta)_{\theta \in T}$ of bounded operators on $L^2(\tilde{M}, \tilde{E})$ such that

- For any Borel ν square integrable section ξ of $\tilde{E}$ over $\tilde{M} \times T$, the map $\theta \mapsto < T_\theta(\xi_\theta), \xi_\theta >$ is Borel measurable.
- For any $\gamma \in \Gamma$, $T_{\gamma\theta} = \gamma T_\theta$.
- The map $\theta \mapsto \| T_\theta \|$ is ν-essentially bounded on T.

Lemma

$W^*_{\rm reg}(G;E)$ is a von Neumann algebra called the regular von Neumann algebra of the foliation with coefficients in E. It acts on $H = L^2(T \times \tilde{M}, \tilde{E}; \nu \otimes d\tilde{m})$.

Proposition

For any $T \in \mathcal{B}_r^E$, $\pi^{\rm reg}(T)$ belongs to $W^*_{\rm reg}(G;E)$.

We denote by G_0 the quotient Borel space of $\tilde{M} \times T$ for the equivalence

$$(\tilde{m}, \theta) \sim (\tilde{m}\alpha, \theta) \text{ for any } \alpha \in \Gamma(\theta).$$

We recall that $\tilde{M} \to M$ is the universal cover and that the action of Γ on T is not locally free. A typical situation is when a normal quotient Γ_1 acts locally freely on T.

Definition

We define $W^*_{\mathrm{triv}}(G;E)$ as the set of classes modulo equality ν-a.e. of families $(T_\theta)_{\theta\in T}$ of bounded operators on $(L^2(\tilde{M}/\Gamma(\theta),\tilde{E})_{\theta\in T}$ such that

- For any Borel (as a section over G_0) ν square integrable section ξ of the Borel field $L^2(\tilde{M}/\Gamma(\theta))$, the map $\theta \mapsto < T_\theta(\xi_\theta), \xi_\theta >$ is Borel measurable.
- $T_{\gamma\theta} = \gamma T_\theta$.
- $\theta \mapsto \|T_\theta\|$ is ν-essentially bounded on T.

Lemma

$W^*_{\mathrm{triv}}(G;E)$ is a von Neumann algebra acting on the Hilbert space $\int^\oplus L^2(\tilde{M}/\Gamma(\theta))d\nu(\theta)$. It is the trivial von Neumann algebra of (G,E).

Remark

The map which assigns to $[\tilde{m}] \in \tilde{M}/\Gamma(\theta)$ the class of $[\tilde{m},\theta] \in L_\theta$ is a diffeomorphism which allows to view $W^*_{\mathrm{triv}}(G;E)$ as the von Neumann algebra of random operators on the leaves of the foliated space (V,F).

Proposition

If $f \in \mathcal{B}_c^E$, *then we define* $\pi_{\rm av}(f) \in W^*_{\rm triv}(G;E)$ *by setting*

$$\pi^{\rm av}(f)_\theta(\xi)([\tilde m]) = \int_{F_\theta} \sum_{\gamma\in\Gamma(\theta)} f[\tilde m, \tilde m'\gamma, \theta]\xi[\tilde m'] d[\tilde m'].$$

Then $\pi^{\rm av}$ *induces a* $*$*-homomorphism of* $\mathcal{B}_m^E$ *in* $W^*_{\rm triv}(G;E)$.

Proposition

There is a faithful normal positive semi-finite trace τ^ν *on* $W^*_{\rm reg}(G;E)$, *given for any* $T \geq 0$ *by*

$$\tau^\nu(T) := \int_T {\rm tr}(\chi_F \circ T_\theta \circ \chi_F) d\nu(\theta).$$

Proposition

We define a faithful normal positive semi-finite trace τ^ν on the trivial von Neumann algebra $W^*_{\mathrm{triv}}(G; E)$, by setting for $T \geq 0$

$$\tau^\nu(T) := \int_T \mathrm{tr}(\chi_F \circ T_\theta \circ \chi_F) d\nu(\theta),$$

Definition

For $f \in \mathcal{B}_c^E$, we set $\tau^\nu_{\mathrm{reg}}(f) := \int_{F\times T} f[\tilde{m}, \tilde{m}, \theta] d\tilde{m} d\nu(\theta)$ and $\tau^\nu_{\mathrm{av}}(f) := \int_{F\times T} \sum_{\gamma\in\Gamma(\theta)} f[\tilde{m}, \tilde{m}\gamma, \theta] d\tilde{m} d\nu(\theta)$

Proposition

$$\tau^\nu \circ \pi_{\mathrm{reg}} = \tau^\nu_{\mathrm{reg}} \text{ and } \tau^\nu \circ \pi_{\mathrm{av}} = \tau^\nu_{\mathrm{av}},$$

- We assume that M is even dimensional and that $D = (D_L)_L$ is a leafwise generalized Dirac operator on $E = E^+ \oplus E^- \to V$.
- We consider the lifted family $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ of generalized Dirac operators on $\tilde{M}$.

Lemma

D (resp. $\tilde{D}$) is τ^ν-measurable w.r.t. $W^*_{triv}(G; E)$ (resp. $W^*_{reg}(G; E)$).

Proposition

The leafwise (resp. Γ-invariant fiberwise) elliptic operator D^+ on (V, F) (resp. on $\tilde{M} \times T \to T$) admits a well defined measured index

$$\mathrm{Ind}^\nu(D^+) = \dim^\nu(\mathrm{Ker}(D^+)) - \dim^\nu(\mathrm{Ker}(D^-)) \quad \in \mathbb{R}.$$

(resp. $\mathrm{Ind}^\nu(\tilde{D}^+) = \dim^\nu(\mathrm{Ker}(\tilde{D}^+)) - \dim^\nu(\mathrm{Ker}(\tilde{D}^-))$*).*

As a consequence of the results of the second lecture:

Theorem

(Connes) Let $\tilde{Q}$ be a Γ-invariant Γ-compactly supported parametrix for $\tilde{D}$ and set

$$\tilde{S}^+ = I - \tilde{Q}^- \tilde{D}^+ \text{ and } \tilde{S}^- = I - \tilde{D}^+ \tilde{Q}^- \text{ so } \tilde{S}^\pm \in C_c^\infty(G; \mathrm{End}(E^\pm)).$$

Then for any $n \geq 1$,

$$\mathrm{Ind}^\nu(\tilde{D}^+) = \tau^\nu((\tilde{S}^+)^n) - \tau^\nu((\tilde{S}_-)^n).$$

A similar formula holds for the operators on the leaves.

Theorem

(Foliated Atiyah's theorem) We have $\mathrm{Ind}^\nu(\tilde{D}^+) = \mathrm{Ind}^\nu(D^+)$ in $\mathbb{R}$.

Remark

This theorem is valid for any Γ invariant foliation on a Γ-closed manifold.

We assume that $\tilde{M} \to M$ is the universal cover and that M is odd dimensional.

Lemma

(Ramachandran) $\tau^\nu(De^{-t^2D^2}) = O(t)$ *and* $\tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2}) = O(t)$.

Proposition

The functions $t \mapsto \tau^\nu(De^{-t^2D^2})$ *and* $t \mapsto \tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})$ *are integrable on* $(0, +\infty)$ *and we define*

$$\eta^\nu_{down}(D) := \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \tau^\nu(De^{-t^2D^2})dt \text{ and}$$

$$\eta^\nu_{up}(D) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})dt.$$

Definition

The measured Cheeger-Gromov invariant of the leafwise generalized Dirac operator D is by definition $\rho^\nu(D) := \eta^\nu_{up}(D) - \eta^\nu_{down}(D)$.

CS Hilbert modules

Proposition

The space $\mathcal{E}_c = C_c^{\infty,0}(\tilde{M} \times T; \tilde{E})$ *is a right* $\mathcal{A}_c$*-module for*

$$(\xi f)(\tilde{m}, \theta) = \sum_{\gamma \in \Gamma} \xi(\tilde{m}\gamma^{-1}, \gamma\theta) f(\gamma\theta, \gamma), \quad \xi \in \mathcal{E}_c, f \in \mathcal{A}_c.$$

Proposition

We define an $\mathcal{A}_c$*-valued non-negative hermitian product on the module* $\mathcal{E}_c$ *by setting*

$$< \xi_1 ; \xi_2 > (\theta, \gamma) := \int_{\tilde{M}} < \xi_1(\tilde{m}, \gamma^{-1}\theta), \xi_2(\tilde{m}\gamma^{-1}, \theta) > d\tilde{m},$$

Definition

- We complete the pre-Hilbert $\mathcal{A}_c$-module $\mathcal{E}_c$ with respect to the reduced C^*-norm and get a Hilbert C^*-module $\mathcal{E}_r$ over $\mathcal{A}_r$.
- We complete $\mathcal{E}_c$ with respect to the maximal C^*-norm and get the Hilbert C^*-module $\mathcal{E}_m$ over $\mathcal{A}_m$.

Proposition

We define an injective $*$-representation π of $\mathcal{B}_c$ in $\mathcal{L}_{\mathcal{A}_c}(\mathcal{E}_c)$ by setting

$$\pi(\varphi)(\xi)(\tilde{m},\theta) := \int_{\tilde{M}} \varphi[\tilde{m},\tilde{m}',\theta]\xi(\tilde{m}',\theta)d\tilde{m}'.$$

Proposition

For any $\varphi \in \mathcal{B}_c^E$, the morphism π extends to a continuous $*$-representations

$$\pi_r : \mathcal{B}_r^E \longrightarrow \mathcal{K}_{\mathcal{A}_r}(\mathcal{E}_r) \text{ and } \chi_m : \mathcal{B}_m^E \longrightarrow \mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m),$$

which are C^*-algebra isomorphisms.

We consider the Hilbert spaces

$$\mathcal{H}_\theta^{reg} := \mathcal{E}_m \otimes_{\pi_\theta^{reg}} \ell^2(\Gamma) \text{ and } \mathcal{H}_\theta^{av} := \mathcal{E}_m \otimes_{\pi_\theta^{av}} \ell^2(\Gamma/\Gamma(\theta)).$$

Lemma

- *There exists an isomorphism of Hilbert spaces, Φ_θ^{reg}, between $\mathcal{H}_\theta^{reg}$ and $L^2(\tilde{M}, \tilde{E})$.*
- *There exists an isomorphism of Hilbert spaces, Φ_θ^{av}, between $\mathcal{H}_\theta^{av}$ and $L^2(L_\theta, E)$.*

$$\Phi_\theta^{av}(\xi \otimes f)(\tilde{m}, \theta) := \sum_{\gamma \in \Gamma} f(\gamma\theta) \left[\xi(\tilde{m}\gamma^{-1}, \gamma\theta)\right], \quad \xi \in C_c^{\infty,0}(\tilde{M} \times T, \tilde{E}) \text{ and } f \in$$

Proposition

For $S \in \mathcal{B}_m^E$ we have

$$\pi_\theta^{av}(S) = \Phi_\theta \circ \left[\pi_m(S) \otimes_{\pi_\theta^{av}} I_{\ell^2(\Gamma/\Gamma(\theta))}\right] \circ \Phi_\theta^{-1}.$$

Dirac operators on Hilbert modules

- Let $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ be a Γ-equivariant family of generalized Dirac operators. Then $\tilde{D}_\theta$ is a formally self-adjoint elliptic operator on $\tilde{M}$.
- Define $(\mathcal{D}\xi)(\tilde{m}, \theta) := \tilde{D}_\theta(\xi(\cdot, \theta))(\tilde{m})$. Then $\mathcal{D}$ is an $\mathcal{A}_c$ linear operator on $\mathcal{E}_c$.
- We denote by $D = (D_{L_\theta})_{\theta \in T}$ the generalized Dirac operator induced by $\tilde{D}$ along the leaves of (V, F).
- We equally denote by D_{L_θ} and $\tilde{D}_\theta$ the closure of the operators. Then these are self-adjoint operators.

Proposition

The closure $\mathcal{D}_r$ and $\mathcal{D}_m$ of $\mathcal{D}$ in $\mathcal{E}_r$ and $\mathcal{E}_m$ respectively, are regular and self-adjoint operators.

Proposition

Let $\psi \in C(\mathbb{R})$. Then for any $\theta \in T$, the operator, acting on $L^2(L_\theta, E)$, given by

$$\Phi_\theta \circ \left[\psi(\mathcal{D}_m) \otimes_{\pi_\theta^{av}} I_{\ell^2(\Gamma/\Gamma(\theta))}\right] \circ \Phi_\theta^{-1},$$

coincides with the operator $\psi(D_{L_\theta})$.

Proposition

Let $\psi \in \mathcal{S}(\mathbb{R})$. Then $\psi(\mathcal{D}_m) \in \mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$ and the operator $\pi_m^{-1}(\psi(\mathcal{D}_m)) \in \mathcal{B}_m^E$ admits a finite τ_{av}^ν trace and also a finite τ_{reg}^ν trace.

Proposition

- $(\tau_{av}^\nu \circ \pi_m^{-1})(\psi(\mathcal{D}_m)) = \tau^\nu \left[(\psi(D_L))_{L \in V/F}\right]$ *where* $(\psi(D_L))_{L\in V/F}$ *is the corresponding element in the trivial von Neumann algebra* $W^*_{\mathrm{triv}}(G, E)$.
- $(\tau_{reg}^\nu \circ \pi_m^{-1})(\psi(\mathcal{D}_m)) = \tau^\nu \left[(\psi(\tilde{D}_\theta))_{\theta \in T}\right]$ *where* $(\psi(\tilde{D}_\theta))_{\theta\in T}$ *is the corresponding element in the regular von Neumann algebra* $W^*_{\mathrm{reg}}(G, E)$.

The foliated index

- We decompose the regular self-adjoint operator $\mathcal{D}_m$ on the Hilbert $C(T) \rtimes_m \Gamma$-module $\mathcal{E}_m$ with respect to the induced $\mathbb{Z}_2$-grading $\mathcal{E}_m = \mathcal{E}_m^+ \oplus \mathcal{E}_m^-$ into

$$\mathcal{D}_m = \begin{pmatrix} 0 & \mathcal{D}_m^- \\ \mathcal{D}_m^+ & 0 \end{pmatrix}$$

- We concentrate on the maximal completion and the maximal closure. Recall that $G = (\tilde{M} \times \tilde{M} \times T)/\Gamma$.

Let $\tilde{Q}$ be a Γ-invariant $\mathbb{Z}_2$-graded Γ-compactly supported parametrix for $\tilde{D}$, i.e.

$$I - \tilde{Q}\tilde{D} \text{ and } I - \tilde{D}\tilde{Q} \text{ belong to } \lambda(C_c^\infty(G; \mathrm{End}(E)).$$

Set $\mathcal{S}_m^+ = I - \mathcal{Q}_m^- \circ \mathcal{D}_m^+$ and $\mathcal{S}_m^- = I - \mathcal{D}_m^+ \circ \mathcal{Q}_m^-$.

Theorem

The operator

$$\tilde{E} := \begin{pmatrix} (\tilde{S}^+)^2 & -\tilde{Q}^- \circ (\tilde{S}^- + (\tilde{S}^-)^2) \\ -\tilde{S}^- \circ \tilde{D}^+ & I_- - (\tilde{S}^-)^2 \end{pmatrix}$$

is an idempotent. The K-theory class $\mathrm{Ind}_\Gamma(\tilde{D}^+) := [\tilde{E}] - [\pi_-] \quad \in K_0(C_c^\infty(G; \mathrm{End}(E))$ *is by definition the K-theory index of $\mathcal{D}_m^+$. This class does not depend on the choice of the parametrix $\tilde{Q}$.*

Remark

If we close the operators in $\mathcal{E}_m$ or in $\mathcal{E}_r$ then $\tilde{E}$ gives by bounded extension idempotents in $\mathcal{K}_{C(T)\rtimes_m\Gamma}(\mathcal{E}_m) \simeq C_m^*(G; \mathrm{End}(E))$ or in $\mathcal{K}_{C(T)\rtimes_r\Gamma}(\mathcal{E}_r) \simeq C_r^*(G; \mathrm{End}(E))$ and the index classes thus obtained are denoted $\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)$ and $\mathrm{Ind}_\Gamma(\mathcal{D}_r^+)$ respectively.

Remark

We shall also denote by $\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)$ *the image under the Morita isomorphism in* $K_0(C_m^*(G))$ *and similarily for the reduced closure. We shall equally denote by* $\mathrm{Ind}_\Gamma(\mathcal{D}_m^+)$ *the corresponding class in* $K_0(C(T) \rtimes_m \Gamma)$.

Proposition

We have

$$\mathrm{Ind}^\nu(D^+) = (\tau^\nu_{av,*} \circ \mathrm{Ind}_\Gamma)(\mathcal{D}_m^+) \text{ and } \mathrm{Ind}^\nu(\tilde{D}^+) = (\tau^\nu_{r,*} \circ \mathrm{Ind}_\Gamma)(\mathcal{D}_m^+).$$

Proposition

(Connes) If C *is a closed* Γ*-invariant de Rham* k*-current on* T*, then we define a cyclic* k*-cocycle and hence a higher index, by setting*

$$\phi_C(f_0\delta_{g_0}, \cdots, f_k\delta_{g_k}) := \delta_e(g_0 \cdots g_k) \times \langle f_0 g_0^* df_1 (g_0 g_1)^* df_2 \cdots (g_0 \cdots g_{k-1})^* df_k, C\rangle$$