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Lecture 4: Higher index theory for foliations and the higher harmonic signature

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Plan of the lecture

- Following Connes-Skandalis, we define gysin maps for foliations and compute the higher K-theory index.
- Gysin maps in Haefliger cohomology.
- In presence of closed invariant currents, deduce scalar higher index formulae.
- Heitsch-Lazarov index formula as a corollary of the Connes-Skandalis theorem.
- The Chern-Connes character of transversely smooth idempotents and the (twisted) higher harmonic signature in Haefliger cohomology.
- Leafwise homotopy equivalence and the induced maps on leafwise L^2-cohomology.
- Homotopy invariance of the (twisted) higher harmonic signature.

Overview of some theorems

We consider leafwise generalized Dirac operators on even smooth foliations of closed manifolds.
Recall...

1. We have extended Atiyah's theorem to the monodromy groupoid G when there exists an invariant measure.
2. We have defined the measured ρ invariant in the case of suspensions and we have seen that the definition works in general.
3. With the Hilbert modules tools, we could reinterprete the measured ρ invariant using determinants and hence $\rightsquigarrow$ Link with K-theory.
4. We have defined K-theory indices in C^*-algebras of G or in smooth subalgebras. These are "abstract" higher indices.
5. We have stated Connes' measured index theorem.

In K theory.

- There exists a (compactly supported) operator Q such that

$$S_0 = I - QD \text{ and } S_1 = I - DQ \quad \in C_c^\infty(\mathcal{G}; \mathrm{End}(E))$$

- The class in $K_0(C_c^\infty(\mathcal{G})$ of the idempotent

$$\begin{pmatrix} S_0^2 & -Q \circ (S_1 + S_1^2) \\ -S_1 \circ D & I - S_1^2 \end{pmatrix}$$

  is an index class denoted $\mathrm{ind}(D)$.

- Using the natural map $K_0(C_c^\infty(\mathcal{G}) \to K_0(C^*(M, F))$, one gets the Connes C^* index class $\mathrm{Ind}(D)$ which is a deep invariant.

In cohomology

- Using a Chern-Connes character in Haefliger cohomology, get $\mathrm{ch}(\mathrm{ind}(D)) \in \mathrm{H}_c^{2*}(M/F)$.
  Ex: For fibrations, we get a cohomology class of the base.
- Using Connes' Chern character, we get a cyclic homology class.
- When the index bundle Ker(D) exists, its Chern character $\mathrm{Ch}(\mathrm{Ker}(D)) \in \mathrm{H}_c^{2*}(M/F)$ is well defined.
- Using the heat approach, one defines "other" index classes
    - Bismut superconnection approach: Heitsch defined $\mathrm{Tr}(e^{-\mathbb{B}_s^2}) \in \mathrm{H}_c^{ev}(M/F)$ and Lott-Goroghovski defined a cyclic homology class.
    - (B.-Heitsch) The Wassermann idempotent is still well defined as a K-theory class of the superexponentially decaying operators.

Theorem (Connes-Skandalis)

For any leafwise elliptic pseudodifferential operator P, we have $\mathrm{Ind}(P) = \mathrm{Ind}_t(P)$ *in* $K_0(C^*\mathcal{G})$*, where* $\mathrm{Ind}_t(P)$ *is a topological index constructed using shrieck maps in Kasparov's theory of* C^**-algebras.*

Definition

A map $f : N \to M/F$ is a cocycle $(V_\alpha, f_{\alpha\beta})$ on N with values in $\mathcal{G}$. The map f is oriented (resp. K-oriented) if the vector bundle $TN \oplus f^*\nu$ is oriented (resp. K-oriented). It is a submersion provided that for each α, $f_{\alpha\alpha} : V_\alpha \to M$ is transverse to F.

Example

- The projection $M \to M/F$ is a submersion which is oriented (resp. K-oriented) iff TF is.
- The projection $p : T^*F \to M/F$ is a K-oriented submersion.
- If N be a smooth transversal, then its inclusion is a K-oriented submersion $N \to M/F$.

Proposition

If N is a smooth transversal to a smooth foliation (X, F_X), then $j^N : N \to X/F_X$ induces an "excision" morphism

$$j_!^N : K_c(N) \longrightarrow K(C_c^\infty(X, F_X)).$$

Proposition

Let $\pi : E \to X$ be a K-oriented vector bundle overa smooth manifold. Then there is a well defined Thom homomorphism

$$\lambda : K_c(X) \longrightarrow K_c(E),$$

given by $[(G^0, G^1, u)] \mapsto [G \otimes S(E), u \otimes 1 + \gamma \otimes c]$.

Proposition

If $f : X \to Y$ is K-oriented, then there is a well defined Gysin map

$$f_! : K_c(X) \longrightarrow K_c(Y).$$

Proposition

Let $i : M \hookrightarrow \mathbb{R}^{2k}$ *and let* $N \to M$ *be the normal bundle to* TF*. Then* N *can is diffeomorphic to a transversal in* $M \times \mathbb{R}^{2k}$ *with the pull-back* $p_1^* TF$*. Therefore, there is a well defined morphism*

$$K_c(N) \longrightarrow K(C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k})).$$

Proposition

The composite map $f^i : T^*F \to M \to N$ *is* K*-oriented and thus induces a morphism* $f_!^i : K_c(T^*F) \to K_c(N)$*.*

Definition (Connes-Skandalis)

The composite map $\alpha_* \circ j_!^N \circ f_!^i : K_c(T^*F) \to K(C^*(M, F))$ is well defined and denoted Ind_t, it is the CS-topological index map. Here $\alpha_* : K(C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k})) \to K(C^*(M, F))$.

Theorem (Connes-Skandalis)

$\mathrm{Ind} = \mathrm{Ind}_t$.

Haefliger Cohomology

- Write $M = \cup_i U_i$ with U_i a foliation charts for F. Choose a transversal $T_i \subset U_i$ such that $T = \cup T_i$ is smooth and complete.
- In $\Omega^k_c(T)$ consider L^k =span$\{\alpha - h^*\alpha\}$ and set $\Omega^k_c(M/F) = \Omega^k_c(T)/L^k$. Then $d : \Omega^k_c(M/F) \to \Omega^{k+1}_c(M/F)$, and Haefliger cohomology $\mathrm{H}^*_c(M/F)$ is the homology of this complex.

Integration over the fiber of F

- Write $\omega = \sum_i \omega_i \in \Omega^{p+k}(M)$, where $\omega_i \in \Omega^*_c(U_i)$ and Integrate each ω_i along the plaques of U_i to get $\int \omega_i \in \Omega^k_c(T_i)$.
- $\int_F \omega$ is by definition the class of $\sum_i \int \omega_i$. So $\int_F : \Omega^{p+k}(M) \to \Omega^k_c(M/F)$.
- $\int_F$ commutes with differentials and induces

$$\int_F : \mathrm{H}^{p+k}(M) \to \mathrm{H}^k_c(M/F).$$

Gysin in Haefliger cohomology

Definition

Let $f : N \to M/F$ be an oriented submersion. Then the pushforward map $f_! : H_c^*(N;\mathbb{R}) \to H_c^*(M/F)$ is defined by $f_!([\omega]) = [\int_f \omega]$, where $\int_f = f_* \circ \int_{F_N}$.

Remark

We have $d_H \circ \int_f = \int_f \circ d_N$.

Proposition

Let $f : N \to M/F$ *be an arbitrary oriented map. There exists a manifold* W *and oriented maps* $i : N \to W$ *and* $\widehat{f} : W \to M/F$ *where* $\widehat{f}$ *is a submersion, and* $f = \widehat{f} \circ i$.

Proposition

If W_1, W_2 *are smooth manifolds, and* $i_j : N \to W_j$ *and* $f_j : W_j \to M/F$ *are oriented smooth maps and* f_j *a submersion, with* $f = f_j \circ i_j$*. Then* $f_{1!} \circ i_{1!} = f_{2!} \circ i_{2!}$.

Definition

We define the Gysin map $f_! : H_c^*(N; \mathbb{R}) \to H_c^*(M/F)$ as the composition $f_! = \widehat{f}_! \circ i_!$ for any factorization $f = \widehat{f} \circ i$ as above.

Remark

For the statement of the index formula, we are only interested in the Gysin map $\pi_! : \mathrm{H}_c^*(T^*F) \to H_c^*(M/F)$.

Chern-Connes character in $\mathrm{H}^*_c(M/F)$

- Decompose $T\mathcal{G}$ into

$$T\mathcal{G} = TF_s \oplus TF_r \oplus \nu_{\mathcal{G}},$$

  and set $\nu_s := TF_r \oplus \nu_{\mathcal{G}} \simeq s^* TM$, the normal bundle to the foliation F_s on $\mathcal{G}$.
- Set $\Omega^*(\mathcal{G}, E) = C^\infty(E \otimes \Lambda \nu_s^*)$.
- $\Omega^*(\mathcal{G}, E)$ is then an $\Omega^*(M)$-module for:

$$M_\omega \phi = p_\nu(s^*(\omega))\phi, \quad \phi \in \Omega^*(\mathcal{G}, E) \text{ and } \omega \in \Omega^*(M).$$

- Denote by ∇^ν the composite operator

$$\Omega^*(\mathcal{G}, E) \xrightarrow{i} C^\infty(E \otimes \Lambda T^*\mathcal{G}) \xrightarrow{\nabla} C^\infty(E \otimes \Lambda T^*\mathcal{G}) \xrightarrow{p_\nu} \mathcal{A}(\mathcal{G}, E).$$

Lemma

Let $\partial_\nu : \mathrm{End}(\Omega^*(\mathcal{G}, E)) \to \mathrm{End}(\mathcal{A}(\mathcal{G}, E))$ *be* $\partial_\nu(T) = [\nabla^\nu, T]$. *Then* ∂_ν *preserves the space* $\mathcal{A}^*_{-\infty}$ *of smoothing operators with values in differential* $*$*-forms.*

Definition

For $T \in \mathcal{A}^k_{-\infty}$, we define the Haefliger $k-$form $\mathrm{Tr}(T)$ by:

$$\mathrm{Tr}(T) = \int_F \mathrm{tr}(K(\bar{x}, \bar{x}))dx.$$

Proposition

The map Tr *is a graded trace which satisfies*

$$\mathrm{Tr} \circ \partial_\nu = d_H \circ \mathrm{Tr}\,.$$

Remark

If ∂_ν^2 were trivial, then we would finish our definition of the Chern-Connes character by setting for any idempotent $e \in M_N(C_c^\infty(\mathcal{G})) = M_N(\mathcal{A}^0_{-\infty})$:

$$\mathrm{Ch}(e) := (\mathrm{Tr} \circ \mathrm{tr}) \left(e \exp \left(\frac{-(\partial_\nu e)^2}{2i\pi} \right) \right).$$

Proposition

There exists a graded algebra structure on $\tilde{A}_{-\infty} = M_2(\mathcal{A}_{-\infty})$ such that the degree 1 differential δ defined by

$$\delta \tilde{T} = \begin{pmatrix} \partial_\nu \tilde{T}_{11} & \partial_\nu \tilde{T}_{12} \\ -\partial_\nu \tilde{T}_{21} & -\partial_\nu \tilde{T}_{22} \end{pmatrix} + \begin{pmatrix} 0 & -\theta \\ 1 & 0 \end{pmatrix} \tilde{T} + (-1)^{\partial \tilde{T}} \tilde{T} \begin{pmatrix} 0 & 1 \\ -\theta & 0 \end{pmatrix}.$$

is a derivation which satisfies $\delta^2 = 0$.

Lemma

For homogeneous $\tilde{T} \in \tilde{A}_{-\infty}$ *define*

$$\Phi(\tilde{T}) = \mathrm{Tr}(\tilde{T}_{11}) - (-1)^{\partial \tilde{T}}\,\mathrm{Tr}(\tilde{T}_{22}\theta).$$

Then $\Phi : \tilde{A}_{-\infty} \to \mathcal{A}^*_c(M/F)$ *is a graded trace which commutes with the differentials.*

Theorem

Let $B = [\tilde{e}_1] - [\tilde{e}_2]$ *be an element of* $K_0(\mathcal{A}^0_{-\infty})$*. The Haefliger form*

$$(\Phi \circ \mathrm{tr})\Big(e_1 \exp\left(\frac{-(\delta e_1)^2}{2i\pi}\right)\Big) - (\Phi \circ \mathrm{tr})\Big(e_2 \exp\left(\frac{-(\delta e_2)^2}{2i\pi}\right)\Big)$$

is closed and its cohomology class depends only on B*. It is denoted by* $\mathrm{Ch}(B)$ *and called the Chern-Connes character of* B*.*

Theorem

Let C be a holonomy invariant $k-$current. For $T_0, \cdots, T_k$ in $\in C_c^\infty(\mathcal{G}, E)$, the formula

$$\tau_C(T_0, \cdots, T_k) = \langle \Phi(T_0 * \delta T_1 * \cdots * \delta T_k), C \rangle \,,$$

defines a Hochschild cocycle. When C is closed, τ_C is a cyclic cocycle.

Definition

Let C be an even closed Haefliger current. We set $\mathrm{Ind}_C(D^+) := \langle \mathrm{Ind}(D^+), [\tau_C] \rangle$. Then $\mathrm{Ind}_C(D^+)$ is the higher C-index of D^+.

Theorem

Let $\pi : T^*F \to M/F$ *and fix an embedding of* M *in a Euclidean space* $\mathbb{R}^{2k}$*. Then*

$$\begin{tikzcd}
K_c^*(T^*F) \arrow[r, "\pi_!^{\mathbb{R}^{2k}}"] \arrow[d, "ch(\cdot)\wedge Td(TF\otimes\mathbb{C})"'] & K_*(C_c^\infty(\mathcal{G}\times\mathbb{R}^{2k})) \arrow[d, "\int_{\mathbb{R}^{2k}}\circ\mathrm{Ch}^{\mathbb{R}^{2k}}"] \\
\mathrm{H}_c^*(T^*F) \arrow[r, "\pi_!"] & \mathrm{H}_c^*(M/F).
\end{tikzcd}$$

Corollary

For any $u \in K_c(T^*F)$,

$$\mathrm{Ch}[\mathsf{Ind}_t(u)] = \pi_{F!}\left(\mathsf{ch}(u)\,\mathsf{Td}(TF\otimes\mathbb{C})\right).$$

Corollary

Assume that TF *is endowed with a* $K-$*orientation* L*. Let* $\alpha : K^0(M) \to K^0(TF)$ *be the Thom isomorphism. For any* $x \in K^0(M)$, $(\mathrm{Ch}\circ\mathsf{Ind}_t\circ\alpha)(x) = \int_F \hat{A}(TF)e^{c_1(L)/2}\,\mathsf{ch}(x)$.

Leafwise homotopy

Assume now that TF is oriented. Let $f : (M, F) \to (M', F')$ be an oriented homotopy equivalence with oriented homotopy inverse g.

Proposition

f induces a well defined map

$$f^* : \mathrm{H}^*_c(L'_{f(x)}; \mathbb{R}) \to \mathrm{H}^*_c(L_x; \mathbb{R}).$$

Moreover, f maps each leaf L onto its image leaf, and $g^* f^* = I$ and $f^* g^* = I$ in the compactly supported cohomology of each leaf.

Proposition

f induces a well defined map $f^* : \mathrm{H}^*_c(M'/F') \to \mathrm{H}^*_c(M/F)$ on Haefliger cohomology.

To be continued...