CONVERGENCE TO STEADY STATES OF SOLUTIONS OF NON-AUTONOMOUS HEAT EQUATIONS IN $\mathbb{R}^N$

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Abstract. Under certain assumptions on $f$ and $g$ we prove that positive, global and bounded solutions $u$ of the non-autonomous heat equation

$$u_t - \Delta u + f(u) = g(t, x)$$

in $\mathbb{R}^N$ ($N \geq 3$) converge to a steady state.

1. The main result

We consider the non-autonomous heat equation

(1.1) \[
\begin{cases}
  u_t - \Delta u + f(u) = g(t, x) & \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\
  u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^N,
\end{cases}
\]

with $N \geq 3$ and

(1.2) \[ f(u) = u - |u|^{p-1}u \quad (u \in \mathbb{R}) \quad \text{for some } 1 < p < \frac{N + 2}{N - 2}, \]

(1.3) \[ g \in L^2(\mathbb{R}_+; L^2 \cap L^q) \quad \text{for some } q > N \quad (L^p := L^p(\mathbb{R}^N)), \]

supp $g(t, \cdot) \subset K$ for all $t \geq 0$ and some $K \subset \mathbb{R}^N$ compact,

(1.4) $g(t, x) \geq 0$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$, and

Under these growth and regularity conditions on $f$, $g$ and $u_0$ the problem (1.1) admits a unique local solution

$$u \in H^1(0, T; L^2) \cap L^2(0, T; H^2) \cap C([0, T]; H^1),$$

i.e. $u(0) = u_0$ and $u$ satisfies the differential equation (1.1) in $L^2$ for almost every $t \in [0, T]$. This follows from the fact that the Nemytski operator associated with $f$ is locally Lipschitz continuous from $H^1 \cap L^\infty$ into $L^2$ and from the regularizing effect of the Laplace operator in $L^2$, i.e. from the $L^2$-maximal regularity of the gaussian semigroup on $\mathbb{R}^N$; see e.g. Brézis [2, Théorème X.11].

Date: December 2, 2005.

2000 Mathematics Subject Classification. 35B40, 35K55.
We address in this note our attention to positive, global, and bounded solutions, i.e. solutions $u \geq 0$ which exist on $\mathbb{R}_+ := [0, \infty)$ and for which the 'energy' $\|u(t)\|_{H^1}$ remains finite. A bootstrap argument shows that such solutions are in fact also uniformly bounded in space and time, i.e. $\sup_{(t,x)} |u(t, x)| < \infty$. We prove the following convergence result.

**Theorem 1.1.** Let $u \in C(\mathbb{R}_+; H^1)$ be a positive solution of (1.1) such that $\sup_{t \in \mathbb{R}_+} \|u(t)\|_{H^1} < \infty$. Assume that there exists $\delta > 0$, such that

$$\sup_{t \in \mathbb{R}_+} (t^{1+\delta} \int_t^\infty \|g(s)\|^2_{L^2(\mathbb{R}^N)} ds < \infty.$$  

Then $\lim_{t \to \infty} u(t) =: w$ exists in $H^1$ and

$$-\Delta w + f(w) = 0.$$  

**Remark 1.2.** Theorem 1.1 remains true for the nonlinearities

$$f(u) = u + \sum_{j=1}^m b_j |u|^{r_j-1}u - |u|^{p-1}u \quad (u \in \mathbb{R})$$

and

$$f(u) = u + \sum_{j=1}^m b_j |u|^{r_j-1}u - \sum_{i=1}^n a_i |u|^{p_i-1}u \quad (u \in \mathbb{R}),$$

where in both examples the $a_i$, $b_j$ are positive constants, $1 < r_j < p < \frac{N+2}{N-2}$ in the first example, and $1 < r_j < p_i \leq \frac{N}{N-2}$ in the second example, for all $i$, $j$.

**Remark 1.3.** Theorem 1.1 remains also true if the initial value $u_0$ is positive, radial for large $x \in \mathbb{R}^N$ and decreasing. More precisely, it suffices that there exists $R \geq 0$ and a nonincreasing function $\tilde{u}_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u_0(x) = \tilde{u}_0(|x|)$ for $|x| \geq R$ and $u_0(x) \geq \tilde{u}_0(R)$ for $|x| < R$.

Similarly, it suffices that $g(t, x) = \tilde{g}(t, |x|)$ for $t \geq 0$, $|x| \geq R$, and $g(t, x) \geq \tilde{g}(t, R)$ for $t \geq 0$, $|x| < R$, where $\tilde{g} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing in the second variable, for every value of the first one; see in particular the proof of Lemma 2.5 below.

For the autonomous case, i.e. for $g = 0$, Theorem 1.1 has been proved by Cortazar, del Pino & Elgueta [7], Feireisl & Petzeltová [10] (for the second function $f$ from Remark 1.2 above), and Busca, Jendoubi & Poláčik [3] for a larger class of nonlinearities $f$. We refer also to Fašangová [8] for the case $N = 1$ and a larger class of nonlinearities $f$. The non-autonomous equation, but on a bounded domain $\Omega \subset \mathbb{R}^N$,
has been studied by Huang & Takáč [13] and the authors [6]. Note that non-positive solutions need not converge in general; for \( N = 1 \) and certain odd initial values this follows from Fašangová & Feireisl, [9].

The proof of Theorem 1.1, carried out in the following section, goes as follows. We first recall that (1.1) admits a Lyapunov function which is decreasing along solutions. Next we prove that any bounded and positive solution is uniformly continuous and has relatively compact range in \( H^1 \). This first crucial step in the proof follows from the principle of concentrated compactness (see Lions [15]) and the method of moving planes. Limit points of \( u \) are stationary solutions, i.e. solutions \( w \in H^2 \) of the stationary problem (1.6). If 0 is the only limit point of \( u \), then \( \lim_{t \to \infty} u(t) = 0 \). If the solution \( u \) admits a limit point \( w \neq 0 \), then the linearization \( -\Delta + f'(w) \) is a Fredholm operator whose kernel has dimension \( N \); see Ni & Takagi [16]. From this we deduce that the underlying energy satisfies the so-called Lojasiewicz-Simon gradient inequality near \( w \). An energy estimate then shows that \( w \) is the only limit point, i.e. that \( \lim_{t \to \infty} u(t) = w \).

2. PROOF OF THEOREM 1.1

In the following, let the energy \( E : H^1(\mathbb{R}^N) \to \mathbb{R} \) be defined by

\[
E(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} F(v), \quad v \in H^1,
\]

where \( F \) is the primitive of \( f \) satisfying \( F(0) = 0 \). The functional is twice continuously Fréchet differentiable. The derivative \( E' : H^1 \to H^{-1} \) is given by

\[
\mathcal{M}(v) := E'(v) = -\Delta v + f(v), \quad v \in H^1.
\]

Lemma 2.1 (Lyapunov function). Let \( u : C([0,T];H^1) \) be any local solution of (1.1). Then the function \( \Phi : [0,T] \to \mathbb{R} \) defined by

\[
\Phi(t) := E(u(t)) + \frac{1}{2} \int_t^\infty \|g(s)\|_{L^2}^2 \, ds, \quad t \in [0,T],
\]

is decreasing and for almost every \( t \in [0,T] \)

\[
\frac{d}{dt} \Phi(t) = -\frac{1}{2} (\|u_t(t)\|_{L^2}^2 + \| - \Delta u(t) + f(u(t))\|_{L^2}^2).
\]

Proof. For more regular solutions satisfying for example the regularity \( u \in H^1(0,T;H^1) \cap L^2(0,T;H^3) \) the claim follows from the continuous Fréchet differentiability of \( E \) and the chain rule for Sobolev functions.
For general solutions $u$, the claim follows upon an approximation by regular solutions. \hfill \Box

**Lemma 2.2.** Let $u \in C(\mathbb{R}_+; H^1)$ be a global solution of (1.1) such that
\[ \sup_{t \in \mathbb{R}_+} \|u(t)\|_{H^1} < \infty. \]

Then $u_t \in L^2(\mathbb{R}_+; L^2)$. \hfill \Box

**Proof.** Since $u$ is bounded with values in $H^1$, the function $E(u(\cdot))$ is bounded with values in $\mathbb{R}$; use the growth condition on $f$ and the Sobolev embedding theorem in order to see this. Hence, the function $\Phi$ from Lemma 2.2 is bounded. By Lemma 2.1, $\Phi$ is decreasing, and therefore $\frac{d}{dt} \Phi$ is integrable on $\mathbb{R}_+$. The claim follows from (2.3). \hfill \Box

**Lemma 2.3 (Uniform continuity).** Let $u \in C(\mathbb{R}_+; H^1)$ be a global solution of (1.1) such that
\[ \sup_{t \in \mathbb{R}_+} \|u(t)\|_{H^1} < \infty. \]

Then $u$ is uniformly continuous with values in $H^1$. \hfill \Box

**Proof.** Since, by assumption (1.3) and (1.4) and a bootstrap argument, the solution $u$ is also uniformly bounded in time and space, we may without loss of generality assume that $f$ is globally Lipschitz continuous; it suffices to change the function $f$ for large arguments. Then it is easy to see that the boundedness of $u$ in $H^1$ implies that $f(u)$ is bounded with values in $H^1$, too. Moreover, it is easy to see that the function $f(u)$ is weakly measurable measurable with values in $H^1$. By Pettis’ theorem [12], $f(u)$ is therefore strongly measurable with values in $H^1$, i.e. $f(u) \in L^\infty(\mathbb{R}_+; H^1)$.

The function $u$ is a solution of (1.1) if and only if $u$ solves the variation of constants formula,
\[ u = T(\cdot)u_0 - T * (f(u) - u) + T * g, \]

where $T(t) = e^{-t e^{\Delta t}}$ is the exponentially stable semigroup (on $L^2$ and on $H^1$) generated by $\Delta - I$. The first term and the third term on the right-hand side of the variation of constants formula belong to $C_0(\mathbb{R}_+; H^1)$ by exponential stability of $T$ and by $L^2$ maximal regularity (the regularizing effect of the Laplacian on $\mathbb{R}^N$). In fact, the $L^2$ maximal regularity and assumption (1.3) imply
\[ T * g \in L^2(\mathbb{R}_+; H^2) \cap H^1(\mathbb{R}_+; L^2), \]

and the latter space embeds continuously into $C_0(\mathbb{R}_+; H^1)$ by interpolation.

The second term on the right-hand side of the variation of constants formula is uniformly continuous with values in $H^1$ since every convolution of a function in $L^1$ (here the semigroup $T \in L^1(\mathbb{R}_+; \mathcal{L}(H^1))$) and a
function in $L^\infty$ (here the function $f(u) - u \in L^\infty(\mathbb{R}_+; H^1)$) is uniformly continuous. 

The following is one key lemma in the proof of Theorem 1.1.

**Lemma 2.4** (Concentrated compactness). Let $u$ be as in Theorem 1.1, and let $(t_n) \not\to \infty$. Then either $\lim_{n \to \infty} \|u(t_n)\|_{H^1} = 0$ or there exists a subsequence of $(t_n)$ (again denoted by $(t_n)$), $m \geq 1$, solutions $w_i \in H^2$, $1 \leq i \leq m$, of the stationary problem (1.3), and sequences $(x^n_i)_{n \geq 0} \subset \mathbb{R}^N$, $2 \leq i \leq m$, such that

\begin{equation}
\lim_{n \to \infty} |x^n_i| = \infty, \quad \lim_{n \to \infty} |x^n_i - x^n_j| = \infty, \quad 2 \leq i < j \leq m, \quad \text{and}
\end{equation}

\begin{equation}
\lim_{n \to \infty} \|u(t_n) - w_1 - \sum_{i=2}^m w_i(\cdot - x^n_i)\|_{H^1} = 0.
\end{equation}

If $m \geq 2$, then $w_i > 0$ for every $2 \leq i \leq m$.

**Proof.** Let $(t_n) \not\to \infty$ be an arbitrary sequence. By Lemma 2.2 and assumption (1.3), respectively, we have $u_t, g \in L^2(\mathbb{R}_+; L^2)$. Hence, for every $\delta > 0$,

\[ \lim_{t \to \infty} \left\| \frac{1}{\delta} \int_t^{t+\delta} u_t(s) \, ds \right\|_{L^2} = \lim_{t \to \infty} \left\| \frac{1}{\delta} \int_t^{t+\delta} g(s) \, ds \right\|_{L^2} = 0. \]

Let $(\delta_n) \searrow 0$ be arbitrary. Then there exists a subsequence of $(t_n)$ (again denoted by $(t_n)$) such that

\[ \lim_{n \to \infty} \left\| \frac{1}{\delta_n} \int_{t_n}^{t_n + \delta_n} u_t(s) \, ds \right\|_{L^2} = \lim_{n \to \infty} \left\| \frac{1}{\delta_n} \int_{t_n}^{t_n + \delta_n} g(s) \, ds \right\|_{L^2} = 0. \]

As a consequence, if we put $u_n := \frac{1}{\delta_n} \int_{t_n}^{t_n + \delta_n} u(s) \, ds$ and $f(u)_n := \frac{1}{\delta_n} \int_{t_n}^{t_n + \delta_n} f(u(s)) \, ds$, then

\[ \limsup_{n \to \infty} \| - \Delta u_n + f(u_n) \|_{L^2} = \limsup_{n \to \infty} \| f(u_n) - f(u)_n \|_{L^2}. \]

However, by Lemma 2.3 and the local Lipschitz continuity of $f$, the functions $u$ and $f(u)$ are uniformly continuous with values in $H^1$ and $L^2$, respectively. Hence, the right-hand side of the last equality is equal to 0, since

\[ \lim_{\delta \to 0} \sup_{t \in \mathbb{R}_+} \left\| \frac{1}{\delta} \int_t^{t+\delta} u(s) \, ds - u(t) \right\|_{H^1} = 0 \quad \text{and} \quad \lim_{\delta \to 0} \sup_{t \in \mathbb{R}_+} \left\| \frac{1}{\delta} \int_t^{t+\delta} f(u(s)) \, ds - f(u(t)) \right\|_{L^2} = 0. \]
By [15, Theorem III.4], the claim of Lemma 2.4 is true if the sequence $u(t_n)$ was replaced by $u_n$. However, by uniform continuity, $\lim_{n \to \infty} \|u(t_n) - u_n\|_{H^1} = 0$, so that the claim is true as it stands.

□

Lemma 2.5 (Compactness). Let $u$ be as in Theorem 1.1, and let $(t_n) \not\to \infty$. Then there exists a subsequence of $(t_n)$ (again denoted by $(t_n)$) and a positive solution $w \in H^2$ of the stationary problem (1.6) such that $\lim_{n \to \infty} \|u(t_n) - w\|_{H^1} = 0$.

Proof. By the compactness of $\text{supp } u_0$ and by the assumption that $\text{supp } g(t, \cdot) \subset K$ for all $t \geq 0$ and some compact $K \subset \mathbb{R}^N$, there exists $\lambda_0 > 0$ such that

\begin{align}
0 &= u_0(\lambda + y, x_2, \ldots, x_N) \leq u_0(\lambda - y, x_2, \ldots, x_N) \\
0 &= g(t, \lambda + y, x_2, \ldots, x_N) \leq g(t, \lambda - y, x_2, \ldots, x_N)
\end{align}

for all $t \geq 0$, $y \geq 0$, $(x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$ and $\lambda \geq \lambda_0$.

Of course, in (2.6) and (2.7) we have also used the assumption that $u_0$ and $g$ are positive.

Fix $\lambda \geq \lambda_0$, and let

$$\mathcal{H}_\lambda := \mathbb{R}_+ \times \{ x \in \mathbb{R}^N : x_1 \geq \lambda \}$$

and define for every $(t, x) \in \mathcal{H}_\lambda$

$$v^1(t, x_1, \ldots, x_N) := u(t, x_1, \ldots, x_N),$$
$$v^2(t, x_1, \ldots, x_N) := u(t, 2\lambda - x_1, x_2, \ldots, x_N)$$
and
$$g^2(t, x_1, \ldots, x_N) := g(t, 2\lambda - x_1, x_2, \ldots, x_N).$$

Then, by definition of $v^1$, $v^2$, $g^2$ and (2.7),

$$v^1_t - \Delta v^1 + f(v^1) = 0 \leq g^2(t) = v^2_t - \Delta v^2 + f(v^2) \text{ on } \text{int } \mathcal{H}_\lambda,$$

and, by (2.6),

$$v^1(0, x) \leq v^2(0, x) \text{ whenever } x \in \mathbb{R}^N, x_1 \geq \lambda.$$

By the parabolic maximum principle,

$$u(t, \lambda + y, x_2, \ldots, x_N) \leq u(t, \lambda - y, x_2, \ldots, x_N)$$
for all $\lambda \geq \lambda_0$, $y \geq 0$, $(x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$, $t \geq 0$.

In particular,

$$\frac{\partial u}{\partial x_1}(t, x) \leq 0 \text{ for all } x_1 \geq \lambda_0, (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, t \geq 0.$$
Repeating the same procedure for arbitrary hyperplanes in $\mathbb{R}^N$, we deduce that there exists $R \geq 0$ such that

$$\nabla u(t, x) \cdot \frac{x}{|x|} \leq 0 \text{ for all } x \in \mathbb{R}^N, |x| \geq R, t \geq 0. \tag{2.9}$$

Any positive solution $w \neq 0$ of the stationary problem (1.6) is strictly positive and $\lim_{|x| \to \infty} w(x) = 0$, [1]. Taking into account of (2.4), the relations (2.5) and (2.9) are compatible only if in (2.5) there is only one function, namely $w_1$. □

Uniqueness of the ground state follows from [14] or [4, Theorem 2.5]. In [4], uniqueness has been proved for a class of nonlinearities which includes also the examples from Remark 1.2.

**Lemma 2.6** (Uniqueness of the ground state). There is at most one positive solution $w \neq 0$ (up to translation) of the stationary problem (1.6).

The following lemma is a consequence of [16, Lemma 4.2, Appendix C]. It follows also from the proof of [10, Proposition 4.1], since our nonlinearity $f$ (and the nonlinearities from Remark 1.2) satisfies the conditions (F1)-(F5) of [4].

**Lemma 2.7.** Let $w \in H^2$ be a positive solution of the stationary problem (1.6). If $w \neq 0$, then the linearization

$$\mathcal{L}(w) : H^1 \to H^{-1}, \quad v \mapsto -\Delta v + f'(w)v$$

is a Fredholm operator of index 0 and $\dim \ker \mathcal{L}(w) = N$.

As a corollary, we obtain that the energy defined in (2.1) satisfies the Lojasiewicz-Simon inequality near every positive solution $w \neq 0$ of the stationary problem (1.6).

**Lemma 2.8** (Lojasiewicz-Simon inequality). Let $w \in H^2$ be a positive solution of the stationary problem (1.6). If $w \neq 0$, then there exist $\sigma, C > 0$ such that for every $v \in H^1$ with $\|v - w\|_{H^1} \leq \sigma$

$$|E(v) - E(w)|^{\frac{1}{2}} \leq C \|v - \Delta v + f(v)\|_{H^{-1}}.$$

**Proof.** By Lemma 2.7, the linearization $\mathcal{L}(w) = -\Delta + f'(w)$ is a Fredholm operator of index 0, and $\dim \ker \mathcal{L}(w) = N$. On the other hand, since $w \neq 0$, the set $\{w(x + \cdot) : x \in \mathbb{R}^N\}$ of all translates of $w$ forms an $N$-dimensional manifold of stationary solutions. In other words, the kernel of $\mathcal{L}(w)$ and the manifold of stationary solutions (in a neighbourhood of $w$) have the same dimension, and $w$ lies in the interior of
the manifold of stationary solutions. The claim thus follows from [17, Lemma 1, p.80], [11, Theorem 2.1], or [5, Corollary 3.12]. □

We are ready to prove our main result.

Proof of Theorem 1.1. By Lemma 2.5, the $\omega$-limit set $\omega(u)$ of the solution $u$ is a non-empty, compact subset of the set of positive stationary solutions. Since it is also connected, and since the set of all positive stationary solutions consists only of the zero solution and of all translates of the unique positive ground state (Lemma 2.6), we have either $\omega(u) = \{0\}$ or there exists $(t_n) \to \infty$ and a solution $w > 0$ of the stationary problem such that $\lim_{n \to \infty} u(t_n) = w$ and $\omega(u) \subset \{w(\cdot - x) : x \in \mathbb{R}^N\}$. In the first case we have $\lim_{t \to \infty} u(t) = 0$ and there remains nothing to prove. So we can assume the second case.

Let $\delta > 0$ be as in assumption (1.5). Note that we may assume without loss of generality that $\delta < 1/2$. Let $\theta < \delta/2 < 1/2$. By Lemma 2.8, there exist constants $\sigma, C > 0$ such that for every $v \in H^1$ with \begin{equation}
|E(v) - E(w)|^{1-\theta} \leq C \|M(v)\|_{H^{-1}}.
\end{equation}
Here and in the following, we put $M(v) := -\Delta v + f(v)$. In order to see that (2.10) is true, it suffices to choose the constant $\sigma > 0$ from Lemma 2.8 so small that $|E(v) - E(w)| \leq 1$ whenever $\|v - w\|_{H^1} \leq \sigma$, and to note that $1 - \theta \geq 1/2$. Let

$$
\Sigma := \{t \geq 1 : \|u(t) - w\|_{H^1} \leq \frac{\sigma}{3}\}.
$$

By Lemma 2.5, the set $\Sigma$ is unbounded. For every $t \in \Sigma$ we define

$$
\tau(t) := \sup\{t' \geq t : \sup_{s \in [t,t']} \|u(s) - w\|_{H^1} \leq \sigma\}.
$$

By continuity, $\tau(t) > t$ for every $t \in \Sigma$.

Let $t_0 \in \Sigma$. We divide the interval $J := [t_0, \tau(t_0))$ into two subsets:

$$
A_1 := \{t \in J : \|\dot{u}(t)\|_{L^2} + \|M(u(t))\|_{L^2} \geq \left(\int_t^{\tau(t_0)} \|g(s)\|_{L^2}^2 \, ds\right)^{1-\theta}\},
$$

$$
A_2 := J \setminus A_1.
$$

Let further

$$
\Phi_0(t) := E(u(t)) - E(w) + \frac{1}{2} \int_t^{\tau(t_0)} \|g(s)\|_{L^2}^2 \, ds, \quad t \in J.
$$
The function $\Phi_0$ differs from the function $\Phi$ defined in (2.2) only by a constant. Hence, for every $t \in J$,
\[
\frac{d}{dt} \Phi_0(t) = \frac{d}{dt} \Phi(t) = -\frac{1}{2} (\|\dot{u}(t)\|_{L^2}^2 + \|M(u(t))\|_{L^2}^2),
\]
so that $\Phi_0$ is a decreasing function. Moreover, for every $t \in J$,
\[
(2.11) \quad \frac{d}{dt}|\Phi_0(t)|^\theta = -\frac{\theta}{2} \text{sgn} \Phi_0(t) |\Phi_0(t)|^{\theta-1}(\|\dot{u}(t)\|_{L^2}^2 + \|M(u(t))\|_{L^2}^2).
\]
Hence, $|\Phi_0|^\theta$ is decreasing if $\Phi_0$ is positive, and increasing if $\Phi_0$ is negative. By (2.10), for every $t \in A_1$,
\[
|\Phi_0(t)|^{1-\theta} \leq |E(u(t)) - E(w)|^{1-\theta} + \left(\frac{1}{2} \int_t^{\tau(t_0)} \|g(s)\|_{L^2}^2 \, ds\right)^{1-\theta} \leq C_2(\|M(u(t))\|_{L^2} + \|\dot{u}(t)\|_{L^2}).
\]
Together with equation (2.11) this implies
\[
\|M(u(t))\|_{L^2} + \|\dot{u}(t)\|_{L^2} \leq C_3 \left|\frac{d}{dt}|\Phi_0(t)|^\theta\right|,
\]
so that $\dot{u}$ is dominated by an integrable function on $A_1$. Moreover,
\[
\int_{A_1} \|\dot{u}(t)\|_{L^2} \, dt \leq C_3 \int_{t_0}^{\tau(t_0)} \left|\frac{d}{dt}|\Phi_0(t)|^\theta\right| \, dt \leq C_3 \left\{|\Phi_0(t_0)|^\theta + |\Phi_0(\tau(t_0))|^\theta\right\}
\]
where we interprete the term involving $\tau(t_0)$ on the right hand side of the second inequality sign as 0 if $\tau(t_0) = \infty$.

If $t \in A_2$, then, by definition of $A_2$ and by assumption (1.5),
\[
(2.13) \quad \|\dot{u}(t)\|_{L^2} + \|M(u(t))\|_{L^2} \leq \left(\int_t^{\tau(t_0)} \|g(s)\|_{L^2}^2 \, ds\right)^{1-\theta} \leq C_4 t^{-(1+\delta)(1-\theta)},
\]
so that $\dot{u}$ is dominated by an integrable function on $A_2$, since $(1+\delta)(1-\theta) \geq 1$. Moreover,
\[
(2.14) \quad \int_{A_2} \|\dot{u}(t)\|_{L^2} \, dt \leq C_5 t_0^{-\delta+2\theta}.
\]
Combining the inequalities (2.12) and (2.14), we see that $\|\dot{u}\|_{L^2}$ is absolutely integrable on the interval $J$ and
\[
\lim_{t_0 \to \infty} \int_{t_0}^{\tau(t_0)} \|\dot{u}(t)\|_{L^2} \, dt = 0.
\]
Note that for every $t \in J$,
\[
\|u(t) - w\|_{L^2} \leq \int_{t_0}^{t} \|\dot{u}(s)\|_{L^2} \, ds + \|u(t_0) - w\|_{L^2}.
\]

From this inequality we obtain that $\tau(t_0) = \infty$ for some $t_0 \in \Sigma$. In fact, assume that this is not true. Then, by definition, $\|u(\tau(t_0)) - w\|_{H^1} = \sigma$ for every $t_0 \in \Sigma$. Let $(t_n)_{n \in \mathbb{N}} \subset \Sigma$ be an unbounded sequence such that $\lim_{n \to \infty} \|u(t_n) - w\|_{H^1} = 0$. By compactness (Lemma 2.5), and by passing to a subsequence, there exists $\tilde{w} \in \omega(u)$ such that $\|\tilde{w} - w\|_{H^1} = \sigma$ and $\lim_{n \to \infty} \|u(\tau(t_n)) - \tilde{w}\|_{H^1} = 0$. Then, by the above inequality,
\[
0 < \|w - \tilde{w}\|_{L^2} \leq \limsup_{n \to \infty} \left\{ \int_{t_n}^{\tau(t_n)} \|\dot{u}(s)\|_{L^2} \, ds + \|u(t_n) - w\|_{L^2} \right\} = 0,
\]
which is a contradiction.

Hence, $\tau(t_0) = \infty$ for $t_0$ large enough. By the above arguments, the function $\|\dot{u}\|_{L^2}$ is thus absolutely integrable on $[t_0, \infty)$, which implies that $\lim_{t \to \infty} u(t)$ exists in $L^2$. By compactness (Lemma 2.5), $\lim_{t \to \infty} u(t) = w$ in $H^1$. The claim is proved. \qed

References

4. C. C. Chen and C. S. Lin, Uniqueness of the ground state of solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, $n \geq 3$, Commun. Partial Differential Equations 16 (1991), 1549–1572.


