WILLMORE BLOW-UPS ARE NEVER COMPACT

RALPH CHILL, EVA FAŠANGOVÁ, AND REINER SCHÄTZLE

Abstract. We prove a Lojasiewicz-Simon gradient inequality for the Willmore functional near Willmore surfaces and apply this inequality to exclude compact blow-ups for the Willmore flow.

1. Introduction

For a smooth immersed closed surface \( f : \Sigma \rightarrow \mathbb{R}^n \) the Willmore functional is defined by

\[
W(f) = \frac{1}{4} \int_\Sigma |H|^2 \, d\mu_f,
\]

where \( H \) denotes the mean curvature vector of \( f \) and \( \mu_f \) is the measure induced by the pull-back metric \( g = f^*g_{euc} \). The Gauß equations and the Gauß-Bonnet Theorem give rise to equivalent expressions:

\[
W(f) = \frac{1}{4} \int_\Sigma |A|^2 \, d\mu_f + 2\pi (1 - p(\Sigma)) = \frac{1}{2} \int_\Sigma |A^0|^2 \, d\mu_f + 4\pi (1 - p(\Sigma)),
\]

where \( A \) denotes the second fundamental form, \( A^0 = A - \frac{1}{2}g \otimes H \) its tracefree part and \( p(\Sigma) \) is the genus of \( \Sigma \). The Willmore functional is scale invariant and moreover invariant under the full Möbius group of \( \mathbb{R}^n \). Critical points of \( W \) are called Willmore surfaces or, more precisely, Willmore immersions.

We always have \( W(f) \geq 4\pi \) with equality only for round spheres; see [13], [14, Theorem 7.2.2] in codimension one, that is \( n = 3 \). On the other hand, if \( W(f) < 8\pi \), then \( f \) is an embedding by an inequality of Li and Yau [7].

By a Willmore flow of the surface \( \Sigma \) we mean a family \((f_t)\) of immersions \( f_t : \Sigma \rightarrow \mathbb{R}^n \) which solves the geometric evolution equation

\[
(1.1) \quad \partial_t f_t + \frac{1}{2}(\Delta_f H + Q(A^0)H) = 0.
\]

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The normal Laplacian $\Delta^\perp$ and the semilinear term $Q(A^0)H$ are defined below. We point out that this evolution equation is a gradient system for the Willmore functional, [4].

It is known that (1.1) is locally well-posed and smoothing for initial values $f_0 \in C^{4,\alpha}(\Sigma)$. However, the question of the maximal existence time $T$ and the behaviour of the Willmore flow near the maximal existence time is in general still open. By [5, Theorem 5.2], the Willmore flow of a sphere in $\mathbb{R}^3$ with initial energy $\leq 8\pi$ exists globally and converges to a round sphere (see [3], [11] for earlier results of this type). On the other hand, a counterexample from [9] shows that the Willmore flow can develop a singularity near the maximal existence time, the initial energy in this counterexample being $> 8\pi$, see also [1].

In [3, Section 4] and [5, pp. 348-349] a blow-up procedure was set up showing that for each sequence $t_j \to T$ there exists a subsequence denoted again $t_j$ and $r_j > 0$, $x_j \in \mathbb{R}^n$ such that

$$f_j := r_j^{-1}(f_{t_j} + c_0 r_j^4 - x_j)$$

converges, for given $c_0 = c_0(n) > 0$ and after appropriate reparametrization, smoothly on compact subsets of $\mathbb{R}^n$ to a Willmore immersion $f_W : \hat{\Sigma} \to \mathbb{R}^n$, where $\hat{\Sigma} \neq \emptyset$ is a complete surface without boundary. Moreover, it is possible to select $t_j \not\to T$ in such a way that $f_W$ is non-trivial in the sense $A_{f_W} \neq 0$, that is $f_W$ does not parametrize a union of planes. For $T < \infty$, we always have $r_j \to 0$ by [4, Theorem 1.2], and the limit of $f_j$, after reparametrization, is called blow-up. For $T = \infty$, we may have $r_j \to 0, r_j = 1$ or $r_j \to \infty$, after rescaling and passing to appropriate subsequences. In case $r_j \to 0$, we still call the limit a blow-up, whereas in case $r_j \to \infty$, we call the limit a blow-down.

In this article, we are interested in the case when $\hat{\Sigma}$ is compact. We know by [3, Lemma 4.3], if one of the components of $\hat{\Sigma}$ is compact, then $\Sigma$ and $\hat{\Sigma}$ are diffeomorphic; in particular, $\hat{\Sigma}$ is compact.

The main result of our article excludes compact blow-ups and blow-downs.

**Theorem 1.1.** For any Willmore flow $(f_t)_t$ of a closed surface $\Sigma$, none of the components of blow-ups or blow-downs is compact.

This main result will follow from a global existence and convergence result for Willmore flows starting near compact Willmore surfaces and having energy above these Willmore surfaces; near is here to be understood with respect to the norm in $W^{2,2} \cap C^1(\Sigma)$. This global existence and convergence result is in turn a consequence of parabolic regularity and the fact that the Willmore functional satisfies the Lojasiewicz-Simon gradient inequality near every Willmore immersion.
As a byproduct of the proof of the main result we obtain that Willmore flows starting near local minimizers of the Willmore functional exist globally and converge to a Willmore surface which is again a local minimizer of the Willmore functional.

**Theorem 1.2.** Let $\Sigma$ be a closed surface and let $f_W : \Sigma \to \mathbb{R}^n$ be a Willmore immersion which locally minimizes the Willmore functional in $C^k$ ($k \geq 2$) in the sense that there exists $\delta > 0$ such that

$$W(f) \geq W(f_W) \text{ whenever } \|f - f_W\|_{C^k} \leq \delta.$$ 

Then there exists $\varepsilon > 0$ such that for any immersion $f_0 : \Sigma \to \mathbb{R}^n$ satisfying

$$\|f_0 - f_W\|_{W^{2,2}\cap C^1} < \varepsilon$$

the corresponding Willmore flow $(f_t)_t$ with initial data $f_0$ exists globally and converges smoothly, after reparametrisation by appropriate diffeomorphisms $\Phi_t : \Sigma \to \Sigma$, to a Willmore immersion $f_\infty$ which also minimizes locally the Willmore functional in $C^k$, i.e.

$$f_t \circ \Phi_t \to f_\infty \text{ as } t \to \infty.$$ 

We point out that if $f_W$ is a local minimizer of the Willmore functional in $C^k$, then it is also a local minimizer in $W^{2,2}\cap C^1$ (see Remark 5.2 below).

For round spheres in codimension one Theorem 1.2 has been proved by Simonett [11], provided that the $W^{2,2}\cap C^1$-norm is replaced by the stronger $h^{2+\alpha}$ norm. Theorem 1.2 applies in particular also to the Willmore surfaces which minimize globally the Willmore functional among those surfaces with fixed genus.

The fact that we can start with initial values which are close to a Willmore surface in the $W^{2,2}\cap C^1$ topology follows from an existence, uniqueness and regularity result for Willmore flows which is proved in the Appendix.

2. **Preliminaries**

Let $\Sigma$ be a closed surface, i.e. a compact, smooth, two-dimensional manifold without boundary. Given an immersion $f : \Sigma \to \mathbb{R}^n$ ($n \geq 3$) we define the pull-back metric $g = g_f$ by

$$\langle X, Y \rangle_g = \langle Df \cdot X, Df \cdot Y \rangle_{\text{euc}}.$$ 

The corresponding Riemannian connection $\nabla$ is given by the representation

$$Df \cdot \nabla_X Y = PD_X(D_Y f),$$

where $P = P_f$ denotes the orthogonal projection of $\mathbb{R}^n$ onto the tangent space $T_{f(p)}f(\Sigma) \subset \mathbb{R}^n$, and $X$ and $Y$ denote abstract tangent vector fields. In local
coordinates, the projection $P$ is given by the formula

$$P x = g^{ij} \langle x, \partial_i f \partial_j f \rangle \partial_j f, \quad x \in \mathbb{R}^n.$$  

The second fundamental form $A = A_f$ is given by

$$A(X, Y) = D_X(D_Y f) - D_f \cdot \nabla_X Y = P^\perp D_X(D_Y f),$$

with $P^\perp = I - P$. The mean curvature vector $H = H_f$ is by definition the trace of $A$,

$$H = A(e_i, e_i),$$

the $(e_i)$ being any orthonormal basis in the tangent space $T_p \Sigma$.

The trace-free part $A^0$ of the second fundamental form is given by

$$A^0(X, Y) = A(X, Y) - \frac{1}{2} \langle X, Y \rangle_g H.$$  

We will also consider the normal connection $\nabla^\perp$ associated with $f$. It acts on normal vector fields $\phi : \Sigma \to \mathbb{R}^n$ (i.e. $\phi = P^\perp \phi$) and is given by the formula

$$\nabla^\perp_X \phi = P^\perp D_X \phi.$$  

In the following we write $\nabla^\perp \phi$ for the linear map $X \mapsto \nabla^\perp_X \phi$ which maps the abstract tangent space into the normal space, and we define the corresponding scalar product

$$\langle \nabla^\perp \phi, \nabla^\perp \psi \rangle := \langle \nabla^\perp \phi(e_i), \nabla^\perp \psi(e_i) \rangle = \langle P^\perp D_e \phi, P^\perp D_e \psi \rangle,$$

where $(e_i)$ is any orthonormal basis in $T_p \Sigma$. In local coordinates, this yields

$$\langle \nabla^\perp \phi, \nabla^\perp \psi \rangle = g^{ij} \langle \partial_i \phi, \partial_j \psi \rangle - g^{ij} g^{kl} \langle \partial_i \phi, \partial_k f \rangle \langle \partial_l f, \partial_j \psi \rangle.$$

For any $k \geq 0$, we let

$$W^{k,2}(\Sigma; \mathbb{R}^n)^\perp := \{ \phi \in W^{k,2}(\Sigma; \mathbb{R}^n) : \phi = P^\perp f \}$$

be the subspace of all vector fields in the Sobolev space $W^{k,2}(\Sigma; \mathbb{R}^n)$ which are normal along $f$. The normal Laplace operator $\Delta^\perp = \Delta^\perp_g$ is the closed linear operator on $L^2(\Sigma; \mathbb{R}^n)^\perp$ defined by

$$\int_{\Sigma} \langle \nabla^\perp \phi, \nabla^\perp \psi \rangle d\mu_f = - \int_{\Sigma} \langle \psi, \nu \rangle d\mu_f,$$

$$\Delta^\perp \phi := \psi,$$

see also [12].
For $k \geq 3$, we denote by $W_{imm}^{k,2}(\Sigma; \mathbb{R}^n)$ the set of all immersions in $W^{k,2}(\Sigma; \mathbb{R}^n)$. This set is open in $W^{k,2}(\Sigma; \mathbb{R}^n)$. The Willmore functional $W : W_{imm}^{k,2}(\Sigma; \mathbb{R}^n) \to \mathbb{R}$ is defined by

$$W(f) := \frac{1}{4} \int_\Sigma |H|^2 \, d\mu_f.$$ 

In [4, Section 2], it was proved that for normal vector fields $\phi \in W^{4,2}(\Sigma; \mathbb{R}^n)$ (i.e. $\phi = P^\perp_f \phi$)

$$W'(f) \cdot \phi = \frac{d}{dt} W(f + t\phi)|_{t=0}$$

$$= \frac{1}{2} \int_\Sigma \langle \Delta^\perp H + Q(A^0)H, \phi \rangle \, d\mu_f$$

$$= : \int_\Sigma \langle \delta W(f), \phi \rangle \, d\mu_f. \tag{2.4}$$

Here, the operator $Q(A^0)$ is acting on vector fields $\phi : \Sigma \to \mathbb{R}^n$ by

$$Q(A^0)\phi = A^0(e_i, e_j)\langle A^0(e_i, e_j), \phi \rangle_{euc},$$

where again $(e_i)$ is any local orthonormal frame. Formula (2.4) justifies that (1.1) is a gradient flow.

3. **Lojasiewicz-Simon gradient inequality**

In this section we prove that the Willmore functional, when considered on the open set of all immersions in $W^{4,2}(\Sigma; \mathbb{R}^n)$, satisfies the so-called Lojasiewicz-Simon gradient inequality near every Willmore immersion. The Lojasiewicz-Simon gradient inequality for real-valued functions goes back to the classical result by Lojasiewicz [8] for real analytic functions on $\mathbb{R}^N$ and to its generalization to the infinite-dimensional case by L. Simon [10]. In the case of the Willmore functional, it is just the inequality (3.2) below. The exponent $\theta$ from the Lojasiewicz-Simon gradient inequality is called Lojasiewicz exponent.

As indicated in the equality (2.4), we abbreviate

$$\delta W(f) := \frac{1}{2}(\Delta^\perp H + Q(A^0)H).$$

**Theorem 3.1** (Lojasiewicz-Simon gradient inequality). Let $f_W : \Sigma \to \mathbb{R}^n$ be a Willmore immersion. Then there exist constants $\theta \in (0, \frac{1}{2}), c_1, c_2 \geq 0$ and $\sigma > 0$ such that for every $f \in W^{4,2}(\Sigma; \mathbb{R}^n)$ with $\|f - f_W\|_{W^{4,2}} \leq \sigma$ one has

$$|W(f) - W(f_W)|^{1-\theta} \leq c_1 \|W'(f)\|_{L^2(\Sigma; \mathbb{R}^n)} \leq c_2 \|\delta W(f)\|_{L^2(\Sigma; \mathbb{R}^n)}. \tag{3.2}$$

In other words, the Willmore functional satisfies the Lojasiewicz-Simon gradient inequality near every Willmore immersion.
Note that Theorem 3.1 remains true if \(f_W\) is not a Willmore immersion, i.e. if \(W'(f_W) \neq 0\). However, in this case, the statement is trivial by simple continuity arguments.

We describe the idea of the proof of Theorem 3.1; the proof itself will be carried out at the end of this section. We will first restrict the Willmore functional to all immersions in \(f \in W^{4,2}(\Sigma; \mathbb{R}^n)\) for which \(f - f_W\) is normal along \(f_W\), and we will show that this restriction satisfies the Lojasiewicz-Simon gradient inequality. In order to prove the inequality, by [2, Corollary 3.11], it suffices to prove that the restriction of the Willmore functional is an analytic function, that its first derivative is analytic from \(W^{4,2}(\Sigma; \mathbb{R}^n)\) into \(L^2(\Sigma; \mathbb{R}^n)\), and that the second variation is a Fredholm operator between these two spaces. These properties of the restriction of the Willmore functional will be essentially proved in two lemmas.

Having the Lojasiewicz-Simon gradient inequality for the restriction of the Willmore functional, Theorem 3.1 will follow by noting that the Willmore functional does not depend on the parametrization of the surface, i.e. \(\mathcal{W}(f) = \mathcal{W}(f \circ \Phi)\) for every immersion \(f\) and every diffeomorphism \(\Phi : \Sigma \rightarrow \Sigma\).

For the definition of analytic functions between two Banach spaces, we refer to [15]. In the following lemma, \(\mathcal{L}(\mathbb{R}^n)\) is the space of all linear operators on \(\mathbb{R}^n\).

**Lemma 3.2.** Let \(\Sigma\) be a closed surface, and let \(X, Y : \Sigma \rightarrow T\Sigma\) be abstract tangent vector fields. The following functions are well-defined and analytic:

(i) The function \(W_{imm}^{4,2}(\Sigma; \mathbb{R}^n) \rightarrow W^{3,2}(\Sigma)\) given by
\[
f \mapsto \langle X, Y \rangle_g.
\]
(ii) The function \(W_{imm}^{4,2}(\Sigma; \mathbb{R}^n) \rightarrow W^{3,2}(\Sigma; \mathcal{L}(\mathbb{R}^n))\) given by
\[
f \mapsto Pf.
\]
(iii) The function \(W_{imm}^{4,2}(\Sigma; \mathbb{R}^n) \rightarrow W^{2,2}(\Sigma; \mathbb{R}^n)\) given by
\[
f \mapsto A(X, Y).
\]
(iv) The function \(W_{imm}^{4,2}(\Sigma; \mathbb{R}^n) \rightarrow W^{2,2}(\Sigma; \mathbb{R}^n)\) given by
\[
f \mapsto H.
\]
(v) The function \(F : W_{imm}^{4,2}(\Sigma; \mathbb{R}^n) \rightarrow W^{2,2}(\Sigma; \mathbb{R}^n)\) given by
\[
f \mapsto F(f) = Q(A^0)H.
\]
(vi) The function \(W_{imm}^{4,2}(\Sigma; \mathbb{R}^n) \rightarrow L^2(\Sigma; \mathbb{R}^n)\) given by
\[
f \mapsto \Delta^\perp H.
\]
(vii) The function \( W^{4,2}_{imm}(\Sigma; \mathbb{R}^n) \rightarrow C(\Sigma) \) given by
\[ f \mapsto \varrho_f, \]
where \( \varrho_f \, d\mu = d\mu_f \) and \( \mu \) is a fixed measure on \( \Sigma \) associated with a Riemannian metric.

Proof. (i) This is immediate from the definition of the metric \( g \) which depends on \( Df \) in a quadratic way, and from the fact that \( W^{3,2} \) is an algebra for the pointwise multiplication.

(ii) This follows from the representation (2.1) of the projection in local coordinates, from (i) and again from the fact that \( W^{3,2} \) is an algebra for the pointwise multiplication.

(iii) This follows from the definition of the second fundamental form \( A \) (second line in (2.2)) and (ii).

(iv) follows from (iii) and the definition of the mean curvature vector as the trace of \( A \).

(v) It suffices to note that the function from (v) is a polynomial in \( g \) and \( A \), and then to use (i) and (iii). In fact, by (i) and (iii), these two functions belong to \( W^{2,2} \) which is (like \( W^{3,2} \)) an algebra for the pointwise multiplication.

(vi) When writing the normal Laplace operator in local coordinates, by using its definition and the identity (2.3), we obtain
\[
\Delta^\perp \phi = P^\perp \left( \frac{1}{\sqrt{|g|}} \nabla_i (\sqrt{|g|} g^{ij} \nabla_j \phi) - g^{ij} g^{kl} \langle \nabla_i \phi, \nabla_k f \rangle \nabla_j \nabla_l f \right).
\]
By the properties (i) and (ii), all the coefficients in this representation are at least continuous and depend analytically on \( f \). The claim follows from this representation and (iv).

(vii) Let \((\Omega_i, \varphi_i)\) be an atlas of \( \Sigma \), and let \((\alpha_i)\) be a partition of unity subordinate to this atlas. For every continuous \( u : \Sigma \rightarrow \mathbb{R} \) we have
\[
\int_{\Sigma} u \, d\mu_f = \sum_i \int_{\varphi_i(\Omega_i)} (\alpha_i \sqrt{|g|} u) \circ \varphi_i^{-1} \, dx.
\]
From this identity we can deduce that any two Riemannian measures are mutually absolutely continuous with corresponding continuous densities which are associated with the function \( \sqrt{|g|} \). The claim thus follows from (i).

Lemma 3.3. For every smooth immersion \( f : \Sigma \rightarrow \mathbb{R}^n \) the operator
\[
(\Delta^\perp)^2 : W^{4,2}(\Sigma; \mathbb{R}^n)^\perp \rightarrow L^2(\Sigma; \mathbb{R}^n)^\perp
\]
is Fredholm of index 0.
Proof. The normal Laplace operator $\Delta^\perp$ is associated with the bilinear form $a : W^{1,2} \perp \times W^{1,2} \perp \to \mathbb{R}$ given by

$$a(\phi, \psi) = \int_\Sigma \langle P^\perp D\phi, P^\perp D\psi \rangle \, d\mu_f,$$

i.e.

$$\int_\Sigma \langle \Delta^\perp \phi, \psi \rangle \, d\mu_f = - \int_\Sigma \langle \nabla^\perp \phi, \nabla^\perp \psi \rangle \, d\mu_f = -a(\phi, \psi)$$

for all sufficiently regular $\phi$ and $\psi$; see also [12]. The form $a$ is bounded, symmetric and $L^2$-elliptic in the sense that

$$a(\phi, \phi) + \omega \|\phi\|^2_{L^2} \geq \eta \|\phi\|^2_{W^{1,2}}$$

for every $\phi \in W^{1,2} \perp$ and some constants $\omega$, $\eta > 0$. In fact, by (2.1), by the Cauchy-Schwarz inequality, and since $\langle \nabla f, \phi \rangle = 0$, for every $\phi \in W^{1,2} \perp$,

$$a(\phi, \phi) + \omega \|\phi\|^2_{L^2} \geq (1 - C \varepsilon) \int_\Sigma \|D\phi\|^2 \, d\mu_f + (\omega - \frac{C}{4 \varepsilon}) \|\phi\|^2_{L^2}$$

if $\varepsilon > 0$ is small and $\omega > 0$ is large enough.

By the Lax-Milgram lemma (or: by Riesz-Fréchet), the operator $-\Delta^\perp + \omega : D(\Delta^\perp) \to L^2 \perp$ is an isomorphism; in particular, by symmetry of the form $a$, $\Delta^\perp$ is self-adjoint. Since $D(\Delta^\perp) \hookrightarrow W^{1,2} \perp$ embeds compactly into $L^2 \perp$, we obtain that $\Delta^\perp : D(\Delta^\perp) \to L^2 \perp$ is Fredholm of index 0. The same is true for the square $(\Delta^\perp)^2 : D((\Delta^\perp)^2) \to L^2 \perp$. It remains to show that $D((\Delta^\perp)^2) = W^{4,2} \perp$.

For this we show first that for every $\phi \in W^{2,2} \perp$ one has

$$(3.3) \quad \Delta^\perp \phi = \Delta \phi + b(D\phi, \phi),$$

where $\Delta = \Delta_{g^f} = g^{ij} \nabla_i \nabla_j$ is the Laplace operator and $b$ is a linear local operator with smooth coefficients.
For every $\phi \in W^{2,2,\perp}$ and every $\psi \in C^\infty(\Sigma; \mathbb{R}^n)$ we have

$$
\int_{\Sigma} \langle \Delta^\perp \phi, \psi \rangle \, d\mu_f = \int_{\Sigma} \langle \Delta^\perp \phi, P^\perp \psi \rangle \, d\mu_f = -\int_{\Sigma} \langle P^\perp D\phi, D^\perp \psi \rangle \, d\mu_f
$$

(3.4)

$$
= -\int_{\Sigma} \langle D\phi, D\psi \rangle \, d\mu_f + \int_{\Sigma} \langle PD\phi, D^\perp \psi \rangle \, d\mu_f + \int_{\Sigma} \langle D\phi, DP^\perp \psi \rangle \, d\mu_f
$$

By definition of the Laplace operator,

$$
\int_{\Sigma} \langle D\phi, D\psi \rangle \, d\mu_f = \int_{\Sigma} g^{ij} \langle \nabla_i \phi, \nabla_j \psi \rangle \, d\mu_f = -\int_{\Sigma} \langle \Delta \phi, \psi \rangle \, d\mu_f.
$$

Since $\langle \nabla_i f, P^\perp \psi \rangle = 0$, we calculate for the second term on the right-hand side of (3.4),

$$
\int_{\Sigma} \langle PD\phi, D^\perp \psi \rangle \, d\mu_f = \int_{\Sigma} g^{ij} g^{kl} \langle \nabla_i \phi, \nabla_k f \rangle \langle \nabla_j \nabla_i f, D^\perp \psi \rangle \, d\mu_f = -\int_{\Sigma} g^{ij} g^{kl} \langle \nabla^l \phi, \nabla_i f \rangle \langle \nabla_j \nabla^l \psi, \nabla_k f \rangle \, d\mu_f
$$

Similarly, since $\langle \phi, \nabla_i f \rangle = 0$, we calculate for the third term on the right-hand side of (3.4),

$$
\int_{\Sigma} \langle D\phi, DP \psi \rangle \, d\mu_f = \int_{\Sigma} g^{ij} \langle \nabla_i \phi, \nabla_j (\nabla_i f \psi, \nabla^k f) \rangle \, d\mu_f = \int_{\Sigma} \langle b_2(D\phi), \psi \rangle \, d\mu_f + \int_{\Sigma} g^{ij} g^{kl} \langle \nabla_i \phi, \nabla_i f \rangle \langle \nabla_j \psi, \nabla_k f \rangle \, d\mu_f
$$

$$
= \int_{\Sigma} \langle b_2(D\phi), \psi \rangle \, d\mu_f - \int_{\Sigma} g^{ij} g^{kl} \langle \phi, \nabla_i f \rangle \langle \nabla_j \psi, \nabla_k f \rangle \, d\mu_f
$$

$$
= \int_{\Sigma} \langle b_3(D\phi, \phi), \psi \rangle \, d\mu_f.
$$
Summing up, (3.4) becomes
\[
\int_{\Sigma} \langle \Delta \phi, \psi \rangle \, d\mu_f = \int_{\Sigma} \langle \Delta \phi + b(D\phi, \phi), \psi \rangle \, d\mu_f,
\]
so that (3.3) follows. The operator \( b \) is linear and local and has smooth coefficients due to the assumption that \( f \) is smooth.

If \( \Delta \phi \in L^2 \) (and \( \phi \in W^{1,2} \)), then \( b(D\phi, \phi) \in L^2 \) and therefore, by (3.3), \( \Delta \phi \in L^2 \). This implies \( \phi \in W^{2,2} \), i.e. \( D(\Delta) = W^{2,2} \).

Similarly, if \( \Delta \phi \in W^{2,2} \) (and \( \phi \in W^{2,2} \)), then \( b(D\phi, \phi) \in W^{1,2} \) and therefore, by (3.3), \( \Delta \phi \in W^{1,2} \), i.e. \( \phi \in W^{3,2} \). But then we actually have \( b(D\phi, \phi) \in W^{2,2} \) and therefore \( \Delta \phi \in W^{2,2} \) so that \( \phi \in W^{4,2} \). This proves the required regularity.

\[\Box\]

**Proof of Theorem 3.1.** In the first step we prove that there exist constants \( \theta \in (0, \frac{1}{2}] \), \( C \geq 0 \) and \( \sigma > 0 \) such that for every \( \phi \in W^{4,2}(\Sigma; \mathbb{R}^n)^{\perp} \) with \( \|\phi\|_{W^{4,2}} \leq \sigma \) one has
\[
|\mathcal{W}(f_W + \phi) - \mathcal{W}(f_W)|^{1-\theta} \leq C \|\mathcal{W}'(f_W + \phi)\|_{L^2(\Sigma; \mathbb{R}^n)},
\]
i.e. we prove that the shifted Willmore functional \( \mathcal{W}(f_W + \cdot) \), when restricted to the space \( W^{4,2} \) of vector fields which are normal to \( f_W \), satisfies the Lojasiewicz-Simon gradient inequality near 0. By [2, Corollary 3.11], it suffices to show that \( \mathcal{W}(f_W + \cdot) \) is analytic in a neighbourhood of 0 in \( W^{4,2} \), that \( \mathcal{W}' \) is analytic with values in \( L^2(\Sigma; \mathbb{R}^n)^{\perp} \), and that \( \mathcal{W}'(f_W) \) is a Fredholm operator of index 0 from \( W^{4,2}(\Sigma; \mathbb{R}^n)^{\perp} \) into \( L^2(\Sigma; \mathbb{R}^n)^{\perp} \).

We firstly note that the Willmore functional is analytic on \( W^{4,2,\text{imm}}(\Sigma; \mathbb{R}^n) \) by Lemma 3.2 (iv) and (vii).

Secondly, we recall from (2.4) that the first variation acts on normal vector fields \( \phi \in W^{4,2}(\Sigma; \mathbb{R}^n) \) (normal to \( f \)) by
\[
\mathcal{W}'(f) \cdot \phi = \frac{1}{2} \int_{\Sigma} \langle \Delta \phi^{\perp} + Q(A^0)H, \phi \rangle \, d\mu_f + Q(A^0)\phi \, d\mu_f.
\]
Hence, by Lemma 3.2 (v), (vi) and (vii), the first variation is analytic when considered as a function from \( W^{4,2,\text{imm}}(\Sigma; \mathbb{R}^n) \) into \( L^2(\Sigma; \mathbb{R}^n) \).

Note also that since \( \mathcal{W}'(f) \cdot \phi = 0 \) for every tangent field \( \phi \in W^{4,2}(\Sigma; \mathbb{R}^n) \) (i.e. \( \phi = P\phi \)) and since \( \Delta \phi^{\perp} + Q(A^0)H \) is a normal vector field, we obtain
\[
\mathcal{W}'(f) \cdot \phi = \mathcal{W}'(f) \cdot (P^{\perp} \phi + P\phi) = \frac{1}{2} \int_{\Sigma} \langle \Delta \phi^{\perp} + Q(A^0)H, P^{\perp} \phi \rangle \, d\mu_f = \frac{1}{2} \int_{\Sigma} \langle \Delta \phi^{\perp} + Q(A^0)H, \phi \rangle \, d\mu_f,
\]
i.e. the relation (3.6) actually holds for every vector field \( \phi \in W^{4,2}(\Sigma; \mathbb{R}^n) \).
We need this in order to calculate the second variation of the Willmore functional. Let $\phi, \psi \in W^{4,2}(\Sigma; \mathbb{R}^n)\perp$ (normal to $f_W$). Then

$$W''(f_W)(\phi, \psi) = \frac{d}{dt} \left. W'(f_W + t\psi) \cdot \phi \right|_{t=0}$$

$$= \frac{1}{2} \frac{d}{dt} \int_\Sigma \langle \Delta_{f_W+t\psi}^+ H_{f_W+t\psi} + Q(A^0_{f_W+t\psi}) H_{f_W+t\psi}, \phi \rangle \, d\mu_{f_W+t\psi} \big|_{t=0}. $$

We calculate, using [4, Lemma 2.2],

$$\frac{d}{dt} \int_\Sigma \langle \Delta_{f_W+t\psi}^+ H_{f_W+t\psi}, P_{f_W+t\psi}^+ \phi \rangle \, d\mu_{f_W+t\psi} \big|_{t=0} =$$

$$= \frac{d}{dt} \int_\Sigma \langle \Delta_{f_W+t\psi}^+ H_{f_W+t\psi}, P_{f_W+t\psi}^+ \phi \rangle \, d\mu_{f_W+t\psi} \big|_{t=0}$$

$$= -\frac{d}{dt} \int_\Sigma \langle \nabla_{f_W+t\psi}^+ H_{f_W+t\psi}, \nabla_{f_W+t\psi}^+ P_{f_W+t\psi}^+ \phi \rangle \, d\mu_{f_W+t\psi} \big|_{t=0}$$

$$= \int_\Sigma \langle (\Delta^+)^2 \psi, \phi \rangle \, d\mu_{f_W} +$$

$$+ \int_\Sigma \langle \Delta^+ (Q(A^0)^2) + \frac{1}{2} \langle H, \psi \rangle H, \phi \rangle \, d\mu_{f_W} -$$

$$- \int_\Sigma \langle A(\cdot, e_i) \langle \nabla_{e_i}^+ \psi, H \rangle + \nabla_{e_i}^+ \psi A(\cdot, e_i), H \rangle, \nabla^+ \phi \rangle \, d\mu_{f_W} -$$

$$- \int_\Sigma \langle (D f \cdot (\nabla^+ \psi)^* + (\nabla^+ \psi)^* \cdot D f) \Delta^+ H, \phi \rangle \, d\mu_{f_W} -$$

$$- \int_\Sigma \langle \nabla^+ H, A(\cdot, e_i) \langle \nabla_{e_i}^+ \psi, \phi \rangle + \nabla_{e_i}^+ \psi A(\cdot, e_i), \phi \rangle \, d\mu_{f_W} +$$

$$+ \int_\Sigma \langle \nabla^+ H, \nabla^+ P^+ \phi \rangle \langle H, \psi \rangle \, d\mu_{f_W},$$

and

$$\frac{d}{dt} \int_\Sigma \langle Q(A^0_{f_W+t\psi}) H_{f_W+t\psi}, \phi \rangle \, d\mu_{f_W+t\psi} \big|_{t=0} =$$

$$= \int_\Sigma \langle F'(f_W) \psi, \phi \rangle \, d\mu_{f_W} - \int_\Sigma \langle Q(A^0) H, \phi \rangle \langle H, \psi \rangle \, d\mu_{f_W};$$

where $F$ is the function from Lemma 3.2 (v).

The main term in the two equalities above is the first term on the right-hand side of (3.7). This bilinear term corresponds to the operator $(\Delta^+)^2$ which, by Lemma 3.3, is a Fredholm operator from $W^{4,2}\perp$ into $L^2\perp$.

Consider the second term on the right-hand side of (3.7). Since $Q(A^0)$ maps $W^{2,2}$ into itself, and since the embedding $W^{4,2} \hookrightarrow W^{2,2}$ is compact, the operator $\psi \mapsto \Delta^+ Q(A^0) \psi$ is compact from $W^{4,2}\perp$ into $L^2\perp$. All the other
terms on the right-hand side of (3.7) depend only on $\psi$, $D\psi$ and $D^2\psi$ and thus they correspond to a compact operator from $W^{4,2} \perp$ into $L^2 \perp$.

The same holds true for the second term on the right-hand side of (3.8). Finally, by Lemma 3.2 (v) and the compactness of the embedding $W^{2,2} \hookrightarrow L^2$, the derivative $F'(f)$ is a compact operator from $W^{4,2} \perp$ into $L^2$.

Summing up, we see that the second derivative $W''(f_W)$ is a compact perturbation of a Fredholm operator from $W^{4,2} \perp$ into $L^2 \perp$, and hence it is a Fredholm operator. By [2, Corollary 3.11], we obtain the inequality (3.5).

In the second step, we prove the claim. Let $\sigma, C, \theta$ be the constants from the Lojasiewicz-Simon inequality (3.5) proved in the first step. There exists $\sigma' > 0$ such that for every $f \in W^{4,2}(\Sigma; \mathbb{R}^n)$ with $\|f - f_W\|_{W^{4,2}} \leq \sigma'$ there exist a diffeomorphism $\Phi : \Sigma \approx \Sigma$ and $\phi \in W^{4,2}(\Sigma; \mathbb{R}^n) \perp$ satisfying $\|\phi\|_{W^{4,2}} \leq \sigma$ and $f \circ \Phi = f_W + \phi$. Then $W(f) = W(f \circ \Phi) = W(f_W + \phi)$ and

$$\|W(f)\|_{L^2} = \|W'(f \circ \Phi)\|_{L^2} = \|W'(f + \phi)\|_{L^2}.$$ 

The claim follows now from (3.5) and the fact that the metrics $g_f$ are uniformly equivalent in a $C^1$-neighbourhood of $f_W$ so that $\|W'(f)\|_{L^2}$ and $\|\delta W(f)\|_{L^2}$ are comparable.

□

4. Global existence

In this section, we apply the Lojasiewicz-Simon inequality to obtain an appropriate global existence and convergence result for the Willmore flow.

**Lemma 4.1.** Let $f_W : \Sigma \to \mathbb{R}^n$ be a Willmore immersion of a closed surface $\Sigma$, and let $k \in \mathbb{N}$, $\delta > 0$. Then there exists $\varepsilon > 0$ such that the following is true:

- suppose that $(f_t)_t$ is a Willmore flow of $\Sigma$ satisfying
  $$\|f_0 - f_W\|_{W^{2,2} \cap C^1} < \varepsilon$$
  and
  $$W(f_t) \geq W(f_W) \text{ whenever } \|f_t \circ \Phi_t - f_W\|_{C^k} \leq \delta,$$
  for some appropriate diffeomorphisms $\Phi_t : \Sigma \approx \Sigma$.

- Then this Willmore flow exists globally, that is, $T = \infty$, and converges, after reparametrization by appropriate diffeomorphisms $\tilde{\Phi}_t : \Sigma \approx \Sigma$, smoothly to a Willmore immersion $f_\infty$, that is,
  $$f_t \circ \tilde{\Phi}_t \to f_\infty \text{ as } t \to \infty.$$

- Moreover, $W(f_\infty) = W(f_W)$ and $\|f_\infty - f_W\|_{C^k} \leq \delta$.

**Proof.** We may assume $k \geq 4$ and, by Proposition A.1 in the Appendix, without loss of generality

$$\|f_0 - f_W\|_{C^k, \alpha} < \varepsilon.$$
for some $\alpha > 0$ and some $\varepsilon > 0$ small enough. There exists a diffeomorphism 
$\Phi_0 : \Sigma \rightarrow \Sigma$ and a vector field $N_0 : \Sigma \rightarrow \mathbb{R}^n$ which is normal along $f_W$ such that 
$$f_0 \circ \Phi_0 = f_W + N_0 =: \tilde{f}_0.$$ 
Moreover, 
$$\|N_0\|_{C^{k,\alpha}} = \|\tilde{f}_0 - f_W\|_{C^{k,\alpha}} \leq C \varepsilon$$ 
for some constant $C$ independent of $\varepsilon$, if $\varepsilon$ is small enough.

Next, we consider the evolution equation 
$$\partial_t \tilde{f}_t + \delta W(\tilde{f}_t) = 0,$$ 
where $\tilde{f}_t = f_W + N_t$ is a smooth family of vector fields $N_t$ which are normal along $f_W$ and where $\partial_t^\perp = P_{\tilde{f}_t}^\perp \partial_t$ is the normal projection (along $\tilde{f}_t$) of the time derivative. Recall that the normal projection $P_{\tilde{f}_t}^\perp = I - P_{\tilde{f}_t}$ is given through (2.1).

Choosing locally a smooth basis $\nu_1, \ldots, \nu_{n-2}$ of $P_{f_W}^\perp \mathbb{R}^n$, writing $N_t = \varphi_{r,t} \nu_r$, and recalling (2.4), we obtain 
$$2 (\partial_t \varphi_{r,t}) P_{\tilde{f}_t}^\perp \nu_r = -\Delta^\perp \mathbf{H} - Q(A^0) \mathbf{H}$$

$$= -g^{ij} P_{\tilde{f}_t}^\perp \nabla_i P_{\tilde{f}_t}^\perp \nabla_j \Delta_s \tilde{f}_t + B(\cdot, N_t, DN_t, D^2 N_t)$$

$$= -g^{ij} g^{kl} P_{\tilde{f}_t}^\perp \nabla_i P_{\tilde{f}_t}^\perp \nabla_j \left( \partial_k \partial_l \tilde{f}_t - \Gamma^m_{kl} \partial_m \tilde{f}_t \right) + B(\cdot, N_t, DN_t, D^2 N_t)$$

$$= -g^{ij} g^{kl} P_{\tilde{f}_t}^\perp \partial_{ijkl} N_t + B(\cdot, N_t, DN_t, D^2 N_t, D^3 N_t)$$

$$= -\left( g^{ij} g^{kl} \partial_{ijkl} \varphi_{r,t} \right) P_{\tilde{f}_t}^\perp \nu_r + B(\cdot, N_t, DN_t, D^2 N_t, D^3 N_t),$$

where $B$ is a smooth function depending on $f_W$ and changing from line to line. 
For $\|\tilde{f}_t - f_W\|_{C^1} \leq \delta$ small enough, $g^{ij} g^{kl}$ is uniformly elliptic, and there exists a constant $C > 0$ such that 
$$\|P_{\tilde{f}_t}^\perp N\|_{C^{k,\alpha}} \geq |N| - |P_{f_W}^\perp N - P_{\tilde{f}_t}^\perp N|$$

$$\geq C |N|$$ 
for $N$ normal along $f_W$, so that $P_{\tilde{f}_t}^\perp \nu_1, \ldots, P_{\tilde{f}_t}^\perp \nu_{n-2}$ forms a basis of $N_{f_{\Sigma}}$. Therefore, 
$$\partial_t \varphi_{r,t} + \frac{1}{2} g^{ij} g^{kl} \varphi_{r,t} = B_r(\cdot, \varphi_t, D\varphi_t, D^2 \varphi_t, D^3 \varphi_t),$$

and by (4.2), 
$$\|\varphi_{r,0}\|_{C^{k,\alpha}} \leq C \varepsilon$$

for $r = 1, \ldots, n - 2$.

For $C \varepsilon < \sigma < \delta$ small enough ($\sigma$ as in the Łojasiewicz-Simon gradient inequality; see Theorem 3.1. Without loss of generality, $\sigma < \delta$), equation (4.3)
has a solution \( (\tilde{f}_t)_t \) on a maximal existence interval \([0, \tilde{T}]\) satisfying in addition (this is part of our definition of the maximal existence time \( \tilde{T} \))

\[ \| \tilde{f}_t - f_W \|_{C^k} \leq \sigma < \delta \quad \text{for all } t \in [0, \tilde{T}]. \]

By parabolic Schauder estimates (see [10]), we get successively from (4.5) and (4.6),

\[ \| \varphi \|_{H^{k+\alpha,(k+\alpha)/4}} \leq C, \]

and in particular

\[ \| \tilde{f}_t - f_W \|_{C^{k,\alpha}} \leq C \quad \text{for all } t \in [0, \tilde{T}]. \]

The flow \((\tilde{f}_t)_t\) and the Willmore flow \((f_t)_t\) are connected in the following way. The flow \((\tilde{f}_t)_t\) is a solution of (4.3) which we write as

\[ \partial_t \tilde{f}_t + \delta \mathcal{W}(\tilde{f}_t) + d\tilde{f}_t \cdot \xi_t = 0 \]

for some smooth family of abstract tangential vector fields \((\xi_t)\). Solving the ordinary differential equation

\[ \partial_t \Phi_t = \xi_t \circ \Phi_t \quad \text{for } t \in [0, \tilde{T}] \quad \text{on } \Sigma, \]

\[ \Phi_0 = \text{id}_\Sigma, \]

we get a smooth family of diffeomorphisms \(\Phi_t\) on \(\Sigma\). We calculate

\[
\begin{align*}
\partial_t (\tilde{f}_t \circ \Phi_t) &= (\partial_t \tilde{f}_t) \circ \Phi_t + (d\tilde{f}_t) \circ \Phi_t \cdot \partial_t \Phi_t \\
&= (\partial_t \tilde{f}_t) \circ \Phi_t + (d\tilde{f}_t) \circ \Phi_t \cdot (\xi_t \circ \Phi_t) \\
&= -\delta \mathcal{W}(\tilde{f}_t) \circ \Phi_t \\
&= -\delta \mathcal{W}(\tilde{f}_t \circ \Phi_t).
\end{align*}
\]

Thus, \((\tilde{f}_t \circ \Phi_t \circ \Phi^{-1})\) is a Willmore flow with initial data \(\tilde{f}_0 \circ \Phi_0 \circ \Phi^{-1} = f_0\).

As the solution of the Willmore flow is unique, we conclude

\[ (4.10) \quad f_t = \tilde{f}_t \circ \Phi_t \circ \Phi^{-1} \quad \text{and} \quad \tilde{T} \leq T. \]

In particular, it suffices to prove that the flow \((\tilde{f}_t)_t\) is global and converges as \(t \to \infty\) to a Willmore immersion \(f_\infty\) having the desired properties.

We recall that the Willmore flow is a gradient flow for the Willmore functional. In particular,

\[ \frac{d}{dt} \mathcal{W}(f_t) = -\| \partial_t f_t \|_{L^2(S; \text{d}\mu_{f_t})}^2 \leq 0, \]

so that \(\mathcal{W}(f_t) = \mathcal{W}(\tilde{f}_t)\) is decreasing. From (4.1) and (4.7), we see that \(\mathcal{W}(\tilde{f}_t) \geq \mathcal{W}(f_W)\) for every \(t \in [0, \tilde{T}]\).

If \(\mathcal{W}(f_t) = \mathcal{W}(f_W)\) for some \(t \in [0, \tilde{T}]\), then \(\mathcal{W}(f_s) = \mathcal{W}(f_W)\) for every \(s \in [t, \tilde{T}]\) and \(f_s = f_t, \tilde{f}_s = \tilde{f}_t\) are stationary for \(s \in [t, \tilde{T}]\) (we remark that the Willmore flow is analytic in time so that the Willmore flow is then stationary.
for all \( s \geq 0 \) by the uniqueness theorem for analytic functions). This implies \( \tilde{T} = \infty = T \) with (4.10), and \( f_s \circ \Phi_s^{-1} \circ \Phi = \tilde{f}_s \rightarrow f_\infty = \tilde{f}_t \) is a Willmore immersion satisfying \( \mathcal{W}(f_\infty) = \mathcal{W}(f_W) \) and \( \| f_\infty - f_W \|_{C^k} < \delta \) by (4.7). In this case, there remains nothing to prove.

So we can assume that \( \mathcal{W}(f_t) > \mathcal{W}(f_W) \) for all \( t \in [0, \tilde{T}] \). Let \( \theta \in (0, \frac{1}{2}] \) be as in the Lojasiewicz-Simon inequality (Theorem 3.1). Then, by (4.7) and the Lojasiewicz-Simon gradient inequality, for every \( t \in [0, \tilde{T}] \),

\[
-\frac{d}{dt}|\mathcal{W}(f_t) - \mathcal{W}(f_W)|^\theta \\
= \theta |\mathcal{W}(f_t) - \mathcal{W}(f_W)|^{\theta-1} \delta \mathcal{W}(\tilde{f}_t) \cdot \partial_t \tilde{f}_t \\
= \theta |\mathcal{W}(f_t) - \mathcal{W}(f_W)|^{\theta-1} \|\mathcal{W}(f_t)\|_{L^2(\Sigma;du_{\tilde{f}_t})} \|\partial_t \tilde{f}_t\|_{L^2(\Sigma;du_{\tilde{f}_t})} \\
\geq \frac{\theta}{C} \|\partial_t \tilde{f}_t\|_{L^2(\Sigma;du_{\tilde{f}_t})}.
\]

As the metrics \( g_{\tilde{f}_t} \) are uniformly equivalent to \( g_{f_W} \), the preceding inequality and (4.4) imply

\[
(4.11) \quad \|\partial_t \tilde{f}_t\|_{L^2(\Sigma)} \leq -C \frac{d}{dt}|\mathcal{W}(f_t) - \mathcal{W}(f_W)|^\theta \text{ for every } t \in [0, \tilde{T}].
\]

Integrating in time yields for every \( t \in [0, \tilde{T}] \),

\[
\|\tilde{f}_t - f_W\|_{L^2(\Sigma)} \leq \|\tilde{f}_0 - f_W\|_{L^2(\Sigma)} + C |\mathcal{W}(\tilde{f}_0) - \mathcal{W}(f_W)|^\theta \\
\leq C \|\tilde{f}_0 - f_W\|_{C^2(\Sigma)}^\theta.
\]

Interpolating with (4.9) yields, by (4.2),

\[
\|\tilde{f}_t - f_W\|_{C^k(\Sigma)} \leq C \|\tilde{f}_t - f_W\|_{C^2(\Sigma)}^{\theta'} \leq C \varepsilon^{\theta'} \leq \frac{\sigma}{2} \text{ for } t \in [0, \tilde{T}],
\]

for some \( 0 < \theta' < \theta \), if \( \varepsilon > 0 \) is small enough. As the solution \( (\tilde{f}_t) \) is maximal with respect to (4.7), we obtain \( \tilde{T} = \infty \), i.e. \( (\tilde{f}_t) \) exists globally and \( \|\tilde{f}_t - f_W\|_{C^k} \leq \sigma \) for every \( t \in \mathbb{R}_+ \). Moreover, by (4.11), and since \( \mathcal{W} \) is decreasing along \( (\tilde{f}_t) \),

\[
\partial_t \tilde{f}_t \in L^1(\mathbb{R}_+;L^2(\Sigma)).
\]

As a consequence,

\[
\lim_{t \to \infty} \tilde{f}_t =: f_\infty \text{ exists in } L^2(\Sigma).
\]

Interpolating again with (4.9), this implies that

\[
\lim_{t \to \infty} \tilde{f}_t = f_\infty \text{ in } C^k(\Sigma),
\]

and by parabolic Schauder estimates the convergence is even smooth, that is,

\[
\lim_{t \to \infty} \tilde{f}_t = f_\infty \text{ in } C^l(\Sigma) \text{ for every } l \in \mathbb{N}.
\]
Since the Willmore flow is a gradient flow, the immersion \( f_\infty \) is a Willmore immersion. By (4.7), \( \| f_\infty - f_W \|_{C^k} \leq \sigma \), the Lojasiewicz-Simon gradient inequality (Theorem 3.1) applies to \( f_\infty \) and yields
\[
|\mathcal{W}(f_\infty) - \mathcal{W}(f_W)|^{1-\theta} \leq c_2 \| \delta \mathcal{W}(f_\infty) \|_{L^2} = 0,
\]
hence \( \mathcal{W}(f_\infty) = \mathcal{W}(f_W) \).

5. Proof of the main results

Proof of Theorem 1.1. Let \([0, T], 0 < T \leq \infty\), be the maximal existence interval of the Willmore flow \((f_t)_t\). We recall from the Introduction that blow-ups and blow-downs were obtained according to [3, Section 4] and [5] by selecting sequences \( t_j \nearrow T, r_j \to 0 \) or \( \infty \), \( x_j \in \mathbb{R}^n \) such that the immersions
\[
f_j := r_j^{-1}(f_{t_j} + c_0 r_j^4 - x_j)
\]
converge smoothly, after appropriate reparametrization, on compact subsets of \( \mathbb{R}^n \) to a Willmore immersion \( f_W : \hat{\Sigma} \to \mathbb{R}^n \), where \( \hat{\Sigma} \neq \emptyset \) is a complete surface without boundary.

We assume, by contradiction, that one of the components of \( \hat{\Sigma} \) were compact. Then by [3, Lemma 4.3], which holds true for any \( r_j > 0 \), we get that \( \hat{\Sigma} \sim \Sigma \) are diffeomorphic and compact and for appropriate diffeomorphisms \( \Phi_j : \Sigma \approx \hat{\Sigma} \)
\[
f_j \circ \Phi_j \to f_W \text{ smoothly on } \Sigma.
\]
Hence for \( j \) large enough
\[
\| f_j \circ \Phi_j - f_W \|_{C^2(\Sigma)} < \varepsilon
\]
with \( \varepsilon > 0 \) as in Lemma 4.1. Putting \( f_{j,t} := r_j^{-1}(f_{t_j+s} r_j^4 - x_j) \circ \Phi_j \), \( (f_{j,t})_t \) is the Willmore flow with initial data \( f_j \circ \Phi_j \) and the maximal existence interval \([0, r_j^{-4}(T - t_j)]\). Clearly, for every \( t \in [0, r_j^{-4}(T - t_j)] \),
\[
\mathcal{W}(f_{j,t}) \geq \lim_{s \to r_j^{-4}(T - t_j)} \mathcal{W}(f_{j,s}) = \lim_{s \to T} \mathcal{W}(f_s) = \lim_{i \to \infty} \mathcal{W}(f_i) = \mathcal{W}(f_W)
\]
by invariance of the Willmore functional and the smooth convergence above. Then Lemma 4.1 implies that \((f_{j,t})_t\) exists globally. Hence, \( T = \infty \) and the original flow \((f_t)_t\) exists globally, too. Moreover,
\[
f_{j,t} \circ \Phi_{j,t} \to f_{j,\infty} \text{ smoothly for } t \to \infty
\]
and appropriate diffeomorphisms \( \Phi_{j,t} : \Sigma \approx \hat{\Sigma} \).

For fixed \( j \), we can write \( t_i + c_0 r_i^4 = t_j + r_j^4 s_i \) with \( s_i \to \infty \) and \( t_i \to T = \infty \), and obtain
\[
r_j^{-1}(f_{t_i + c_0 r_i^4} - x_j) \circ \Phi_{j,s_i} = f_{j,s_i} \circ \Phi_{j,s_i} \to f_{j,\infty} \text{ smoothly for } i \to \infty.
\]
In particular, the diameters converge:
\[ \text{diam } f_{t_i+c_0 r_i^t}(\Sigma) \to r_j \text{ diam } f_{j,\infty}(\Sigma) \in ]0,\infty[ \text{ for } i \to \infty. \]

On the other hand, as \( r_i^{-1}(f_{t_i+c_0 r_i^t} - x_i) \circ \Phi_i \to f_W \) smoothly for \( i \to \infty \), we obtain
\[ r_i^{-1} \text{diam } f_{t_i+c_0 r_i^t}(\Sigma) \to \text{diam } f_W(\Sigma) \in ]0,\infty[ \text{ for } i \to \infty. \]
This contradicts to \( r_i \to 0 \) or \( \infty \) as \( i \to \infty \), and therefore none of the components of \( \tilde{\Sigma} \) is compact. \( \square \)

**Remark 5.1.** As pointed out in the introduction for \( T = \infty \), we may have \( r_j \to 0, r_j = 1 \) or \( r_j \to \infty \), after rescaling and passing to appropriate subsequences.

If in the case \( r_j = 1 \) the limit surface \( \tilde{\Sigma} \) contains a compact component, we get as in the previous proof by [3, Lemma 4.3] that
\[ (f_t - x_j) \circ \Phi_j \to f_W \text{ smoothly on } \Sigma \]
for some \( t_j \not= T = \infty \) and appropriate diffeomorphisms \( \Phi_j : \Sigma \to \Sigma \). Then as above \( \mathcal{W}(f_t) \geq \mathcal{W}(f_W) \) for all \( t \geq 0 \) by invariance of the Willmore functional and the smooth convergence, and Lemma 4.1 implies that the Willmore flows \( (f_t - x_j)_t \) converge, for \( j \) large enough, and after reparametrization by appropriate diffeomorphisms \( \Phi_{j,t} : \Sigma \to \mathbb{R}^n; \)
\[ (f_t - x_j) \circ \Phi_{j,t} \to f_{j,\infty} \text{ for } t \to \infty. \]
Moreover, \( f_{j,\infty} \to f_W \) smoothly as \( j \to \infty \).

From this we conclude first that \( x_j \) is bounded, and we may assume \( x_j = 0 \). Next, we see that the whole flow \( (f_t)_t \) converges after reparametrization for \( t \to \infty \) to a Willmore immersion. \( \square \)

**Proof of Theorem 1.2.** If \( f_W \) is a local minimizer as in Theorem 1.2, condition (4.1) from Lemma 4.1 is clearly satisfied throughout the flow. Then Lemma 4.1 yields global existence and convergence as in the claim with \( \mathcal{W}(f_\infty) = \mathcal{W}(f_W) \) and \( \|f_\infty - f_W\|_{C^k} \leq \delta \). In particular, \( f_\infty \) is a Willmore immersion and minimizes the Willmore functional locally in \( C^k \). \( \square \)

**Remark 5.2.** If \( f_W \) is a local minimizer of the Willmore functional in \( C^k(\Sigma) \), then it is also a local minimizer in \( W^{2,2} \cap C^1(\Sigma) \). In fact, assume that \( \gamma > 0 \) is such that
\[ \|f - f_W\|_{C^k} < \gamma \implies \mathcal{W}(f) \geq \mathcal{W}(f_W). \]
Let \( \tau > 0 \) be as in Proposition 6.1 below, and assume that \( \|f_0 - f_W\|_{W^{2,2} \cap C^1} < \tau \). Let \( (f_t)_t \) be the Willmore flow with initial data \( f_0 \). Then \( \mathcal{W}(f_0) \geq \mathcal{W}(f_t) \) for all \( t \) in the maximal existence interval and \( \|f_t \circ \Phi - f_W\|_{C^k} < \gamma \) for some \( t \) and some diffeomorphism \( \Phi : \Sigma \to \Sigma \). Hence,
\[ \mathcal{W}(f_0) \geq \mathcal{W}(f_t) = \mathcal{W}(f_t \circ \Phi) \geq \mathcal{W}(f_W). \]
6. Appendix

6.1. Regularization. In this appendix, we prove a short time regularization
result for the Willmore flow which enables us to assume closeness of our initial
data only in $W^{2,2} \cap C^1$.

Proposition 6.1. For any smooth immersion $f_\ast : \Sigma \to \mathbb{R}^n$ of a closed
surface $\Sigma$ and any $k \in \mathbb{N}, \gamma > 0$, there exists a $\tau > 0$ such that for any
smooth immersion $f_0 : \Sigma \to \mathbb{R}^n$ with

$$\| f_0 - f_\ast \|_{(W^{2,2} \cap C^1) (\Sigma)} < \tau$$

the corresponding Willmore flow $(f_t)_t$ with initial data $f_0$ satisfies for some
time $0 < t \leq \gamma$ in the existence interval and some diffeomorphism $\Phi : \Sigma \to \Sigma$

$$\| f_t \circ \Phi - f_\ast \|_{C^k(\Sigma)} < \gamma.$$

Proof. Suppose that the assertion is not true for some $\gamma > 0$. Then there
exists a sequence $f_m \to f_\ast$ in $(W^{2,2} \cap C^1)(\Sigma)$ such that the corresponding
Willmore flows $(f_{m,t})_t$ with initial data $f_{m,0} = f_m$ satisfy

$$\| f_{m,t} \circ \Phi - f_\ast \|_{C^k(\Sigma)} \geq \gamma$$

for all $0 < t \leq \gamma$ in the existence interval of $(f_{m,t})_t$ and all diffeomorphisms $\Phi : \Sigma \to \Sigma$.

We choose $0 < \varrho \leq 1$ satisfying

$$\int_{f^{-1}(B_{2\varrho})} |A_{f_m}|^2 \, d\mu_{f_m} \leq \varepsilon/2 \quad \text{for any } B_{2\varrho} \subseteq \mathbb{R}^n$$

for some $\varepsilon > 0$ chosen below. For $m$ large enough, we have $\| f_m - f_\ast \|_{L^\infty(\Sigma)} < \varrho$ which implies that $f^{-1}_m (B_{2\varrho}(x)) \subseteq f^{-1}_\ast (B_{2\varrho}(x))$ for any $x \in \mathbb{R}^n$, and as $f_m \to f_\ast$ in $(W^{2,2} \cap C^1)(\Sigma)$, we see $g_{f_m} \to g_{f_\ast}$ uniformly on $\Sigma$ and $|A_{f_m}|^2_{g_{f_m}} \sqrt{g_{f_m}} \to |A_{f_\ast}|^2_{g_{f_\ast}} \sqrt{g_{f_\ast}}$ in $L^1(\Sigma)$. This yields for every $m$ large
enough, which we will assume from now on always, and every $x \in \mathbb{R}^n$

$$\int_{f^{-1}_m (B_{2\varrho}(x))} |A_{f_m}|^2_{g_{f_m}} \, d\mu_{f_m} \leq \int_{f^{-1}(B_{2\varrho}(x))} |A_{f_m}|^2_{g_{f_m}} \, d\mu_{f_m} \leq$$

$$\leq \int_{f^{-1}_m (B_{2\varrho}(x))} |A_{f_\ast}|^2_{g_{f_\ast}} \, d\mu_{f_\ast} + \left\| |A_{f_m}|^2_{g_{f_m}} \sqrt{g_{f_m}} - |A_{f_\ast}|^2_{g_{f_\ast}} \sqrt{g_{f_\ast}} \right\|_{L^1(\Sigma)} \leq \varepsilon.$$
By [4, Theorem 1.2], for $\varepsilon \leq \varepsilon_0(n)$ small enough, the Willmore flows $(f_{m,t})_t$ exist at least on an interval $[0, T]$ with $T = c_0(n)g^4 \leq c_0(n) \leq 1$ and moreover they satisfy

$$\int_{f_{m,t}^{-1}(B_\varepsilon)} |A_{f_{m,t}}|^2 \, \mu_{f_{m,t}} \leq C\varepsilon$$

for any $B_\varepsilon \subseteq \mathbb{R}^n$ and any $t \in [0, T]$.

Then, by the interior estimate from [3, Theorem 3.5], we get for $C\varepsilon < \varepsilon_0(n)$ small enough that

$$\| \nabla^k A_{f_{m,t}} \|_{L^\infty(\Sigma)} \leq C_k\varepsilon^{1/2}t^{-3/4}$$

for $t \in [0, T]$, where $\nabla^k$ denotes $k$-times applying the normal covariant derivative in the normal bundle along $f_{m,t}$. We conclude

$$|\partial_t f_{m,t}| = \frac{1}{2} \left| \Delta_{f_{m,t}} H_{f_{m,t}} + Q(A^0_{f_{m,t}}) H_{f_{m,t}} \right| \leq C\varepsilon^{1/2}t^{-3/4}$$

and

$$\| f_{m,t} - f_m \|_{L^\infty(\Sigma)} \leq \int_0^t \| \partial_s f_{m,s} \|_{L^\infty(\Sigma)} \, ds \leq C\varepsilon^{1/2}t^{1/4}.$$

Choosing $t_m \in (0, T]$ with $t_m \to 0$ and

$$C\varepsilon^{1/2}t_m^{1/4} > \| f_m - f_* \|_{L^\infty(\Sigma)}$$

for $m$ large enough, we see

$$\| f_{m,t} - f_* \|_{L^\infty(\Sigma)} \leq C\varepsilon^{1/2}t^{1/4}$$

for $t \in [t_m, T]$.

Together with $\| A_{f_{m,t}} \|_{L^\infty(\Sigma)} \leq C\varepsilon^{1/2}t^{1/4}$ by (6.3), we can write $f_{m,t} \circ \Phi_{m,t} = f_* + N_{m,t} := \tilde{f}_{m,t}$ with normal fields $N_{m,t}$ along $f_m$ and

$$\| \tilde{f}_{m,t} - f_* \|_{L^\infty(\Sigma)} \leq C\varepsilon^{1/2}t^{1/4},$$

$$\| \tilde{f}_{m,t} - f_* \|_{C^1(\Sigma)} \leq C\varepsilon^{1/4} < \delta$$

for $t_m \leq t \leq T$, if $\delta = \delta(f_*)$ is appropriate and $\varepsilon$ small enough. For $\delta = \delta(f_*)$ small enough, we see for the smooth pull-back metric $g_* = f_*^* g_{\text{euc}}$ that

$$\frac{1}{2} g_* \leq g_{f_{m,t}} \leq 2g_*$$

for $t \in [t_m, T]$.

The reparametrized Willmore flow $\partial_t \tilde{f}_{m,t} + \delta W(\tilde{f}_{m,t}) = 0$ for $t_m \leq t \leq T$ still satisfies (6.2), hence (6.3). Choosing locally a smooth basis $\nu_1, \ldots, \nu_{n-2}$ of $N_f, \Sigma$ and writing $N_{m,t} = \varphi_{r,m,t} \nu_r$, we see in local charts for $k \geq 0$

$$\partial^k \tilde{f}_{m,t} = \partial^k f_* + \sum_{l=0}^k \partial^l \varphi_{r,m,t} \ast \partial^{k-l} \nu_r,$$
where, as in [3], \( \varphi \ast \psi \) denotes any form depending on \( \varphi \) and \( \psi \) in a universal, bilinear way.

As \( P_{f,m,t}^\perp \nu_1, \ldots, P_{f,m,t}^\perp \nu_{n-2} \) forms a basis of \( N_{f,m,t} \Sigma \), we get for \( \delta = \delta(f_*) \) small enough

\[
|\partial^k \varphi_{m,t}| \leq C|\partial^k \varphi_{r,m,t} P_{f,m,t}^\perp \nu_r| \leq \\
\leq C_{f_*}(|P_{f,m,t}^\perp \partial^k \tilde{f}_{m,t}| + |\partial^{k-1} \varphi_{m,t}| + \ldots |\varphi_{m,t}| + 1) \leq \\
\leq C_{f_*}(|P_{f,m,t}^\perp \partial^k \tilde{f}_{m,t}| + |\partial^{k-1} \tilde{f}_{m,t}| + \ldots |\tilde{f}_{m,t}| + 1).
\]

Recalling \( \partial_{ij} \tilde{f}_{m,t} = A_{f,m,t,ij} + \Gamma_{f,m,t,ij}^l \partial_l \tilde{f}_{m,t} \) in local charts and \( P_{f,m,t}^\perp \partial \tilde{f}_{m,t} = 0 \), we see

\[
P_{f,m,t}^\perp \partial^k+2 \tilde{f}_{m,t} = P_{f,m,t}^\perp \partial^k A_{f,m,t} + \sum_{l=0}^{k-1} \partial^k \Gamma \ast P_{f,m,t}^\perp \partial^{k+1-l} \tilde{f}_{m,t},
\]

hence, as \( \Gamma = \Gamma(\partial f, \partial^2 f) \),

\[
|\partial^k+2 \tilde{f}_{m,t}| \leq C_{f_*}(|\partial^{k+2} \varphi_{m,t}| + \ldots |\varphi_{r,m,t}| + 1) \leq \\
\leq C_{f_*}(|P_{f,m,t}^\perp \partial^{k+2} \tilde{f}_{m,t}| + |\partial^{k+1} \tilde{f}_{m,t}| + \ldots |\tilde{f}_{m,t}| + 1) \leq \\
C_{f_*} \Lambda_{k+1} (|P_{f,m,t}^\perp \partial^k A_{f,m,t}| + 1),
\]

where \( |\partial^{k+1} \tilde{f}_{m,t}| + \ldots + |\tilde{f}_{m,t}| \leq \Lambda_{k+1} \). In particular for \( k = 0 \), by (6.3) and (6.4),

\[
|\partial^2 \tilde{f}_{m,t}|, |\Gamma| \leq C_{f_*}(1 + \varepsilon^{1/2} t^{-1/4})
\]
in local charts. Now we proceed as in the last step of the proof of [4, Theorem 1.2] and obtain inductively by (6.3) using (6.5)

\[
|\partial^k A_{f,m,t}|, |\partial^{k+2} \tilde{f}_{m,t}| \leq C_{f_*} \varepsilon \tau \text{ for } t_m \leq \varepsilon t \leq t \leq T.
\]

From the parabolic evolution equation (4.3) for \( \varphi_{m,t} \), we obtain as in (4.8) with interior estimates in time

\[
\| \tilde{f}_{m,t} \|_{H^{k+n, (k+n)/4}(\Sigma \times [2\tau,T])} \leq C_{f_*} \varepsilon \tau.
\]

Recalling \( t_m \to 0 \) we see that \( (\tilde{f}_{m,t})_t \to (\tilde{f}_t) \) smoothly on compact subsets of \([0,T]\) and \( \tilde{f} \) is a smooth Willmore flow solving \( \partial^\perp_t \tilde{f}_t + \delta W(\tilde{f}_t) = 0 \) on \([0,T]\) and satisfying

\[
\int_{B_\varepsilon} |A_{\tilde{f}_t}|^2 \, d\mu_{\tilde{f}_t} \leq C\varepsilon \text{ for any } B_\varepsilon \subseteq \mathbb{R}^n,
\]

\[
\| \tilde{f}_t - f_* \|_{L^\infty(\Sigma)} \leq C\varepsilon^{1/2} t^{1/4},
\]

\[
\| \tilde{f}_t - f_* \|_{C^1(\Sigma)} \leq \delta,
\]

for all \( t \in [0,T] \).
Now we consider the Willmore flow \( f_t = f_s + N_t \) with \( N_t \) normal along \( f_s \) and \( \partial^t_t f_t + \delta W(f_t) = 0 \) with initial data \( f_0 := f_s \) which exists on a time interval \([0, T_0]\) with \( 0 < T_0 \leq \infty \). As \( \tilde{f}_t, f_t \to f_s \) in \( L^\infty(\Sigma) \) when \( t \to 0 \), we obtain from the uniqueness proposition below (Proposition 6.2) that \( \tilde{f}_t = f_t \) for \( 0 < t < \min(T, T_0) \). As \((f_t)_t\) is smooth, there exists \( 0 < t < \min(T, T_0, \gamma) \) such that

\[
\| f_t - f_s \|_{C^k(\Sigma)} < \gamma.
\]

As \( f_{m,t} \circ \Phi_{m,t} = \tilde{f}_{m,t} \to \tilde{f}_t = f_t \) smoothly, we see for \( m \) large enough

\[
\| f_{m,t} \circ \Phi_{m,t} - f_s \|_{C^k(\Sigma)} < \gamma
\]

which contradicts (6.1), and the proposition follows.

6.2. Uniqueness. It remains to state and prove the uniqueness proposition used in the above proof.

**Proposition 6.2.** Let \( f_* : \Sigma \to \mathbb{R}^n \) be a smooth immersion of a closed surface \( \Sigma \) and \( f_m = f_* + N_m : \Sigma \times [0, T] \to \mathbb{R}^n, m = 1, 2, \) be two Willmore flows with \( N_m \) normal along \( f_* \) and \( \partial^t_t f_{m,t} + \delta W(f_{m,t}) = 0 \) satisfying

\[
\| f_{m,t} - f_* \|_{C^1(\Sigma)} \leq \varepsilon, \\
\int_{B_\varrho} |A_{f_{m,t}}|^2 \, d\mu_{f_{m,t}} \leq \varepsilon \text{ for any } B_\varrho \subseteq \mathbb{R}^n,
\]

for all \( t \in [0, T] \) and some \( \varrho > 0 \).

If \( \varepsilon < \varepsilon_0(n, f_*) \) is small enough, then

\[
(6.7) \quad \| f_{2,t} - f_{1,t} \|_{L^2(\Sigma)} \leq C f_* C_{f_*} \varepsilon^{1/2} T - C f_* \varepsilon^{1/2} \| f_{2,\tau} - f_{1,\tau} \|_{L^2(\Sigma)}
\]

for \( 0 < \tau \leq t \leq \min(c(n) \varrho^4, T) \). If, moreover,

\[
f_{2,t} - f_{1,t} \to 0 \text{ weakly in } L^2(\Sigma) \text{ for } t \to 0
\]

then the flows coincide:

\[
f_{2,t} = f_{1,t} \text{ for } t \in [0, T].
\]

**Proof.** We may assume that \( T \leq c_0 \varrho^4 \) for some \( c_0 < c(n) \) small enough and after rescaling that \( \varrho = 1 \). We write \( g_m, \mu_m, A_m, \ldots \) for the metric, measure, second fundamental form, ... induced by \( f_m \) for \( m = 1, 2 \) and for metric terms induced by \( g_* := f_* \text{euc} \) for \( m = * \). The normal projections along \( f = f_m, f_* \) are denoted by \( \mathcal{P}_f^1 \).

By the interior estimate from [3, Theorem 3.5], we get for \( \varepsilon < \varepsilon_0(n), c_0 < c(n) \) small enough that

\[
(6.8) \quad \| \nabla^{m,1,k}_{A_m,t} \|_{L^\infty(\Sigma)} \leq C_k \varepsilon^{1/2} t^{-k/4 - (k+1)/4},
\]
where $\nabla^{m,\perp,k}$ denotes $k$–times applying the normal covariant derivative with respect to $g_{f_m}$ in the normal bundle along $f_m$. We conclude for $\varepsilon < \varepsilon(f_\ast)$ small enough that

\begin{equation}
\frac{1}{2} g_{f_{m,t}} \leq g_\ast \leq 2 g_{f_{m,t}}, \tag{6.9}
\end{equation}

and

\begin{equation}
|\partial_t f_{m,t}| \leq 2 |\partial_t^\perp f_{m,t}| = \left| \Delta_{g_{m,t}}^\perp \mathbf{H}_{m,t} + Q(A_{m,t}^0) H_{m,t} \right| \leq C \varepsilon^{1/2} t^{-3/4}. \tag{6.10}
\end{equation}

In particular $f_{2,t} - f_{1,t}$ converges strongly in $L^\infty(\Sigma)$ for $t \to 0$, and if $f_{2,t} - f_{1,t} \to 0$ weakly in $L^2(\Sigma)$ for $t \to 0$, then

\begin{equation}
\| f_{2,t} - f_{1,t} \|_{L^\infty(\Sigma)} \leq \int_0^t \| \partial_s (f_{2,s} - f_{1,s}) \|_{L^\infty(\Sigma)} \, ds \leq C \varepsilon^{1/2} t^{1/4}. \tag{6.11}
\end{equation}

As in (6.6) we get from (6.8)

\begin{equation}
|\partial^2 f_{m,t}|, |\partial g_{m,t}|, |\Gamma^m - \Gamma^\ast| \leq C_{f_\ast} (1 + \varepsilon^{1/2} t^{-1/4}) \tag{6.12}
\end{equation}

in local charts. Since the difference of two Christoffel symbols is a tensor, the second estimate is invariant not depending on the chart or the metrics $g_{f_m}$ or $g_\ast$ by (6.9).

Putting $h := f_2 - f_1$, we conclude

\begin{equation}
|\Gamma^2 - \Gamma^1| \leq C |\partial^2 h| + |\nabla h| |\partial^2 f| \leq C |\nabla^{\ast,2} h| + C_{f_\ast} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h|, \tag{6.13}
\end{equation}

where the estimate of the term on the left by the term on the right is independent of the chart, as it involves only tensors. Next

\begin{align}
A_{2,ij} - A_{1,ij} &= \nabla^2_i \nabla^j f_2 - \nabla^2_j \nabla^i f_1 = \\
&= \nabla^i \nabla^j h - (\Gamma^2_{ij} - \Gamma^1_{ij}) \nabla^k f_2 + (\Gamma^1_{ij} - \Gamma^2_{ij}) \nabla^k f_1 = \\
&= \nabla^i \nabla^j h + (\Gamma^2 - \Gamma^1) \ast \nabla h + (\Gamma^2 - \Gamma^1) \ast \nabla f_1,
\end{align}

hence

\begin{equation}
|A_2 - A_1| \leq C |\nabla^{\ast,2} h| + C_{f_\ast} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h|. \tag{6.13}
\end{equation}

Multiplying the evolution equations $\partial_t f_{m,t} + \delta \mathcal{W}(f_{m,t}) = 0$ by $h_t = f_{2,t} - f_{1,t}$ and integrating with respect to $\mu_m$, we get

\begin{align}
\int_\Sigma 2 \langle \partial_t^\perp f_{m,t}, h \rangle \, d\mu_m + \int_\Sigma \langle \nabla^\perp h, \mathbf{H}_{m,t} \rangle \, d\mu_m + \int_\Sigma \langle Q(A_{m}^0) \mathbf{H}_{m,t}, h \rangle \, d\mu_m = \\
=: I^m_1 + I^m_2 + I^m_3 = 0 \quad \text{for } m = 1, 2. \tag{6.14}
\end{align}
For the first term, we calculate

\[ I_1^2 - I_1^1 = \int \Sigma 2(\partial_t f_2, h) \, d\mu_2 - \int \Sigma 2(\partial_t f_1, h) \, d\mu_1 = \]

\[ = \int \Sigma 2(\langle \partial_t f_2, P_{f_2} h \rangle - \langle \partial_t f_1, P_{f_1} h \rangle) \, d\mu_1 + \int \Sigma 2(\partial_t^+ f_1, h) \left( \frac{d\mu_2}{\mu} - \frac{d\mu_1}{\mu} \right) = \]

\[ = \int \Sigma 2(\partial_t h, P_{f_1} h) \, d\mu_1 - \int \Sigma 2(\partial_t f_2, (P_{f_2} - P_{f_1}) h) \, d\mu_1 + \int \Sigma 2(\partial_t^+ f_2, h) \left( \frac{d\mu_2}{\mu} - \frac{d\mu_1}{\mu} \right) \geq \]

\[ \geq \int \Sigma 2(\partial_t h, P_{f_1} h) \, d\mu_1 - C_{f_2} \varepsilon^{1/2} t^{-3/4} \int \Sigma |\nabla h| |h| \, d\mu_*. \]

By \[4, \text{Lemma 2.2 (2.16)}\] and a change of variable

\[ \partial_t \, d\mu = \left( -\langle \mathbf{H}, \partial_t^+ f \rangle + \text{div}_g \partial_t^+ \tan f \right) \, d\mu, \]

where \( \partial_t^+ f = \partial_t f - \partial_t^\perp f \) is the tangential part of \( \partial_t f \) along \( f \). We proceed

\[ \frac{d}{dt} \int \Sigma |P_{f}^+ h|^2 \, d\mu = \]

\[ = \int \Sigma 2(\partial_t^+ P_{f}^+ h, P_{f}^+ h) \, d\mu + \int \Sigma |P_{f}^+ h|^2 \left( -\langle \mathbf{H}, \partial_t^+ f \rangle + \text{div}_g \partial_t^+ \tan f \right) \, d\mu. \]

\[ = \int \Sigma 2(\partial_t^+ P_{f}^+ h, P_{f}^+ h) \, d\mu - \int \Sigma |P_{f}^+ h|^2 \langle \mathbf{H}, \partial_t^+ f \rangle \, d\mu - \int \Sigma |\nabla (|P_{f}^+ h|^2) |\partial_t^+ \tan f \, d\mu \leq \]

\[ \leq \int \Sigma 2(\partial_t^+ P_{f}^+ h, P_{f}^+ h) \, d\mu + C_{f_2} \varepsilon^{1/2} t^{-3/4} \int \Sigma |\nabla h| |h| \, d\mu_* + C_{f_2} (1 + \varepsilon t^{-1}) \int \Sigma |h|^2 \, d\mu_. \]

We observe

\[ \partial_t^+ P_{f}^+ h = \partial_t^\perp h - g^{ij} \langle h, \partial_i f \rangle \partial_t^\perp \partial_j f \]

and

\[ \partial_t^\perp \partial_j f = \nabla_j^\perp \partial_t f = \nabla_j^\perp \left( \partial_t^\perp f + g^{kl} \langle \partial_k f, \partial_k f \rangle \partial_t \partial_l f \right) = \nabla_j^\perp \partial_t^\perp f + g^{kl} \langle \partial_k f, \partial_k f \rangle A_{jl} = \]

\[ = -\frac{1}{2} \nabla_j^\perp \left( \Delta^\perp \mathbf{H} + Q(A^0) \mathbf{H} \right) + g^{kl} \langle \partial_k f, \partial_k f \rangle A_{jl}, \]

hence by (6.8)

\[ |\partial_t^\perp \partial_j f| \leq C \varepsilon^{1/2} t^{-1}. \]
Therefore
\[
\int \Sigma 2(\partial^+_t P^+_f h, P^+_f h)\ d\mu \leq \int \Sigma 2(\partial^+_t h, P^+_f h)\ d\mu + C\varepsilon^{1/2}t^{-1} \int |h|^2\ d\mu.
\]
Plugging into (6.16), we obtain
\[
\frac{d}{dt} \int \Sigma |P^+_f h|^2\ d\mu \leq \int \Sigma 2(\partial^+_t h, P^+_f h)\ d\mu + C_f \varepsilon^{1/2}t^{-3/4} \int |\nabla h| |h|\ d\mu + C_f(1+\varepsilon^{1/2}t^{-1}) \int |h|^2\ d\mu. 
\]
and with (6.15)
\[
(6.17)
\]
\[
I_2^2 - I_1^2 \geq \frac{d}{dt} \int \Sigma |P^+_f h|^2\ d\mu - C_f \varepsilon^{1/2}t^{-3/4} \int |\nabla h| |h|\ d\mu - C_f(1+\varepsilon^{1/2}t^{-1}) \int |h|^2\ d\mu.
\]
By the equation of Mainardi-Codazzi and \( \nabla_j \nabla_i f = A_{ij} \), we see
\[
\nabla^+_k \nabla_j \nabla_i f = \nabla^+_k A_{ij} = \nabla^+_j A_{ki} = \nabla^+_j \nabla_k \nabla_i f
\]
and
\[
I_2 = \int \Sigma g^{ij} g^{kl} \langle \nabla^+_i \nabla^+_j \nabla^+_k \nabla^+_l f, h \rangle\ d\mu = \int \Sigma g^{ij} g^{kl} \langle \nabla^+_i \nabla^+_j f, \nabla^+_k \nabla^+_l P^+_f h \rangle\ d\mu.
\]
We calculate
\[
\nabla^+_j \nabla^+_i P^+_f h = \nabla^+_j \nabla^+_i \left( h - g^{rs}(h, \partial_r f)\partial_s f \right) = \nabla^+_j \left( \nabla^+_i h - g^{rs}(h, \partial_r f)A_{ls} \right) =
\]
\[
= \nabla^+_j \left( \nabla_i h - g^{rs}(\nabla_i h, \partial_r f)\partial_s f \right) - \nabla^+_j \left( g^{rs}(h, \partial_r f)A_{ls} \right) =
\]
\[
= P^+_1 \nabla_j \nabla_i h - g^{rs}(\nabla_j h, \partial_r f)A_{is} - g^{rs}(\nabla_i h, \partial_r f)A_{js} - g^{rs}(h, \partial_r f)\nabla^+_j A_{ls}.
\]
Again using the equation of Mainardi-Codazzi \( \nabla^+_j A_{ls} = \nabla^+_s A_{jl} \), we get
\[
I_2 = \int \Sigma g^{ij} g^{kl} \langle \nabla_i \nabla_j f, P^+_j \nabla_i h \rangle\ d\mu - 2 \int \Sigma g^{ij} g^{kl} g^{rs}(A_{ik}, A_{js})\langle \partial_r f, \nabla_i h \rangle\ d\mu -
\]
\[
- \int \Sigma g^{ij} g^{kl} g^{rs}(A_{ik}, A_{is})\langle A_{jr}, h \rangle\ d\mu - \int \Sigma g^{ij} g^{kl} g^{rs}(A_{ik}, \nabla^+_s A_{jl})\langle \partial_r f, h \rangle\ d\mu =
\]
\[
= \int \Sigma g^{ij} g^{kl} \langle \nabla_i \nabla_j f, \nabla_i \nabla_j h \rangle\ d\mu - 2 \int \Sigma g^{ij} g^{kl} g^{rs}(A_{ik}, A_{js})\langle \nabla_i f, \nabla_i h \rangle\ d\mu -
\]
\[
- \int \Sigma g^{ij} g^{kl} g^{rs}(A_{ik}, A_{is})\langle A_{jr}, h \rangle\ d\mu + \frac{1}{2} \int \Sigma |A|^2(\nabla_i h)\ d\mu + \frac{1}{2} \int \Sigma g^{rs} |A|^2(\nabla_i f, \nabla_i h)\ d\mu =
\]
\[ \int \sum g^{ij} g^{kl} \langle \nabla_k \nabla_i f, \nabla_j \nabla h \rangle \, d\mu + \int \left( A * A * \nabla f * \nabla h + A * A * A * h \right) \, d\mu. \]

Recalling the cubic form of \(Q(A^0)H\) in the second fundamental form and again that \(\nabla_k \nabla_i f = A_ki\) is normal along \(f\), we obtain (6.18)

\[ I_2 + I_3 = \int \sum g^{ij} g^{kl} \langle \nabla_k \nabla_i f, \nabla_j \nabla h \rangle \, d\mu + \int \left( A * A * \nabla f * \nabla h + A * A * A * h \right) \, d\mu. \]

Calling these integrals \(J_2\) and \(J_3\), we proceed using (6.8), (6.11) - (6.13)

\[ J^2_2 - J^1_2 = \int \sum g^{ij} g^{kl} \langle \nabla_k \nabla_i f_2, \nabla_j \nabla^2 h \rangle \, d\mu_2 - \int \sum g^{ij} g^{kl} \langle \nabla_k \nabla_i f_1, \nabla_j \nabla^2 h \rangle \, d\mu_1 = \]

\[ \geq \int \sum g^{ij} g^{kl} \langle \nabla_k \nabla_i f_2 - (T^r_{ki} - T^*_{ki}) \nabla_r f_2, \nabla_j \nabla^2 h - (T^s_{ij} - T^*_{ij}) \nabla_s h \rangle \, d\mu_2 - \]

\[ - \int \sum g^{ij} g^{kl} \langle \nabla_k \nabla_i f_1 - (T^r_{ki} - T^*_{ki}) \nabla_r f_1, \nabla_j \nabla^2 h - (T^s_{ij} - T^*_{ij}) \nabla_s h \rangle \, d\mu_1 - \]

\[ - C \int \sum \langle \nabla h, |A_2| \nabla^2 h + (\Gamma^2 - \Gamma^*) * \nabla h \rangle \, d\mu_* \geq \]

\[ \geq c_{0.2s} \int \sum |\nabla^2 h|^2 \, d\mu_* - C \int \langle A_2, (\Gamma^2 - \Gamma^*) * \nabla h \rangle - \langle A_1, (\Gamma^1 - \Gamma^*) * \nabla h \rangle \, d\mu_* - \]

\[ - C \int \langle \nabla^2 h \rangle \, d\mu_* - C \varepsilon^{1/2} t^{-1/4} \int |\nabla^2 h|^2 \, d\mu_* \geq \]

\[ \geq c_{0.2s} \int \sum |\nabla^2 h|^2 \, d\mu_* - C f (1 + \varepsilon t^{-1/2}) \int |\nabla h|^2 \, d\mu_* - \]

\[ - C \int |A_2 - A_1| \, d\mu_* - C \int |A_1| |\Gamma^2 - \Gamma^*| \, d\mu_* - C \int |\nabla h| \, d\mu_* - \]

\[ - C \int |\Gamma^2 - \Gamma^*| \, d\mu_* - C \int |\nabla h| \, d\mu_* - \]

\[ - C \int |\Gamma^2 - \Gamma^*| \, d\mu_* - \]

\[ \int |\langle \nabla f_1 - \nabla f_s, \nabla^2 h \rangle| \, d\mu_* - C \int |\langle \nabla f_1, \nabla^2 h \rangle| \, d\mu_* \geq \]
\[
\geq c_{0, f_s} \int_{\Sigma} |\nabla^* h|^2 \, d\mu_s - C_{f_s} (1 + \varepsilon t^{-1/2}) \int_{\Sigma} |\nabla h|^2 \, d\mu_s - \\
-C_{f_s} (1 + C\varepsilon^{-1/2} t^{-1/4}) \int_{\Sigma} \left( C|\nabla^* h| + C_{f_s} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h| \right) |\nabla h| \, d\mu_s - \\
-C \| \nabla f_1 - \nabla f_* \|_{L^\infty(\Sigma)} \int_{\Sigma} \left( C|\nabla^* h| + C_{f_s} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h| \right) |\nabla^* h| \, d\mu_s - \\
- \int_{\Sigma} \left( C|\nabla^* h| + C_{f_s} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h| \right) |\langle \nabla f_s, \nabla^* h \rangle| \, d\mu_s \geq
\]

(6.19)

\[
\geq c_{0, f_s} \int_{\Sigma} |\nabla^* h|^2 \, d\mu_s - C_{f_s} (1 + \varepsilon t^{-1/2}) \int_{\Sigma} |\nabla h|^2 \, d\mu_s - C_{f_s} \int_{\Sigma} |\langle \nabla f_s, \nabla^* h \rangle|^2 \, d\mu_s,
\]

as \( \| \nabla f_1 - \nabla f_* \|_{L^\infty(\Sigma)} \leq \varepsilon \). For the last term, we recall that \( h = f_2 - f_1 = N_2 - N_1 \perp N_{f*} \Sigma \), hence \( \langle h, \nabla f_* \rangle = 0 \). By differentiation, we get

\[
\langle \nabla_i h, \nabla_k f_* \rangle + \langle h, A_{s,ik} \rangle = 0,
\]

\[
\langle \nabla_j^* \nabla_i^* h, \nabla_k f_* \rangle + \langle \nabla_i h, A_{s,jk} \rangle + \langle \nabla_j h, A_{s,ik} \rangle + \langle h, \nabla_i^* A_{s,ik} \rangle = 0
\]

and by smoothness of \( f_s \)

\[
|\langle \nabla f_s, \nabla^* h \rangle| \leq C_{f_s} (|h| + |\nabla h|).
\]

Plugging into (6.19) yields

(6.20)

\[
J_2^2 - J_1^2 \geq c_{0, f_s} \int_{\Sigma} |\nabla^* h|^2 \, d\mu_s - C_{f_s} (1 + \varepsilon t^{-1/2}) \int_{\Sigma} |\nabla h|^2 \, d\mu_s - C_{f_s} \int_{\Sigma} |h|^2 \, d\mu_s.
\]

Next using (6.8), (6.13), we get for any \( \delta > 0 \)

\[
|J_2^2 - J_1^2| \leq \\
\leq C \int_{\Sigma} |A_2 - A_1| \, |A| \, |\nabla h| \, d\mu_s + C \int_{\Sigma} |A|^2 \, |\nabla h|^2 \, d\mu_s + \\
+C \int_{\Sigma} |A_2 - A_1| \, |A|^2 \, |h| \, d\mu_s + C \int_{\Sigma} |A|^3 \, |\nabla h| \, |h| \, d\mu_s \leq \\
\leq C\varepsilon^{1/2} t^{-1/4} \int_{\Sigma} \left( C|\nabla^* h| + C_{f_s} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h| \right) |\nabla h| \, d\mu_s + \\
+C\varepsilon t^{-1/2} \int_{\Sigma} \left( C|\nabla^* h| + C_{f_s} (1 + \varepsilon^{1/2} t^{-1/4}) |\nabla h| \right) |h| \, d\mu_s +
\]

\[
\leq c_{0, f_s} \int_{\Sigma} |\nabla^* h|^2 \, d\mu_s - C_{f_s} (1 + \varepsilon^{1/2} t^{-1/4}) \int_{\Sigma} |\nabla h|^2 \, d\mu_s - C_{f_s} \int_{\Sigma} |h|^2 \, d\mu_s.
\]
\[ +C \varepsilon t^{-1/2} \int \Sigma |\nabla h|^2 \, d\mu_* + C \varepsilon^2 t^{-1} \int \Sigma |h|^2 \, d\mu_* \leq \]
\[
\leq \delta \int \Sigma |\nabla h|^2 \, d\mu_* + C_{f, \varepsilon} (1 + \varepsilon t^{-1/2}) \int \Sigma |\nabla h|^2 \, d\mu_* + C_{\delta} \varepsilon^2 t^{-1} \int \Sigma |h|^2 \, d\mu_*.
\]

(6.21)

Combining (6.14), (6.17), (6.18), (6.20), (6.21), we obtain

\[
\frac{d}{dt} \int \Sigma |P_{f_1} h|^2 \, d\mu_1 + c_{0, f_1} \int \Sigma |\nabla h|^2 \, d\mu_* \leq \]
\[
\leq C_{f_1} (1 + \varepsilon^{1/2} t^{-1/2}) \int \Sigma |h|^2 \, d\mu_* + C_{f_1} (1 + \varepsilon^{1/2} t^{-1}) \int \Sigma |h|^2 \, d\mu_* = \]
\[
= C_{f_1} (1 + \varepsilon^{1/2} t^{-1/2}) \int \Sigma \langle -\Delta h, h \rangle \, d\mu_* + C_{f_1} (1 + \varepsilon^{1/2} t^{-1}) \int \Sigma |h|^2 \, d\mu_* \leq \]
\[
\leq \delta \int \Sigma |\nabla h|^2 \, d\mu_* + C_{f_1, \delta} (1 + \varepsilon^{1/2} t^{-1}) \int \Sigma |h|^2 \, d\mu_*,
\]

and

\[
\frac{d}{dt} \int \Sigma |P_{f_1} h|^2 \, d\mu_1 + c_{0, f_1} \int \Sigma |\nabla h|^2 \, d\mu_* \leq C_{f_1} (1 + \varepsilon^{1/2} t^{-1}) \int \Sigma |h|^2 \, d\mu_1.
\]

As $|P_{f_1} h| \geq C|h|$ for $\varepsilon < \varepsilon(f_*)$ small enough, integration from $0 < \tau < t \leq T$ yields

\[
\int \Sigma |h|^2 \, d\mu_1 \leq C \exp \left( \int_{\tau}^{t} C_{f_1} (1 + \varepsilon^{1/2} s^{-1}) \, ds \right) \int \Sigma |h\tau|^2 \, d\mu_1 \leq \]
\[
\leq C_{f_1, \varepsilon}^{1/2} \tau^{-C_{f_1} \varepsilon^{1/2}} \int \Sigma |h\tau|^2 \, d\mu_1
\]

and (6.7) follows. If $f_{2, t} - f_{1, t} \rightarrow 0$ weakly in $L^2(\Sigma)$, we get from (6.10)

\[
\| f_{2, t} - f_{1, t} \|_{L^2(\Sigma, g_{\tau})} \leq C_{f_1} C_{f_2}^{1/2} \tau^{-C_{f_1} \varepsilon^{1/2}} \rightarrow 0 \quad \text{for } \tau \rightarrow 0,
\]

when $C(f_*)^{1/2} < 1/2$, and therefore $f_2 - f_1 = 0$. \[\Box\]
References