An Iterative BEM for the Inverse Problem of Detecting Corrosion in a Pipe

Xin, Yang†, Mourad Choulli‡, Jin Cheng∗

† Institute of Mathematics, Fudan University, Shanghai 200433, China
e-mail: weighsun@hotmail.com
‡ Département de Mathématiques Université de Metz
Ile du Saulcy, 57045 Metz cedex, France
e-mail: choulli@math.univ-metz.fr
∗ Department of Mathematics, Fudan University, Shanghai 200433, China
e-mail: jcheng@fudan.edu.cn

Abstract

In this paper, we consider an inverse problem of determining the corrosion occurring in an inaccessible interior part of a pipe from the measurements on the outer boundary. The problem is modelled by Laplace’s equation with an unknown term γ in the boundary condition on the inner boundary. Based on the Maz’ya iterative algorithm, a regularized BEM method is proposed for obtaining approximate solutions for this inverse problem. The numerical results show that our method can be easily realized and is quite effective.

Keywords: Corrosion, Iterative Method, Cauchy problem, Laplace equation

Corresponding author: Jin Cheng
Department of Mathematics, Fudan University
Shanghai 200433, China
Tel: +86-21-6564-3880 (O)
Fax: +86-21-6564-6073
Email: jcheng@fudan.edu.cn

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XIN YANG, MOURAD CHOUlli, AND JIN CHENG

ABSTRACT. In this paper, we consider an inverse problem of determining the corrosion occurring in an inaccessible interior part of a pipe from the measurements on the outer boundary. The problem is modelled by Laplace’s equation with an unknown term $\gamma$ in the boundary condition on the inner boundary. Based on the Maz’ya iterative algorithm, a regularized BEM method is proposed for obtaining approximate solutions for this inverse problem. The numerical results show that our method can be easily realized and is quite effective.

1. INTRODUCTION

Detecting the corrosion inside a pipe is one of the most important topics in the engineering, especially for the safety administration of the nuclear power station. There are several ways to do this. In this paper, we will discuss the mathematical theory and numerical algorithm for a method of detecting the corrosion by electrical fields. More exactly, we consider an inverse problem of determining the corrosion occurring in an inaccessible interior part of a pipe from the measurements on the outer boundary. Our goal is to determine information about the corrosion that possibly occurs on an interior surface of the pipe, which is an 'inaccessible' part, and we collect electrostatic data on the part of the exterior surface of the pipe, which is an 'accessible' part.

In the case that the thickness of the pipe is sufficient small compared with the radius of the pipe and the Cauchy data are given on the whole outer boundary, this inverse problem can be treated by Thin Plate Approximation method (TPA). The algorithm and numerical analysis can be found in [7]. But this algorithm works
only under the assumption that the thickness is small enough compared with the radius of the pipe. The case, in which Cauchy data are given on the part of the outer boundary and the smallness assumption is abandoned, has not been studied and it is obvious that is of great important for the practice problems.

The main difficulty for this inverse problem is ill-posedness of the inverse problem. The measured data are given only on part of the outer boundary and we want to determine an unknown function in the inner boundary. Because of the ill-posedness, the errors in measured data will be enlarged in the numerical treatment if we do not treat it suitably. In this paper, based on the Maz’ya iterative method, we propose a new BEM algorithm for this inverse problem. It can easily realized. The numerical results show the efficiency of this method.

This paper is organized as the follows:

1. Formulation of the inverse problem
2. The iterative boundary element method
3. Numerical examples
4. Conclusions

2. FORMULATION OF THE INVERSE PROBLEM

Suppose that a domain \( \Omega = \{ x \mid r_1 < |x| < r_2 \} \subset \mathbb{R}^2 \) (see figure 1) and the boundaries \( \Gamma_1 = \{ x \mid |x| = r_1 \} \), \( \Gamma_2 = \{ x \mid |x| = r_2 \} \).

Assume that \( \Omega \) is a metallic body with constant conductivity. In the domain \( \Omega \), we consider the electrostatic field. The electric potential \( u \) satisfies the Laplace’s
equation in $\Omega$, i.e.,

\begin{equation}
\Delta u = 0, \quad \text{in} \quad \Omega.
\end{equation}

Let $\Gamma_0$ be the open set of outer boundary $\Gamma_2$ of $\Omega$ which is an 'accessible' part. On $\Gamma_0$, the Dirichlet data and the Neumann data of the electric potential $u$ are given, i.e.,

\begin{align}
(2.2) \quad u(x) &= \phi(x) \quad x \in \Gamma_0, \\
(2.3) \quad u_n(x) &= \psi(x) \quad x \in \Gamma_0
\end{align}

where $u_n$ is the outer normal derivative of $u$ on the boundary.

We denote the rest part of the exterior boundary of $\Omega$ to be $\tilde{\Gamma}_2$,

$$\tilde{\Gamma}_2 = \Gamma_2 \setminus \Gamma_0.$$

We assume that the corrosion only happened on the interior boundary of the domain $\Omega$ and the corrosion can be described by a non-negative function $\gamma$ in the boundary condition on the interior boundary. That is

\begin{equation}
(2.4) \quad u_n + \gamma u = 0, \quad \text{on} \quad \Gamma_1
\end{equation}

where $\gamma \geq 0$ represents the corrosion damage.

The inverse problem we discuss in this paper is to find the unknown coefficient $\gamma$ from the Cauchy data $\phi$ and $\psi$ on $\Gamma_0$.

We will treat this inverse problem by the following steps:

**Step 1:** Get the Cauchy data on the interior circle by solving the Cauchy problem for Laplace equations.

We use the iterative boundary element method to solve the Cauchy problem:

\begin{equation}
\begin{cases}
\Delta u(x) = 0, \quad x \in \Omega \\
u(x) = \phi(x), \quad x \in \Gamma_0 \\
u_n(x) = \psi(x), \quad x \in \Gamma_0
\end{cases}
\end{equation}

our goal is to get the Cauchy data on $\Gamma_1$:

$$u(x) = \phi_1(x), \quad x \in \Gamma_1, \quad u_n(x) = \psi_1(x), \quad x \in \Gamma_1.$$ 

**Step 2:** Get the impedance $\gamma$ from the Cauchy data on interior circle.
By the boundary condition
\[ u_n + \gamma u = 0, \quad x \text{ on } \Gamma_1, \]
we can obtain the coefficient \( \gamma \):
\[ \gamma = -\frac{u_n}{u} \bigg|_{\Gamma_1} = -\frac{\psi}{\phi_1} \quad \text{if } \phi_1 \neq 0 \]

Remark 2.1. It can be proved that the measure of the zero set \( \{ \phi_1 = 0 \} \) can not be non-zero. Therefore our method is valid for almost all \( x \in \Gamma_1 \).

3. The iterative boundary element method for this Cauchy problem

In this section, based on the results in [9], [10], [11], we will give the iterative boundary element method for the Cauchy problem in the Step 1. We can prove the convergence rate by only assuming some regularity for the solution on the boundary. The some numerical simulation results for the Cauchy problem are also presented.

3.1. Description of the algorithm. In [11], V.A. Kozlov, V.G. Maz’ya and A.V. Fomin proposed the algorithm as following:
1. Specify an initial boundary guess \( u_0 \) on \( \Gamma_1 \) and \( \Gamma_2 \).
2. Solve the well-posed mixed boundary value problem:
\[
\begin{cases}
\Delta U^{(0)}(x) = 0, & x \in \Omega \\
U^{(0)}_n = \psi & x \in \Gamma_0 \\
U^{(0)} = u_0 & x \in \Gamma_1 \cup \Gamma_2
\end{cases}
\]
we can determine \( U^{(0)}(x) \) for \( x \in \Omega \) and \( q_0 = U^{(0)}_n(x) \) for \( x \in \Gamma_1 \cup \Gamma_2 \).
3. (i). Suppose that the approximation \( q_k \) is obtained, we can solve the mixed boundary value problem:
\[
\begin{cases}
\Delta U^{(2k+1)} = 0 & x \in \Omega \\
U^{(2k+1)} = \phi & x \in \Gamma_0 \\
U^{(2k+1)}_n = q_k & x \in \Gamma_1 \cup \Gamma_2.
\end{cases}
\]
Then we can determine \( U^{(2k+1)}(x) \) for \( x \in \Omega \) and \( u_{k+1} = U^{(2k+1)}(x) \) for \( x \in \Gamma_1 \cup \Gamma_2 \).
(ii) By $u_{k+1}$, we can obtain $U^{(2k+2)}(x)$ for $x \in \Omega$ and $q_{k+1} = U^{(2k+2)}(x)$ for $x \in \Gamma_1 \cup \tilde{\Gamma}_2$ by solving the mixed boundary value problem:

\[
\begin{cases}
\Delta U^{(2k+2)} = 0 & x \in \Omega \\
U_n^{(2k+2)} = \psi & x \in \Gamma_0 \\
U^{(2k+2)} = u_{k+1} & x \in \Gamma_1 \cup \tilde{\Gamma}_2
\end{cases}
\]

4. Repeat step 3 for $k \geq 0$ until a prescribed stopping criterion is satisfied.

The stopping criterion we used in this paper is $\|u_{k+1} - u_k\|_{L^2(\Gamma_1 \cup \Gamma_2)} \leq \varepsilon$, where $\varepsilon$ is a small positive number.

**Remark 3.1.** The mixed boundary value problems (3.2), (3.3) are well-posed problems.

In this paper, we will solve the mixed boundary value problems (3.2), (3.3) by the boundary element method (BEM). As for BEM, it can be found in many books. Here we just refer to the books [1] and references in it.

In the following, we only give the outline of the iterative BEM.

Consider the following mixed boundary value problem in two-dimension case.

\[
\begin{cases}
\Delta u = 0 & in \ \Omega \\
u = f & on \Gamma_D \\
u_n = g & on \Gamma_N
\end{cases}
\]

As we known, the foundational integral formula of the harmonic function

\[
u(M_i) = \int_{\Gamma} (u^* \frac{\partial u}{\partial \nu} - u \frac{\partial u^*}{\partial \nu}) d\Gamma,
\]

where $u^* = \frac{1}{2\pi} \ln \frac{1}{r_{M_i}}$ represents the foundational solution of the Laplace equation.

And the boundary integral formula is:

\[
c_i u(M_i) = \int_{\Gamma} (u^* \frac{\partial u}{\partial \nu} - u \frac{\partial u^*}{\partial \nu}) d\Gamma,
\]

Equation (3.6) can be discretized as follows:

\[
c_i u_i + \sum_{j=1}^{N} \int_{\Gamma_j} uq^* d\Gamma - \sum_{j=1}^{N} \int_{\Gamma_j} q d\Gamma = 0
\]
The $u$ and $q$ values inside the (3.7) are constant within each element and consequently can be taken out of the integrals. This gives

\begin{equation}
(3.8)
c_i u_i + \sum_{j=1}^{N} (\int_{\Gamma_j} q^* d\Gamma) u_j - \sum_{j=1}^{N} (\int_{\Gamma_j} u^* d\Gamma) q_j = 0
\end{equation}

With the given boundary condition, we can rearrange the equation (3.8) with all the unknowns on the left-hand side and a vector on the right-hand side obtained by multiplying matrix elements by the known values. This gives

\begin{equation}
(3.9)
c_i u_i + \sum_{j=1}^{m} (\int_{\Gamma_j} q^* d\Gamma) u_j - \sum_{j=m+1}^{N} (\int_{\Gamma_j} u^* d\Gamma) q_j = \sum_{j=m+1}^{N} (\int_{\Gamma_j} q^* d\Gamma) u_j - \sum_{j=1}^{m} (\int_{\Gamma_j} u^* d\Gamma) q_j
\end{equation}

The whole set of equations can be expressed in matrix form as

\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\begin{bmatrix}
q_D \\
u_D \\
q_N \\
u_N
\end{bmatrix} =
\begin{bmatrix}
B \\
A
\end{bmatrix}
\begin{bmatrix}
\phi \\
\psi
\end{bmatrix}
\]

where $u_D$, $q_d$ represent the Dirichlet and Neumann data on $\Gamma_D$ and $u_N$, $q_N$ represent the Dirichlet and Neumann data on $\Gamma_N$.

The step 3 of our iterative method can be presented as:

(i) solving following linear equations:

\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\begin{bmatrix}
\psi_{k+1} \\
u_{k+1}
\end{bmatrix} =
\begin{bmatrix}
B \\
A
\end{bmatrix}
\begin{bmatrix}
\phi \\
q_k
\end{bmatrix}
\]

and get $u_{k+1}$ that needed in the next equations.

(ii) With $u_{k+1}$, we can get $q_{k+1}$ by solving

\[
\begin{bmatrix}
\phi_{k+1} \\
q_{k+1}
\end{bmatrix} = \begin{bmatrix}
A \\
B
\end{bmatrix}
\begin{bmatrix}
\psi \\
u_{k+1}
\end{bmatrix}
\]

Our boundary element method is transformed to a problem of solving the linear equations twice in every iterative. It can be easily realized by the technique of Matrix computing.

3.2. Convergence analysis. In this section we give the convergence analysis under some regularity assumption on the unknown potential $u$ on the boundary $\Gamma_1$.

First of all, we simplify the subproblem 1 as the following Cauchy Problem for Laplace equation:
Let $\Omega \subset \mathbb{R}^2$ be an open bounded set and $\Gamma_1, \Gamma_2$ be two parts of the boundary $\partial \Omega$, satisfying $\Gamma_1 \cup \Gamma_2 = \partial \Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

\begin{equation}
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u \cdot u = f & \text{on } \Gamma_1 \\
u \cdot \nu = g & \text{on } \Gamma_2
\end{cases}
\end{equation}

(3.10)

where $\nu$ is the unit outer derivative vector.

Given the Cauchy data $(f, g) \in H^{1/2}(\Gamma_1) \times H^{1/2}_0(\Gamma_1)'$, we assume that there exists a $H^1$-solution of the problem (3.10). We are mainly interested in the determination of the Neumann trace.

Following work is to introduce a operator $T : H^{1/2}_0(\Gamma_2)' \longrightarrow H^{1/2}_0(\Gamma_2)'$ represent the above iterative. Refer to [8].

We can simplify our iterative method as

\begin{equation}
\begin{cases}
\Delta \omega = 0 & \text{in } \Omega; \quad \omega|_{\Gamma_1} = f; \quad \omega_{\nu|_{\Gamma_2}} = \phi \\
\Delta \nu = 0 & \text{in } \Omega; \quad \nu_{\nu|_{\Gamma_1}} = g; \quad \nu|_{\Gamma_2} = \psi
\end{cases}
\end{equation}

We define the operators $L_n : H^{1/2}_0(\Gamma_2)' \longrightarrow H^1(\Omega)$ and $L_d : H^{1/2}(\Gamma_2) \longrightarrow H^1(\Omega)$ by:

$$L_n(\phi) := \omega \in H^1(\Omega)$$

$$L_d(\psi) := \nu \in H^1(\Omega)$$

Define the Neumann trace operator $\gamma_n : H^1(\Omega) \longrightarrow H^{1/2}_0(\Gamma_2)'$, $\gamma_n(u) := u_{\nu|_{\Gamma_2}}$ and the Dirichlet trace operator $\gamma_d : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma_2)$, $\gamma_d(u) := u|_{\Gamma_2}$

So we can rewrite the iterative as:

\begin{equation}
\begin{cases}
\omega = L_n(\phi_k); \quad \psi = \gamma_d(\omega) \\
\nu = L_d(\psi_k); \quad \phi_{k+1} = \gamma_n(\nu)
\end{cases}
\end{equation}

If we define $T := \gamma_n \circ L_d \circ \gamma_d \circ L_n$, we conclude that $T$ is an affine operator on $H^{1/2}_0(\Gamma_2)$, which satisfies

$$\phi_{k+1} = T(\phi_k) = T^{k+1}(\phi_0)$$

That means we are able to describe the iterative with powers of the operator $T$.

As $L_n$ and $L_d$ are both affine, we can write

$$L_n(\cdot) = L_n^1(\cdot) + \omega f, \quad L_d(\cdot) = L_d^1(\cdot) + \nu g$$
where the $H^1(\Omega, F)$-functions $\omega_f$ and $\nu_g$ depend only of $f$ and $g$ respectively.

With these definitions we have

$$
\phi_{k+1} = T(\phi_k) = \gamma_n \circ L_d^I \circ \gamma_d \circ L^I_d(\phi_k) + \gamma_n \circ L_d^I \circ \gamma(\omega_f) + \gamma_n(\nu_g)
$$

$$
= T^{k+1}_I(\phi_0) + \sum_{j=0}^k T^j_I(z_{f,g}).
$$

From [8], we know the operator $T_I$ is positive, self adjoint, injective, regular asymptotic in $H^{1/2}_{00}$ and non expansive. In [8] the convergence of this iterative method is presented. Under the source condition which is not so obvious for the engineers. Here we use only regularity assumptions in the convergence analysis.

Since our problem is in an annular domain, the following theorems are discussed in annular domain. But the results can be extended into general domain.

Firstly, we define the Sobolev spaces of periodic functions:

$$
H^s_{per}(-\pi, \pi) := \{ \phi(y) = \sum_{j \in \mathbb{Z}} \phi_j e^{ijk} \mid \sum_{j \in \mathbb{Z}} (1 + j^2)^s \phi_j^2 < \infty \}, s \in \mathbb{R}
$$

Before we give the theorems, we introduce the following logarithmic-type source conditions:

$$
f(\lambda) = \begin{cases} (\ln(\exp(1)\lambda^{-1}))^{-p} & \lambda > 0 \\
0 & \lambda = 0 \end{cases}
$$

**Theorem 3.2.** Set $\Omega$ be an annular domain, $\Omega \subset \mathbb{R}^2$, Let $(f, g)$ be consistent Cauchy data and assume that the solution $\overline{\phi}$ of the Cauchy problem (3.10) satisfies

$$
\overline{\phi} - \phi_0 \in H^1_{per}
$$

where $\phi_0 \in H$ is some initial guess. Let $\mu > 2$, $(f_\epsilon, g_\epsilon)$ some given noisy data with $\|z_\epsilon - z_{f,g}\| \leq \epsilon$, $\epsilon > 0$ and $k(\epsilon, z_\epsilon)$ the stopping rule determined by the discrepancy principle

$$
k(\epsilon, z_\epsilon) = \min\{k \in \mathbb{N} \mid \|z_\epsilon - (I - T_I)\phi_k^\epsilon\| \leq \mu \epsilon\}.
$$

Then there exists a constant $C$, depending on $\phi_0$ only, such that

i) $\|\overline{\phi} - \phi_k^\epsilon\| \leq C(\ln k)^{-1}$

ii) $\|z_\epsilon - (I - T_I)\phi_k^\epsilon\| \leq Ck^{-1}(\ln k)^{-1}$

for all iteration index $k$ satisfying $1 \leq k \leq k(\epsilon, z_\epsilon)$.
**Theorem 3.3.** Set \( k_\epsilon = k(\epsilon, z_\epsilon) \). Under the assumption of Theorem 3.2 we have

i) \( k_\epsilon (\ln(k_\epsilon))^{-1} = O(\epsilon^{-1}) \)

ii) \( \| \varphi - \varphi_{k_\epsilon}^\epsilon \| \leq O((\ln \sqrt{\epsilon})^{-1}) \)

The next lemma is most important for the proof of the theorems.

**Lemma 3.4.** Set \( \Omega \) be an annular domain, \( \Omega \subset \mathbb{R}^2 \), the solution \( \varphi \) of the Cauchy problem (3.10) in this domain satisfies,

\[
\varphi - \phi_0 \in H^1_{\text{per}}
\]

where \( \phi_0 \in H \) is some initial guess and \( H^1_{\text{per}} \) is the Sobolev spaces of periodic functions defined as (3.11). This regularity assumption is equivalent to choose some \( \psi \in H^0_{\text{per}} \) satisfying

\[
\varphi - \phi_0 = f(I - T_I)\psi.
\]

where \( f \) is the logarithmic-type source conditions (3.12).

**Proof.** For the simplicity, we consider the Cauchy problem (3.10) in the annular domain

\[
\Gamma_1 = \{(R, \theta); \theta \in (-\pi, \pi)\}, R > 1
\]

\[
\Gamma_2 = \{(1, \theta); \theta \in (-\pi, \pi)\}
\]

where \( f(\theta) = \sum_{j=1}^{N} a_j \sin(j\theta) \), \( g(\theta) = \sum_{j=1}^{N} b_j \sin(j\theta) \).

Given the Neumann data

\[
\phi_0(\theta) = \sum_{j=1}^{N} \phi_{0,j} \sin(j\theta)
\]

we can get:

\[
(T_I \phi_0)(\theta) = \sum_{j=1}^{\infty} \lambda_j \phi_{0,j} \sin j\theta
\]

where

\[
\lambda_j = \frac{(R^j - R^{-j})^2}{(R^j + R^{-j})^2}
\]

For \( \varphi - \phi_0 \in H^1_{\text{per}} \), there exists \( a_j, j = 1 \ldots N \) satisfying \( \sum_{j=1}^{N} a_j^2 < \infty \)

\[
\varphi - \phi_0 = \sum_{j=1}^{N} a_j j^{-1} \sin jy
\]
AN INVERSE PROBLEM OF DETECTING CORROSION

So we get

\[ \sum_{j=1}^{N} (1 + j^2) a_j^2 j^{-2} < \infty \]

To the logarithmic-type source conditions (3.12), the source condition is to find some \( \psi \in H^0_{\text{per}} \), satisfying

\[ \overline{\phi} - \phi_0 = f(I - T_l) \psi, \]

So our problem comes into finding this \( \psi \).

Set \( \psi = \sum_{j=1}^{N} b_j \sin jy \), then

\[ b_j = \frac{a_j}{jf(1 - \lambda_j)} \]

from the estimate

\[
\ln \left( \frac{\exp(1)}{1 - \lambda_j} \right) \geq 1 - \ln(\exp(1)[1 - \frac{R^j - R^{-j}}{R^j + R^{-j}}]) \\
= -\ln \left( \frac{2R^{-j}}{R^j + R^{-j}} \right) \\
\geq 2j\ln R - 1
\]

\[
\ln \left( \frac{\exp(1)}{1 - \lambda_j} \right) \leq 1 + \ln \left( \frac{1}{1 - \frac{R^j - R^{-j}}{R^j + R^{-j}}} \right) \\
= 1 + \ln \left( \frac{R^j + R^{-j}}{2R^{-j}} \right) \\
\leq 2j\ln R + 1 - \ln 2
\]

we have

\[ 2j\ln R - 1 \leq \frac{1}{f(1 - \lambda_j)} \leq 2j\ln R + 1 - \ln 2 \]

And with \( \sum_{j=1}^{N} a_j^2 < \infty \), we can obtain \( \sum_{j=1}^{N} b_j^2 < \infty \), i.e., \( \psi \in H^0_{\text{per}} \). \( \square \)

**Lemma 3.5.** Let \((f, g)\) be consistent Cauchy data and assume that the solution \( \overline{\phi} \) of the fixed point equation satisfies the source condition

\[ \overline{\phi} - \phi_0 = f(I - T_l) \psi, \quad \text{for some } \psi \in H \]

where \( \phi_0 \in H \) is some initial guess and \( f \) is the function defined in (3.12) with \( p \geq 1 \). Let \( \mu > 2 \), \((f_\epsilon, g_\epsilon)\) some given noisy data with \( \|z_\epsilon - z_{f, g}\| \leq \epsilon, \epsilon > 0 \) and
\( k(\epsilon, z_\epsilon) \) the stopping rule determined by the discrepancy principle. Then there exists a constant \( C \), depending on \( p \) and \( \| \psi \| \) only, such that

\[
\begin{align*}
  & i) \quad \| \bar{\phi} - \phi^k_{\epsilon} \| \leq C (\ln k)^{-p} \\
  & ii) \quad \| z_\epsilon - (I - T_l) \phi^k_{\epsilon} \| \leq C k^{1 - \epsilon} (\ln k)^{-p}
\end{align*}
\]

for all iteration index \( k \) satisfying \( 1 \leq k \leq k(\epsilon, z_\epsilon) \)

**Lemma 3.6.** Set \( k_\epsilon = k(\epsilon, z_\epsilon) \). Under the assumption of Lemma 3.5, we have

\[
\begin{align*}
  & i) \quad k_\epsilon (\ln k_\epsilon)^p = O(\epsilon^{-1}) \\
  & ii) \quad \| \bar{\phi} - \phi^k_{\epsilon, \epsilon} \| \leq O((-\ln \sqrt{\epsilon})^{-p})
\end{align*}
\]

The proof of lemma 3.5, lemma 3.6 can be found in [4].

By using the lemmas above, it is easy to give the proof. Theorem 3.2 can be proved by lemma 3.4 and lemma 3.5. Theorem 3.3 can be proved by lemma 3.4 and lemma 3.6.

3.3. **Numerical experiment for the Mazya iteration.** In this section, we will test the previous algorithm to calculate a few examples with Matlab. For simplicity, we set the domain \( \Omega \) with interior radius 1 and outer radius 1 + \( b \) in the following experiments. The number of the boundary element is \( n \). The quadratic elements will be used. we take \( n \) nodes on the outer circle and also \( n \) nodes on the interior circle. And set the number of nodes whose data are given to be \( m \). we consider harmonic function:

\[ u(x, y) = \log [(x - 0.5)^2 + (y - 0.5)^2] \]

we use the algorithm in the previous section to get the unknown data on the boundary, then use harmonic basic integral formulation to calculate the data on the circle with the radius \( 1 + a (a \leq b) \). In the following numerical experiment the noise level is \( \delta \). The figures on the left show the exact solution comparing with the approximate solution, the dot line represents the approximate solution and the real line represents the exact solution. And the figures on the right side are the curves of the absolute errors. We used the stopping rule as \( \| u_{k+1} - u_k \|_{L^2(\Gamma_1 \cup \Gamma_2)} \leq 10^{-3} \).

**Example 1.** In this experiment we take \( n = 100, \, 200, \, m = 50, \, b = 1, \, a = 0.5 \) and \( \delta = 0.01 \) respectively.
If we want to have the higher precision, we should use more element in each iterative.

**Example 2.** In this experiment we set $n = 100$, $m = 50$, $b = 1$, $a = 0.5$ and $\delta = 0.01$, 0.001 respectively.

$n=100, \ m=30, \ \delta = 0.01$: 
The subproblem 1 is ill-posed in the Hadamard sense, i.e., the solution does not depend continuous on the data. The numerical result shows this fact.

**Example 3.** In this experiment we set \( n = 100, m = 50, a = 0.1, 0.25, 0.5, b=1 \) and \( \delta = 0.01: \) ( \( a = 0.5 \) is shown in pervious example) 

a=0.25:
From the numerical simulation, it can be seen that the precision will decrease as $a$ decreases.

4. Numerical results for the inverse problem

In this section we will use the iterative algorithm to treat our inverse problem and give some numerical examples.

In the following, we choose the ring domain as

$$\Omega = \{(x, y)|1 \leq \sqrt{x^2 + y^2} \leq 2\}.$$
Example 1. In this example, we want to recover the continuous piecewise linear function:

\[
\gamma(\theta) = \begin{cases} 
1 & \text{when } \theta < 1 \\
4\theta - 3 & \text{when } 1 \leq \theta < 1.5 \\
-2\theta + 6 & \text{when } 1.5 \leq \theta < 2.5 \\
1 & \text{when } 2.5 \leq \theta < 4.5 \\
3\theta - \frac{25}{2} & \text{when } 4.5 \leq \theta < 5.5 \\
-6\theta + 37 & \text{when } 5.5 \leq \theta < 6 \\
1 & \text{otherwise}
\end{cases}
\]

We will use the data \(\phi, \psi\) which are obtained by solving the direct problem by the boundary element:

\[
\begin{align*}
\Delta U &= 0 & x \in \Omega \\
U_n &= -1 & x \in \Gamma_0 \cup \Gamma_1 \\
U_n + \gamma U &= 0 & x \in \Gamma_2
\end{align*}
\]

These are the result figures.

(1) Set \(m=100, n=100\)

(figure 4.7) (figure 4.8)
Example 2. We consider the harmonic function:

\[ u(x, y) = y^3 - x^2 y + x^2 - y^2 + 6. \]

where the polar coordinate form is

\[ u(r, \theta) = r^3(\sin^3 \theta - \cos^2 \theta \sin \theta) + r^2 \cos 2\theta + 6. \]

It is easy to known the coefficient \( \gamma \) on the inner circle is

\[ \gamma(\theta) = \frac{3(\sin^3 \theta - \cos^2 \theta \sin \theta) + 2 \cos 2\theta}{\sin^3 \theta - \cos^2 \theta \sin \theta + \cos 2\theta + 6}. \]

The numerical results are shown below. The one on the left side is the comparison between the approximate \( \gamma \) and the exact \( \gamma \). The right one is the absolute error curve.

(i) Set \( m=100, n=100 \)

(ii) Set \( m=50, n=100 \)
(ii) Set \( m = 50, n = 100 \)

\[ \begin{array}{c}
\text{approximate } \gamma \\
\text{exact } \gamma \\
\end{array} \]

\[ \begin{array}{c}
\text{approximate } \gamma \\
\text{exact } \gamma \\
\end{array} \]

(figure 4.3) (figure 4.4)

(iii) Set \( m = 100, n = 100 \) and add 5\% random noisy on the Cauchy data.

\[ \begin{array}{c}
\text{approximate } \gamma \\
\text{exact } \gamma \\
\end{array} \]

\[ \begin{array}{c}
\text{approximate } \gamma \\
\text{exact } \gamma \\
\end{array} \]

(figure 4.5) (figure 4.6)

The numerical results show that our method is sensitive with respect to the noise. This is consistency with the ill-posedness of the inverse problem we discuss.

In the case that we only give the Cauchy data on part of the outer boundary, usually we can only get the local solution. It seems difficult to the reasonable numerical solutions.

5. CONCLUSIONS

In this paper, we have investigated an inverse problem of detecting corrosion in a pipe. The problem is modelled by Laplace’s equation with the unknown coefficient in the boundary condition. We propose a new BEM iterative method and test the algorithm by numerical experiments.
REFERENCES


INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

E-mail address: weighsun@hotmail.com

Département de Mathématiques Université de Metz, Ile du Saulcy, 57045 Metz cedex, France

E-mail address: choulli@math.univ-metz.fr

DEPARTMENT OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

E-mail address: jcheng@fudan.edu.cn