SCHÉMAS AUX DIFFÉRENCES SUR LA “CUBED-SPHERE”

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Outline

1. The Cubed-Sphere Grid
2. One-dimensional Hermitian compact operators on the Cubed-Sphere
3. High-order accurate discrete differential operators on the Cubed-Sphere
4. Applications
The Cubed-Sphere grid

The cubed-sphere

The Cubed-Sphere is a spherical grid made of six identical squared patches matching the faces of a cube.

Figure: **Left:** Cubed-Sphere grid. **Right:** Topology and spherical periodicity.
Equianglar coordinate system

On each patch the local coordinate system consists of the equatorial angle $\xi$ and the latitudinal angle $\eta$.

Figure: Left: Regular cartesian grid using $(\xi_F, \eta_F)$ coordinates of the patch FRONT. Right: Projection of the Grid onto the FRONT face of the cube in the plane $x = 1$. 
The metric tensor is full

$$G = \frac{1}{\delta^4} (1 + X^2)(1 + Y^2) \begin{bmatrix} 1 + X^2 & -XY \\ -XY & 1 + Y^2 \end{bmatrix}, \quad (1)$$

$X = \tan(\xi)$, $Y = \tan(\eta)$ are the gnomonic coordinates, image of the $(\xi, \eta)$ coordinates on the faces of the cube, and

$$\delta = \sqrt{1 + X^2 + Y^2}.$$

The covariant and contravariant local bases $(g_\xi, g_\eta)$ and $(g^\xi, g^\eta)$ have a simple expression in terms of $(X, Y)$.
Contravariant base

The contravariant vectors are deduced by

\[
\begin{align*}
g^\xi &= G^{11} g_\xi + G^{12} g_\eta, \\
g^\eta &= G^{21} g_\xi + G^{22} g_\eta,
\end{align*}
\]

where

\[
G^{-1} = \begin{bmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{bmatrix}
\]  

Explicit form

Example: the contravariant base \((g^\xi, g^\eta)\) is given for the patch \textit{Front} by

\[
\begin{align*}
g^\xi &= \frac{1}{x(1 + X^2)} \begin{bmatrix} -X \\ 1 \\ 0 \end{bmatrix}, \\
g^\eta &= \frac{1}{x(1 + Y^2)} \begin{bmatrix} -Y \\ 1 \\ 0 \end{bmatrix}
\end{align*}
\]
Metric of the cubed-sphere (2)

Contravariant base

The contravariant vectors are deduced by

\[
\begin{align*}
\mathbf{g}^\xi &= G^{11} \mathbf{g}_\xi + G^{12} \mathbf{g}_\eta, \\
\mathbf{g}^\eta &= G^{21} \mathbf{g}_\xi + G^{22} \mathbf{g}_\eta,
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where

\[
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Explicit form

Example: the contravariant base \((\mathbf{g}^\xi, \mathbf{g}^\eta)\) is given for the patch *Front* by

\[
\mathbf{g}^\xi = \frac{1}{x(1 + X^2)} \begin{bmatrix} -X \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{g}^\eta = \frac{1}{x(1 + Y^2)} \begin{bmatrix} -Y \\ 1 \\ 0 \end{bmatrix}
\]
The simplest choice of data representation on the Cubed-Sphere is called a **grid function**. On each cartesian grid $I, II, III, IV, V$, and $VI$ it consists of point values $v_{i,j}$ (classical finite-differences).

A grid function on the Cubed-Sphere consists of the 6 sets of data

$$w^k_{i,j}, \quad -M \leq i, j \leq M, \quad k = I, II, III, IV, V, VI,$$

with $M = N/2$, $N =$ size of the local grid.

The spherical periodicity is expressed as

- Matching the values along the 12 edges.
- Matching the values at the 8 corners.
Non-local finite difference derivative

The **Hermitian discrete derivative** of a sequence \( (u_j)_{j \in \mathbb{Z}} \) is the sequence \( (u_{x,j})_{j \in \mathbb{Z}} \) defined by

\[
\sigma_x u_{x,j} = \delta_x u_j, \quad j \in \mathbb{Z},
\]

where \( \sigma_x, \delta_x \) are the finite difference operators

\[
\sigma_x u_{x,j} = \frac{1}{6} u_{x,j-1} + \frac{2}{3} u_{x,j} + \frac{1}{6} u_{x,j+1}, \quad \delta_x u_j = \frac{u_{j+1} - u_{j-1}}{2h}, \quad h > 0
\]

Truncation error

The Hermitian derivative is pointwise fourth-order accurate (analog to the pointwise cubic spline derivative),

\[
u_{x,j} = u'(x_j) + O(h^4)
\]
An irregular periodic grid is given by $\alpha_j = \varphi(x_j) \ 0 \leq j \leq N - 1$ with $x_N = x_0$. The grid $x_j = jh$ is an equispaced discretisation of $[0, 1]$ with $1-$ periodicity.

The Hermitian derivative $u_{\alpha,j}$ at $\alpha_j$ is defined by

$$u_{\alpha,j} = \frac{(u \circ \varphi)_{x,j}}{\varphi_{x,j}}, \ 0 \leq j \leq N - 1,$$

(9)

where

- $\varphi_{x,j}$ is the Hermitian derivative of $\varphi(x)$ at point $x_j$.
- $(u \circ \varphi)_{x,j}$ is the Hermitian derivative of $(u \circ \varphi)(x)$ at point $x_j$. 

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Calculation principle

- Calculate Hermitian derivatives along great circles (geodesics) of the Cubed-Sphere.
- Reconstruct from a set of suitably selected Hermitian derivatives a fourth-order discrete spherical gradient at each grid point.
Assembling data along great circle coordinate lines of the patches FRONT and BACK

Figure: ● Patches FRONT and BACK: copying data.
● Patches EAST and WEST: interpolationing data using cubic splines along iso-ξ lines.
Set $I_{\alpha}$ of great circles associated to the patches FRONT and BACK

Figure: Network $I_{\alpha}$: Iso-$\eta$ coordinate great circles of patches FRONT and BACK
Set $I_\beta$ of great circles associated to the patch FRONT and BACK

Figure: Network $I_\beta$: Iso-$\xi$ coordinate great circles of patches FRONT and BACK
Covering the cubed-sphere with great circles

Six sets of great circles

- System \((I_\alpha)\): based on \(\xi\) coordinate lines of face I,
- System \((I_\beta)\): based on \(\eta\) coordinate lines of face I,
- System \((II_\alpha)\): based on \(\xi\) coordinate lines of face II,
- System \((II_\beta)\): based on \(\eta\) coordinate lines of face II,
- System \((V_\alpha)\): based on \(\xi\) coordinate lines of face V,
- System \((V_\beta)\): based on \(\eta\) coordinate lines of face V.

Covering the cubed-sphere

Main idea: the calculation of one-dimensional derivatives along all the great circles of these six sets is enough to evaluate an approximate gradient at all points of the grid!
STEP 1:

Calculating Hermitian derivatives

One-dimensional Hermitian derivatives are computed along great circles of networks $I_\alpha$ and $I_\beta$.

Intrinsic calculation along geodesics

Hermitian derivatives are calculated with respect to angle $\alpha \in [0, 2\pi]$ (resp. $\beta \in [0, 2\pi]$), which is the curvilinear abscissa along each great circle of network $I_\alpha$ (resp. $I_\beta$).

Main observation

This approach avoids dealing with local interpatch treatment to cope with the discontinuity of the metrics.
Calculating gradient on patch FRONT and BACK

STEP 2:

Calculating the discrete spherical gradient

Pointwise evaluation of the discrete gradient $\nabla_s u$ on patches FRONT and BACK using

$$\nabla_s u = \frac{\partial u}{\partial \xi} \big|_\eta g^\xi + \frac{\partial u}{\partial \eta} \big|_\xi g^\eta,$$

where $(g^\xi, g^\eta)$ is the local contravariant basis at $(\xi, \eta)$.

Chain rule in one dimension

The partial derivatives $\frac{\partial u}{\partial \xi} \big|_\eta$ and $\frac{\partial u}{\partial \eta} \big|_\xi$ are deduced from the Hermitian derivatives using the chain rule as

$$\begin{cases}
\frac{\partial u}{\partial \xi} \big|_\eta &= \frac{\partial u}{\partial \alpha} \big|_\eta \frac{\partial \alpha}{\partial \xi} \big|_\eta, \\
\frac{\partial u}{\partial \eta} \big|_\xi &= \frac{\partial u}{\partial \beta} \big|_\xi \frac{\partial \beta}{\partial \eta} \big|_\xi.
\end{cases}$$
Final explicit formula

Spherical gradient on patches FRONT and BACK

\[ \nabla_{s,h} u_{i,j} = u_{\alpha,i,j} \underbrace{\left( \cos \eta_j \frac{1 + \tan^2 \xi_i}{1 + \cos^2 \eta_j \tan^2 \xi_i} \right)}_{\alpha-\text{Hermitian deriv.}} g_{\xi,i,j} + u_{\beta,i,j} \underbrace{\left( \cos \xi_i \frac{1 + \tan^2 \eta_j}{1 + \cos^2 \xi_i \tan^2 \eta_j} \right)}_{\beta-\text{Hermitian deriv.}} g_{\eta,i,j} \]  

(12)

Full gradient calculation on the Cubed-Sphere

Analog calculations for the pair of patches WEST/EAST and NORTH/SOUTH.
**Interest of the present approach**

1. **“Intrinsic” calculation**
   - All finite difference approximations are obtained using great circles curvilinear abscissa.

2. **No interpatch interpolation**
   - There is no ghost point procedure to extend the patch: we stick to the periodic hermitian approximation: easy change of hermitian formula without heavy recoding.

3. **Standard finite difference degrees of freedom**
   - No pole problem, no “overlapping” grid problems. Also: no special functions such as spherical harmonics or spherical wavelets.
Test 1: Accuracy of the calculated gradient

Approximate gradient of $u(x, y, z) = \sin(10\pi x) \sin(2\pi y) \sin(6\pi z)$

<table>
<thead>
<tr>
<th>N</th>
<th>$\epsilon_\infty$</th>
<th>rate</th>
<th>N=16</th>
<th>rate</th>
<th>N=32</th>
<th>rate</th>
<th>N=64</th>
<th>rate</th>
<th>N=128</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>23.475</td>
<td>2.99</td>
<td>2.949</td>
<td>4.61</td>
<td>0.121</td>
<td>4.07</td>
<td>7.205(-3)</td>
<td>3.91</td>
<td>4.774(-4)</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1250</td>
<td>625</td>
<td>312.5</td>
<td>156.25</td>
<td>78.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>386</td>
<td>1538</td>
<td>6146</td>
<td>24578</td>
<td>93306</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table: Convergence rate of the hermitian gradient of the oscillating function $u(x, y, z) = \sin(10\pi x) \sin(2\pi y) \sin(6\pi z)$ restricted to the unit sphere.
We consider the function the gradient of the function
\[ u(x, y, z) = \exp(x) + \exp(y) + \exp(z) \]
restricted to the unit sphere.

<table>
<thead>
<tr>
<th>N</th>
<th>rate</th>
<th>N=16</th>
<th>rate</th>
<th>N=32</th>
<th>rate</th>
<th>N=64</th>
<th>rate</th>
<th>N=128</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_\infty )</td>
<td>5.323(-4)</td>
<td>4.07</td>
<td>1.912(-5)</td>
<td>4.00</td>
<td>1.191(-6)</td>
<td>4.00</td>
<td>7.432(-6)</td>
<td>3.72</td>
</tr>
<tr>
<td>( \Delta \xi ) in km</td>
<td>1250</td>
<td>625</td>
<td>312.5</td>
<td>156.25</td>
<td>78.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nb of grid points</td>
<td>386</td>
<td>1538</td>
<td>6146</td>
<td>24578</td>
<td>93306</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Convergence rate of the hermitian gradient of the function
\[ u(x, y, z) = \exp(x) + \exp(y) + \exp(z) \]
restricted to the unit sphere.
Advection equation on the sphere

- Test problem: propagation of a cosine-bell at constant solid spherical velocity around the earth.
- Serves as a preliminary test to evaluate the accuracy of numerical methods for the shallow water system.
- Reference results widely reported in the literature by various spatial approximations: spectral, FV MUSCL, SE, DG, etc, on various grids: longitude/latitude, icosahedral, Cubed-Sphere, etc.

Setting of the problem

$h(x, t)=$ height of the bell.

\[
\begin{align*}
\frac{\partial}{\partial t} H(x, t) + c \cdot \nabla_s H(x, t) &= 0 \\
H(x, 0) &= H_0(x).
\end{align*}
\]  

(13)

The velocity is a “constant” solid velocity.
The Cosine-Bell at initial position

Figure: Initial position of the cosine-bell. Cubed-sphere grid with $N = 64$. 
Spatial approximation

Centered finite-difference scheme

Purpose: assess the accuracy of the new approximate gradient. We thus use the centered semi-discrete fourth-order scheme.

\[
\frac{dH_{i,j}^k(t)}{dt} + c \cdot \nabla_{s,h} H_{i,j}^k(t) = 0, \quad -M \leq i, j \leq M, \quad I \leq k \leq VI. \tag{14}
\]

Filtering

No upwinding is necessary to handle a linear wave propagation problem. A simple high-frequency filter is enough to damp high-frequency dispersive oscillations of a centered scheme.

Tenth-order filter

Suppression of the numerical dispersive effect supported by the “1/ − 1” mode. Filtering is a well-known practice in finite-difference computations.
Fourth-order time-stepping scheme

Runge-Kutta RK4 scheme

For the time-dependent system

\[
\frac{d}{dt} V(t) = AV(t), \quad A \in \mathbb{M}_N(\mathbb{R}),
\]

the standard explicit RK4 scheme is

\[
\begin{cases}
    k_0 = AV^n \\
    k_1 = A(V^n + \frac{1}{2} \Delta t k_0) \\
    k_2 = A(V^n + \frac{1}{2} \Delta t k_1) \\
    k_3 = A(V^n + \Delta t k_2) \\
    V^{n+1} = V^n + \Delta t \left( \frac{1}{6} k_0 + \frac{1}{3} k_1 + \frac{1}{3} k_2 + \frac{1}{6} k_3 \right).
\end{cases}
\]
Figure: History of the three relative errors $L^1$, $L^2$, $L^\infty$ for the cosine-bell advection problem for a cubed-sphere grid of size $N = 40$. 
Cosine-bell test case: numerical results (2)

<table>
<thead>
<tr>
<th>CFL</th>
<th>Direction</th>
<th>L₁ error</th>
<th>L₂ error</th>
<th>Lₘ error</th>
<th>Maximum</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>α = 0deg</td>
<td>5.43601(-2)</td>
<td>3.26394(-2)</td>
<td>2.65404(-2)</td>
<td>2.50503(-2)</td>
<td>-2.65404(-2)</td>
</tr>
<tr>
<td></td>
<td>α = 45deg</td>
<td>5.11205(-2)</td>
<td>2.93375(-2)</td>
<td>2.24284(-2)</td>
<td>2.11063(-2)</td>
<td>-2.24284(-2)</td>
</tr>
<tr>
<td>0.5</td>
<td>α = 0deg</td>
<td>4.0020(-2)</td>
<td>2.25259(-2)</td>
<td>1.91928(-2)</td>
<td>2.87524(-2)</td>
<td>1.91193(-2)</td>
</tr>
<tr>
<td></td>
<td>α = 45deg</td>
<td>3.45757(-2)</td>
<td>1.86116(-2)</td>
<td>1.43016(-2)</td>
<td>1.27016(-2)</td>
<td>-1.43016(-2)</td>
</tr>
</tbody>
</table>

**Table:** Relative error for the cosine-bell advection problem with the fourth-order centered hermitian scheme (14) combined with the RK4 time-stepping scheme. The mesh size is $40 \times 40 \times 6$ after 12 days.
Cosine-bell test case: numerical results (3)

Isolines

Figure: Contours after one rotation with $\alpha = \pi/4$. Left: numerical solution. Right: error. The cubed-sphere uses $N = 40$. The propagation is oriented towards north-east.
Test 3: Deformational test-case (R. Nair et al.)

Advection equation with time-dependent velocity

New series of test-cases: divergent free and non-divergent free series of spherical flows. Important for the accurate transport on the sphere.

Setting of the problem

\( H(x, t) = \text{height of the bell.} \)

\[
\begin{aligned}
\partial_t H(x, t) + c(x, t) \cdot \nabla_s H(x, t) &= 0 \\
H(x, 0) &= H_0(x).
\end{aligned}
\]  

(17)

Divergent-free velocity \( c(x, t) = \nabla \psi(x, t) \)

\[
\begin{aligned}
\psi(x, t) &= k \sin^2(\lambda/2) \cos^2(\theta) \cos(\pi t/T), \quad k > 0, \\
c(x, t) &= \nabla_s \psi(x, t)
\end{aligned}
\]  

(18)
Deformational test-case (2)

(a) Initial condition at $t = 0$
(b) Solution at $t = T/2$
(c) Final position at $t = T$
(d) Error $h_{cal} - h_{ex}$ at $t = T$

Figure: Non divergent flow test-case. The final time is $T = 5$. The number of unknowns is 21602 for a resolution of 1.5$^{\text{deg}}$ around the equator. The number of time-steps is 600 with an approximate CFL number of 0.75.
Table: Relative error for the non divergent flow test-case with the fourth-order centered hermitian scheme (14) combined with the RK4 time-stepping scheme. The grid $N = 60$ has 21602 ($=6 \times 60^2 + 2$) degrees of freedom, with a spatial resolution of $1.5^{\text{deg}}$ along the equator. The CFL number is approximately 0.75.
Comparison with a DG3 scheme

<table>
<thead>
<tr>
<th>scheme</th>
<th>Nb dofs / Nb. of time steps</th>
<th>L₁ error</th>
<th>L₂ error</th>
<th>L_∞ error</th>
<th>M'(h)</th>
<th>m'(h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HB-CS scheme</td>
<td>21602 dofs / 640 time steps</td>
<td>0.0084</td>
<td>0.0154</td>
<td>0.0255</td>
<td>0.0007</td>
<td>-0.0213</td>
</tr>
<tr>
<td>DG3 scheme</td>
<td>38400 dofs / 2400 time steps</td>
<td>0.0071</td>
<td>0.0124</td>
<td>0.0216</td>
<td>-0.0004</td>
<td>-0.0175</td>
</tr>
</tbody>
</table>

Table: Comparison between the HB-CS scheme with a DG3 scheme on the cubed-sphere for a similar resolution
Approximate spherical divergence

Principle

For $F$ a tangential vector field, use on each patch the formula

$$\text{div}_S F = \frac{1}{\sqrt{G}} \left[ \frac{\partial}{\partial \xi} (\sqrt{G} F \cdot g^\xi) |_\eta + \frac{\partial}{\partial \eta} (\sqrt{G} F \cdot g^n) |_\xi \right]$$  \hspace{1cm} (19)

“Intrinsic” calculation using Hermitian derivatives

Calculating the divergence on patches FRONT and BACK using Hermitian derivatives along great circles of the networks $I_\alpha$ and $I_\beta$.

$$\frac{\partial}{\partial \xi} (\sqrt{G} F \cdot g^\xi) |_\eta = \frac{\partial}{\partial \alpha} (\sqrt{G} F \cdot g^\xi) |_\eta \frac{\partial \alpha}{\partial \xi} |_\eta,$$  \hspace{1cm} (20)

$$\frac{\partial}{\partial \eta} (\sqrt{G} F \cdot g^n) |_\xi = \frac{\partial}{\partial \beta} (\sqrt{G} F \cdot g^n) |_\xi \frac{\partial \beta}{\partial \eta} |_\xi.$$  \hspace{1cm} (21)
Accuracy of the calculated divergence

Numerical test

Tangential vector functions of the form

\[ F = c(x)(n(x) \times \phi), \quad \text{div}_S F = \nabla_S c(x) \cdot (n(x) \times \phi) \]

<table>
<thead>
<tr>
<th>N=8</th>
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<th>rate</th>
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<th>rate</th>
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<tr>
<td>(\varepsilon_\infty)</td>
<td>33.064(0)</td>
<td>3.83</td>
<td>2.321(0)</td>
<td>4.42</td>
<td>1.087(-1)</td>
<td>4.10</td>
<td>6.311(-3)</td>
<td>4.02</td>
</tr>
<tr>
<td>(\Delta \xi \text{ in km})</td>
<td>1250</td>
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Table: Error between the calculated and the exact spherical divergence.

\(c(x) = \sin(4\pi x) + \sin(\pi y) + \sin(6\pi z), \phi = [1, 1, 1]^T.\)
Spherical Laplacian

Spherical Laplacian:

\[ \Delta_s u = \nabla_s \cdot (\nabla_s u), \]  

(22)

where \( \nabla_s \cdot \) and \( \nabla_s \) are the spherical divergence and the spherical gradient.

Coordinate expression of the Laplacian

In the coordinate system \((\xi, \eta)\) the spherical divergence operator of the vector field \( F \) is expressed as

\[ \nabla \cdot F = \frac{1}{\sqrt{\tilde{G}}} \left( \frac{\partial}{\partial \xi} (\sqrt{\tilde{G}} F \cdot g^\xi) + \frac{\partial}{\partial \eta} (\sqrt{\tilde{G}} F \cdot g^\eta) \right). \]  

(23)

where \( \tilde{G} \) is

\[ \tilde{G} = \sqrt{\det G}. \]  

(24)
Spherical Laplacian:

\[ \Delta_s u = \nabla_s \cdot (\nabla_s u), \]  \hspace{1cm} (22)

where \( \nabla_s \cdot \) and \( \nabla_s \) are the spherical divergence and the spherical gradient.

Coordinate expression of the Laplacian

In the coordinate system \((\xi, \eta)\) the spherical divergence operator of the vector field \(F\) is expressed as

\[ \nabla \cdot F = \frac{1}{\sqrt{\bar{G}}} \left( \frac{\partial}{\partial \xi} (\sqrt{\bar{G}} F \cdot g^\xi) + \frac{\partial}{\partial \eta} (\sqrt{\bar{G}} F \cdot g^\eta) \right). \]  \hspace{1cm} (23)

where \(\bar{G}\) is

\[ \bar{G} = \sqrt{|\det G|}. \]  \hspace{1cm} (24)
Compact Laplacian on each patch

From the gradient $\nabla_s u$, we deduce on each patch the grid functions $u_1(\xi, \eta)$ and $u_2(\xi, \eta)$,

$$u_1(\xi, \eta) = \sqrt{G} \nabla_s u \cdot g^\xi, \quad u_2(\xi, \eta) = \sqrt{G} \nabla_s u \cdot g^\eta$$  \hspace{1cm} (25)

Using the discrete gradient above an approximation of the two functions $u_1$ and $u_2$ on each patch is

$$(u_1, h)^k_{i,j} = \sqrt{G} \left( \nabla_{s,h} u^k_{i,j} \right) \cdot g^\xi_{i,j}, \quad (u_2, h)^k_{i,j} = \sqrt{G} \left( \nabla_{s,h} u^k_{i,j} \right) \cdot g^\eta_{i,j}.$$  \hspace{1cm} (26)

for $-M \leq i, j \leq M$ and $I \leq k \leq V I$. 
Boundary conditions

Here there is no use of periodicity around great circles. Instead the discrete hermitian Laplacian is coordinate dependent. Thus we need boundary conditions along the four edges of each patch to define local hermitian derivatives. We adopt the one-sided approximate derivative

$$u'(x) = \delta^+_x u_j - \frac{1}{2} (\delta^+_x)^2 u_j + \frac{1}{3} (\delta^+_x)^3 u_j - \frac{1}{4} (\delta^+_x)^4 u_j + O(h^4),$$  \hspace{1cm} (27)

where the forward difference operator $\delta^+_x$ is

$$\delta^+_x u_j = \frac{u_{j+1} - u_j}{h}. $$  \hspace{1cm} (28)

The fourth-order approximate derivative is

$$u_{x,i} = \frac{1}{12h} (-25u_i + 48u_{i+1} - 36u_{i+2} + 16u_{i+3} - 3u_{i+4} )$$  \hspace{1cm} (29)
Discrete hermitian Laplacian

The discrete Laplacian is

$$\Delta_h u^k_{i,j} = \frac{1}{\sqrt{G}} \left( (u_{1,h})^k_{\xi,i,j} + (u_{2,h})^k_{\eta,i,j} \right).$$

This is a formula dependent on the coordinate system $(\xi, \eta)$ on each patch.
The spherical harmonic with index \((n, m)\), \(-n \leq m \leq n\), \(0 \leq n\) is defined in spherical coordinates \((\theta, \lambda)\) by

\[
f^m_n(x) = \bar{P}^{|m|}_n(\sin \theta)e^{im\lambda}.
\] (31)

The function \(\bar{P}^{|m|}_n(z)\) is the associated Legendre polynomial of order \(n, m\) with standard normalization given by

\[
\int_{-1}^{1} \bar{P}^{|m|}_n(x)^2 \, dx = 1.
\] (32)

Main property: the spherical harmonics are the eigenfunctions of the spherical Laplacian.
Eigenfunction numerical test

The eigenvalue associated with $f^m_n(x)$ is

$$\lambda_n = -n(n + 1).$$

(33)

The test of accuracy of the hermitian Laplacian is

$$e(f^m_n) = \max_{-M \leq i, j \leq M, I \leq k \leq VI} \left| \Delta_{h,s} f^m_n - \lambda_n f^m_n \right|,$$

(34)
### Numerical results

Table: Convergence rate of the Laplacian of several spherical harmonics functions. The reported values are given in (34).

<table>
<thead>
<tr>
<th></th>
<th>N=8</th>
<th>rate</th>
<th>N=16</th>
<th>rate</th>
<th>N=32</th>
<th>rate</th>
<th>N=64</th>
<th>rate</th>
<th>N=128</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e(f^1_1)$</td>
<td>8.6508(-3)</td>
<td>3.74</td>
<td>6.4381(-4)</td>
<td>4.18</td>
<td>3.5512(-5)</td>
<td>4.34</td>
<td>1.7496(-6)</td>
<td>3.35</td>
<td>1.7066(-7)</td>
<td></td>
</tr>
<tr>
<td>$e(f^2_3)$</td>
<td>2.3477(-1)</td>
<td>3.70</td>
<td>1.8073(-2)</td>
<td>4.06</td>
<td>1.0805(-3)</td>
<td>3.68</td>
<td>8.3954(-5)</td>
<td>3.51</td>
<td>7.3302(-6)</td>
<td></td>
</tr>
<tr>
<td>$e(f^4_8)$</td>
<td>1.9702(1)</td>
<td>3.54</td>
<td>1.6991(0)</td>
<td>3.24</td>
<td>1.7884(-1)</td>
<td>3.98</td>
<td>1.1325(-2)</td>
<td>4.00</td>
<td>7.0537(-4)</td>
<td></td>
</tr>
<tr>
<td>$e(f^9_{14})$</td>
<td>2.0246(2)</td>
<td>2.92</td>
<td>2.6661(1)</td>
<td>3.06</td>
<td>3.1909</td>
<td>4.00</td>
<td>1.9809(-1)</td>
<td>3.96</td>
<td>1.2696(-2)</td>
<td></td>
</tr>
<tr>
<td>$\Delta \xi_{\text{max}}$ in km</td>
<td>1250</td>
<td>625</td>
<td>312.5</td>
<td>156.25</td>
<td>78.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nb of grid points</td>
<td>386</td>
<td>1538</td>
<td>6146</td>
<td>24578</td>
<td>93306</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Further observations and future work

Higher-order accuracy

Higher-order accuracy achievable using higher-order Hermitian derivatives.

Fast computing issues

Using fast computing tools such as local FFT seems attractive. Useful for implicit time-stepping schemes.

Wave problems on the Cubed-Sphere

Centered fourth-order finite-difference schemes for wave problems on the spherical earth. Short term objective: Laplace Tidal Equations.

High-order accurate spherical gradients for the 4th-order MUSCL scheme

Having at hand high-order gradients, hessians, etc, is a crucial issue for the design of stable fourth-order MUSCL method for the full SW equations on the sphere.
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Some perspectives

Models

- Investigation of complex linear wave problems on the sphere (atmosphere and ocean).
- Numerical computation of eigenmodes of linear operators of interest in climatology/oceanography.

Scientific computing

- Essential issue in climatology: local grid refinement on the cubed-sphere.
- Parallel implementation of the great circles approach seems attractive.

Numerical analysis

Prove some convergence results of the great circles approach to compute the spherical gradient.
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