HERMITIAN COMPACT SCHEMES FOR THE NAVIER-STOKES EQUATIONS

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Joint work with
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1. The Pure Streamfunction Formulation of the Navier-Stokes equations
2. Compact finite-difference schemes for biharmonic problems
3. Fast resolution procedure
4. Compact finite-difference schemes for the Navier-Stokes equation
Navier-Stokes equations in 2D

**Velocity-pressure formulation:**

Find $u(x, t) \in \mathbb{R}^2$, $p(x, t) \in \mathbb{R}$ solutions of

\[
\begin{aligned}
    u_t + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0, \quad x \in \Omega \subset \mathbb{R}^2, \quad t > 0 \\
    \text{div} \, u &= 0, \quad x \in \Omega, \quad t > 0 \\
    u &= 0, \quad x \in \partial \Omega, \quad t > 0 \\
    u(x, 0) &= u_0(x), \quad x \in \Omega
\end{aligned}
\]

\((NS)\)

**Streamfunction formulation:**

$u = (-\psi_y, \psi_x) = \nabla \perp \psi$, $\nabla \wedge u = \Delta \psi$ The streamfunction $\psi$ evolves according to

\[
\partial_t (\Delta \psi) + (\nabla \perp \psi) \cdot \nabla (\Delta \psi) - \nu \Delta^2 \psi = 0, \quad x \in \Omega, \quad t > 0
\]

(Landau-Lifschitz, Fluid Dynamics).

The boundary conditions are given for all points $(x, y) \in \partial \Omega$,

\[
\begin{cases}
    \psi(x, y, t) = 0 \quad \text{no-leak condition + gauge condition} \\
    \frac{\partial \psi}{\partial n}(x, y, t) = 0 \quad \text{tangential velocity given}
\end{cases}
\]

Initial data: $\psi_0(x, y) = \psi(x, y, t)|_{t=0}$, $(x, y) \in \Omega$. 

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Hermitian Compact Schemes for the Navier-Stokes Equations
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(1)

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### Navier-Stokes equations in 2D

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\end{align*}
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  (Landau-Lifschitz, Fluid Dynamics).

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Suppose given \((u_i)_{i\in\mathbb{Z}}\). The hermitian derivative is \((u_{x,i})_{i\in\mathbb{Z}}\) given by

\[
\frac{1}{6} u_{x,i-1} + \frac{2}{3} u_{x,i} + \frac{1}{6} u_{x,i+1} = \frac{u_{i+1} - u_{i-1}}{2h}, \quad i \in \mathbb{Z}
\]  

(2)

Finite Difference form

Can be rewritten as

\[
\sigma_x u_{x,i} = \delta_x u_i, \quad i \in \mathbb{Z}
\]  

(3)

where \(\sigma_x, \delta_x\) are

\[
\sigma_x u_i = \frac{1}{6} u_{i-1} + \frac{2}{3} u_i + \frac{1}{6} u_{i+1}, \quad \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h}
\]  

(4)

Fourth order accuracy

\[
u_{x,i} = u'(x_i) + O(h^4)
\]  

(5)

Connection to cubic splines

\[
u_{x,i} = u'_s(x_i)
\]  

(6)

where \(u_s(x)\) is the cubic spline approximation to \(u(x)\).
Hermitian Derivative Operator

**Definition**

Suppose given \((u_i)_{i \in \mathbb{Z}}\). The hermitian derivative is \((u_x,i)_{i \in \mathbb{Z}}\) given by

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u_{x,i} = u'(x_i) + O(h^4)
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\[
u_{x,i} = u_s'(x_i)
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where \(u_s(x)\) is the cubic spline approximation to \(u(x)\).
Three-Point Biharmonic Operator

Definition

Suppose given \((u_i)_{i \in \mathbb{Z}}\) and \((u_{x,i})_{i \in \mathbb{Z}}\) the corresponding hermitian derivative. The Three-Point Biharmonic \((\delta^4_x u_i)_{i \in \mathbb{Z}}\) is \((\delta^2_x u_i = (u_{i+1} + u_{i-1} - 2u_i)/h^2)\),

\[
\delta^4_x u_i = \frac{12}{h^2} \left( \delta_x u_{x,i} - \delta^2_x u_i \right)
\]

(7)

Fourth order accuracy

\[
\delta^4_x u_i = u^{(4)}(x_i) + O(h^4)
\]

(8)

Connection to cubic splines

Denote by \(u_s(x)\) the cubic spline interpolation of the data \((u_i)_{0 \leq i \leq N}\) with endpoints derivatives \(u_{x,0}, u_{x,N}\). For gridfunctions \((u_i)_{0 \leq i \leq N}, (v_i)_{0 \leq i \leq N}\) with \(u_0 = u_N = v_0 = v_N = 0\),

\[
(\delta^4_x u, v)_h = \int_0^1 u''_s(x)v''_s(x)dx
\]

(9)

where \((u, v)_h = h \sum_{i=1}^{N-1} u_i v_i\).
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A compact scheme for the biharmonic problem in 1D

One-dimensional biharmonic problem

Solve on $I = [0, 1]$

$$\begin{align*}
\delta^4_x u(x) &= f(x), \quad 0 < x < 1 \\
u(0) &= u'(0) = u(1) = u'(1) = 0
\end{align*} \quad (10)$$

Compact scheme

The approximate problem is: find $u = [u_0, u_1, \ldots, u_{N-1}, u_N]$ solution of

$$\begin{align*}
\delta^4_x u_j &= \frac{12}{h^2} \left( \delta_x u_{x,j} - \delta^2_x u_j \right) = f(x_j), \quad 1 \leq j \leq N - 1 \\
\frac{1}{6} u_{x,j-1} + \frac{2}{3} u_{x,j} + \frac{1}{6} u_{x,j+1} &= \delta_x u_j, \quad 1 \leq j \leq N - 1 \\
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\] (11)
**Theorem**

Let \( \tilde{u} \) be the approximate solution of the biharmonic problem \( u^{(4)}(x) = f(x) \) with Dirichlet B.C. . Let \( u(x) \) be the exact solution and \( u^* \) its evaluation at grid points. The error \( e = \tilde{u} - u^* = [u_1, \cdots, u_{N-1}] \) satisfies

\[
|e|_h \leq C h^4
\]  

(12)

where \( C \) depends only on \( f \).

**Proof**

Not straightforward result, due to the boundary conditions ! Method of proof: careful analysis of the structure of the matrix of \( \delta^4_x \) on a bounded domain \([0, \cdots, N]\).

**Accuracy**

The pointwise truncation error on a bounded domain cannot be deduced from the fourth order accuracy in the "free" space. Here the pointwise truncation of \( \delta^4_x \) is 1 at \( i = 1, 2, \cdots, N - 1 \).

**Energy method**

Energy method (as in FEM) provide only a suboptimal error estimate.
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Properties of $\delta^4_x$

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The nine-point Biharmonic Operator for the 2D bih. problem

A compact Biharmonic operator

Biharmonic operator:
\[
\Delta^2 \psi = \partial_x^4 \psi + \partial_y^4 \psi + 2 \partial_x^2 \partial_y^2 \psi
\]  \hspace{1cm} (13)

Approximation by:
\[
\Delta_h^2 \psi_{i,j} = \delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j} + 2 \delta_x^2 \delta_y^2 \psi_{i,j}
\]  \hspace{1cm} (14)

where the discrete gradient \( \nabla_h \psi = (\psi_{x,i,j}, \psi_{y,x,y}) \) is defined by the hermitian relations

\[
\begin{align*}
\frac{1}{6} \psi_{x,i-1,j} + \frac{2}{3} \psi_{x,i,j} + \frac{1}{6} \psi_{x,i+1,j} &= \delta_x \psi_{i,j}, \quad 1 \leq i \leq N - 1 \\
\frac{1}{6} \psi_{y,i,j-1} + \frac{2}{3} \psi_{y,i,j} + \frac{1}{6} \psi_{y,i,j+1} &= \delta_y \psi_{i,j}, \quad 1 \leq j \leq N - 1
\end{align*}
\]  \hspace{1cm} (15)

Stephenson Biharmonic

This operator is the same than the one introduced by J.W. Stephenson (Jour. Comp. Phys. 1984).

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\frac{1}{6} \psi_{y,i,j-1} + \frac{2}{3} \psi_{y,i,j} + \frac{1}{6} \psi_{y,i,j+1} &= \delta_y \psi_{i,j} , \quad 1 \leq j \leq N - 1
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Stephenson scheme for the 2D Biharmonic Problem

Continuous Biharmonic problem

\[
\begin{cases}
\Delta^2 \psi(x, y) = f(x, y), 
& (x, y) \in \Omega \\
\psi(x, y) = g_1(x, y), 
& (x, y) \in \partial \Omega \\
\frac{\partial \psi}{\partial n}(x, y) = g_2(x, y), 
& (x, y) \in \partial \Omega
\end{cases}
\]

Discrete Biharmonic problem in a square

Solve the system in \( \psi_{i,j} \), \( 0 \leq i, j \leq N \)

\[
\Delta_h^2 \psi_{i,j} = f^*(x_i, y_j), 
& 1 \leq i, j \leq N - 1
\]

subject to the boundary conditions

\[
\begin{cases}
\psi_{i,j} = g_1^*(x_i, y_j), 
& \{i = 0, N, \ 0 \leq j \leq N\} \quad \text{or} \quad \{j = 0, N, \ 0 \leq i \leq N\}, \\
\psi_{x,i,j} = -g_2^*(x_i, y_j), 
& i = 0, \ 0 \leq j \leq N, \\
\psi_{x,i,j} = g_2^*(x_i, y_j), 
& i = N, \ 0 \leq j \leq N, \\
\psi_{y,i,j} = -g_2^*(x_i, y_j), 
& j = 0, \ 0 \leq i \leq N, \\
\psi_{y,i,j} = g_2^*(x_i, y_j), 
& j = N, \ 0 \leq i \leq N.
\end{cases}
\]
Stephenson scheme for the 2D Biharmonic Problem

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\left\{ \begin{array}{l}
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\frac{\partial \psi}{\partial n}(x, y) = g_2(x, y), \quad (x, y) \in \partial \Omega 
\end{array} \right.
\]

(16)

Discrete Biharmonic problem in a square

Solve the system in \( \psi_{i,j} \), \( 0 \leq i, j \leq N \)

\[
\Delta^2_{h} \psi_{i,j} = f^*(x_i, y_j), \quad 1 \leq i, j \leq N - 1
\]

(17)

subject to the boundary conditions

\[
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\psi_{x,i,j} = g_2^*(x_i, y_j), \quad i = N, \quad 0 \leq j \leq N, \\
\psi_{y,i,j} = -g_2^*(x_i, y_j), \quad j = 0, \quad 0 \leq i \leq N, \\
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\end{array} \right.
\]

(18)
Stencil of the nine-point Bih. operator
Properties of the Stephenson scheme for the 2D Bih. Problem

No artificial BC on the vorticity $\Delta \psi$

Only the natural BC on $\psi$ are required by the scheme. In the Dirichlet case, it is $\psi, \frac{\partial \psi}{\partial n}$.

Second order accuracy

The operator $\Delta^2_h$ is second order accurate. The one-dimensional operators $\delta^4_x \psi, \delta^4_y \psi$ are 4th order accurate (in the “free” setting). The second order accuracy is due only to the mixed term $\delta^2_x \delta^2_y \psi$. 
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Matrix operator of $\delta^2_x$ and $\delta^4_x$

Matrix operators

One has $-\delta^2_x = T/h^2$ with

$$T = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & -1 & 2 & -1 \\
0 & \ldots & 0 & -1 & 2
\end{bmatrix} \in \mathbb{M}_N^{-1}(\mathbb{R}) \quad (19)$$

The symmetric positive definite matrix $P$ is deduced from $T$ by

$$P = 6I - T, \quad (20)$$

The nine-point Biharmonic

$$\Delta^2_h = \frac{1}{h^4} \left[ 6P^{-1}T^2 \otimes I + 6I \otimes P^{-1}T^2 + 2T \otimes T \right]$$

$$+ \frac{36}{h^4} \left[ v_1, v_2 \right] \left[ \begin{array}{c} v_1^T \\ v_2^T \end{array} \right] \otimes I_N^{-1} + \frac{36}{h^4} I_N^{-1} \otimes \left[ v_1, v_2 \right] \left[ \begin{array}{c} v_1^T \\ v_2^T \end{array} \right]. \quad (21)$$

$$\left\{ \begin{array}{l}
v_1 = (\alpha - \beta)^{1/2} P^{-1} \left( \frac{\sqrt{2}}{2} e_1 - \frac{\sqrt{2}}{2} e_{N-1} \right) \in \mathbb{R}^{N-1} \\
v_2 = (\alpha + \beta)^{1/2} P^{-1} \left( \frac{\sqrt{2}}{2} e_1 + \frac{\sqrt{2}}{2} e_{N-1} \right) \in \mathbb{R}^{N-1}
\end{array} \right. \quad (22)$$
Matrix operator of \( \delta^2_x \) and \( \delta^4_x \)

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\end{cases}
\] (22)
Shermann-Morrison formula

The matrix of $\Delta_2^2 h$ is a low-rank perturbation (due to the BC) of a diagonal operator (in a spectral basis), which represents the biharmonic in the “free space”:

$$A = B + \frac{36}{h^4} RR^T,$$

(23)

The Sherman-Morrison formula gives

$$\bar{A}^{-1} = \bar{B}^{-1} - 36B^{-1}R\left[I_4(N-1) + 36R^TB^{-1}R\right]^{-1}R^T\bar{B}^{-1}.$$

(24)

Fast resolution procedure

A fast solver is deduced in 8 steps. The key steps are:

- Using the FFT to compute $BU = F$ (system in $\mathbb{R}^{(N-1)^2}$).
- Using the PCG to solve

$$\left(I_4(N-1) + 36R^TB^{-1}R\right)V = G,$$  

(system in $\mathbb{R}^{4(N-1)}$).

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Fast solver

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(25)
A fourth order biharmonic operator

Fourth order Biharmonic

It is possible to modify the mixed term in the Stephenson operator to obtain a 4th order accurate scheme. Simply replace $\delta_x^2 \delta_y^2 u$ by

$$\tilde{\delta}_x^2 \delta_y^2 \psi_{i,j} = 3 \delta_x^2 \delta_y^2 \psi_{i,j} - \delta_x^2 \delta_y \psi_{y,i,j} - \delta_y^2 \delta_x \psi_{x,i,j} = \partial_x^2 \partial_y^2 \psi_{i,j} + O(h^4). \quad (26)$$

Fast solver for the fourth order Biharmonic

The fast solver follows the same principle than for the second order Biharmonic
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Fast solver for the fourth order Biharmonic

The fast solver follows the same principle than for the second order Biharmonic
## Computing efficiency

### Table: Indicative CPU time on a Laptop

<table>
<thead>
<tr>
<th>N</th>
<th>N=128</th>
<th>N=256</th>
<th>N=512</th>
<th>N=1024</th>
<th>N=2048</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU&lt;sub&gt;tot&lt;/sub&gt;</td>
<td>0.11s</td>
<td>0.45s</td>
<td>1.84s</td>
<td>7.91s</td>
<td>34.63s</td>
</tr>
<tr>
<td>CPU&lt;sub&gt;∞&lt;/sub&gt;</td>
<td>0.093s</td>
<td>0.39s</td>
<td>1.47s</td>
<td>6.46s</td>
<td>27.72s</td>
</tr>
<tr>
<td>CPU&lt;sub&gt;tot&lt;/sub&gt; / (N&lt;sup&gt;2&lt;/sup&gt; Log(N))</td>
<td>1.37(-6)</td>
<td>1.24(-6)</td>
<td>1.16(-6)</td>
<td>1.09(-6)</td>
<td>1.07(-6)</td>
</tr>
</tbody>
</table>
Fourth order accuracy for $\psi$, $\nabla \psi$, $\Delta \psi$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$| \psi - \psi_h |_{\infty, h}$</th>
<th>$| \psi_x - \psi_{x,h} |_{\infty, h}$</th>
<th>$| \psi - \psi_{y,h} |_{\infty, h}$</th>
<th>$| \Delta \psi - \Delta h \psi_{h} |_{\infty, h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 16$</td>
<td>3.42(-5)</td>
<td>1.00(-4)</td>
<td>1.00(-4)</td>
<td>3.99(-4)</td>
</tr>
<tr>
<td>conv. rate</td>
<td>4.04</td>
<td>4.01</td>
<td>4.01</td>
<td>4.00</td>
</tr>
<tr>
<td>$N = 32$</td>
<td>2.08(-6)</td>
<td>6.21(-6)</td>
<td>6.21(-6)</td>
<td>2.48(-5)</td>
</tr>
<tr>
<td>conv. rate</td>
<td>4.01</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td>$N = 64$</td>
<td>1.29(-7)</td>
<td>3.87(-7)</td>
<td>3.87(-7)</td>
<td>1.55(-6)</td>
</tr>
<tr>
<td>conv. rate</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td>$N = 128$</td>
<td>8.06(-9)</td>
<td>2.41(-8)</td>
<td>2.41(-8)</td>
<td>9.68(-8)</td>
</tr>
<tr>
<td>conv. rate</td>
<td>3.99</td>
<td>3.99</td>
<td>3.99</td>
<td>3.83</td>
</tr>
<tr>
<td>$N = 256$</td>
<td>5.04(-10)</td>
<td>1.51(-9)</td>
<td>1.51(-9)</td>
<td>6.77(-9)</td>
</tr>
<tr>
<td>conv. rate</td>
<td>3.74</td>
<td>4.02</td>
<td>4.02</td>
<td>-0.22</td>
</tr>
<tr>
<td>$N = 512$</td>
<td>3.76(-11)</td>
<td>9.27(-11)</td>
<td>9.07(-11)</td>
<td>7.90(-9)</td>
</tr>
<tr>
<td>conv. rate</td>
<td>-0.13</td>
<td>0.19</td>
<td>0.19</td>
<td>0.59</td>
</tr>
<tr>
<td>$N = 1024$</td>
<td>4.12(-11)</td>
<td>8.09(-11)</td>
<td>8.09(-11)</td>
<td>5.22(-8)</td>
</tr>
</tbody>
</table>

Table: Error and convergence rate for Test Case 1 with the fourth order scheme
Second order scheme for the Navier-Stokes equation

Navier-Stokes equation in streamfunction

\[
\partial_t \Delta \psi + (\nabla \perp \psi) \cdot \nabla (\Delta \psi) \rightleftharpoons \nu \Delta^2 \psi = 0 \quad \text{, } x \in \Omega \quad \text{, } t > 0
\] (27)

+ Dirichlet B.C on $\psi$.

Approximation in space (method of lines)

\[
\psi(x_i, y_j, t) \rightleftharpoons \tilde{\psi}_{i,j}(t), \text{ solution of }
\]

\[
\partial_t \Delta_h \tilde{\psi}_{i,j} - \tilde{\psi}_{y,i,j} \Delta_h \tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j} \Delta_h \tilde{\psi}_{y,i,j} - \nu \Delta^2_h \tilde{\psi}_{i,j} = 0 \quad \text{, } x \in \Omega \quad \text{, } t > 0
\] (28)

+ Dirichlet B.C on $\tilde{\psi}_{i,j}, \tilde{\psi}_{x,i,j}, \tilde{\psi}_{y,i,j}$.

Fully centered second order scheme

The operator in space are just translated on the discrete grid using:

- Second order Laplacian, second order Biharmonic Five-point Laplacian:

\[
\Delta \psi(x_i, y_j) \rightleftharpoons \Delta_h \tilde{\psi}_{i,j}, \quad \Delta^2 \psi(x_i, y_j) \rightleftharpoons \Delta^2_h \tilde{\psi}_{i,j}
\] (29)

- Second order convective term

\[
(\nabla \perp \psi(x_i, y_j)) \cdot \nabla (\Delta \psi(x_i, y_j)) \rightleftharpoons -\tilde{\psi}_{y,i,j} \Delta_h \tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j} \Delta_h \tilde{\psi}_{y,i,j}
\] (30)
Second order scheme for the Navier-Stokes equation

Navier-Stokes equation in streamfunction

\[ \partial_t \Delta \psi + (\nabla^\perp \psi) \cdot \nabla (\Delta \psi) - \nu \Delta^2 \psi = 0 \ , \ x \in \Omega \ , \ t > 0 \]  \hspace{1cm} (27)

+ Dirichlet B.C on \( \psi \).

Approximation in space (method of lines)

\( \psi(x_i, y_j, t) \approx \tilde{\psi}_{i,j}(t) \), solution of

\[ \partial_t \Delta_h \tilde{\psi}_{i,j} - \tilde{\psi}_{y,i,j} \Delta_h \tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j} \Delta_h \tilde{\psi}_{y,i,j} - \nu \Delta_h^2 \tilde{\psi}_{i,j} = 0 \ , \ x \in \Omega \ , \ t > 0 \]  \hspace{1cm} (28)

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- Second order convective term
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\[ \partial_t \Delta \psi + (\nabla^\perp \psi) \cdot \nabla (\Delta \psi) - \nu \Delta^2 \psi = 0, \quad x \in \Omega, \quad t > 0 \]  
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Approximation in space (method of lines)

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Navier-Stokes equation in streamfunction

\[ \partial_t \Delta \psi + (\nabla^\perp \psi) \cdot \nabla (\Delta \psi) - \nu \Delta^2 \psi = 0, \quad x \in \Omega, \quad t > 0 \]  \hspace{1cm} (27)

+ Dirichlet B.C on \( \psi \).

Approximation in space (method of lines)

\[ \psi(x_i, y_j, t) \simeq \tilde{\psi}_{i,j}(t), \text{ solution of} \]

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Numerical analysis

**Theorem**

Let $T > 0$. Then there exist constants $C, h_0 > 0$, depending possibly on $T, \nu$ and on the exact solution $\psi$, such that, for all $0 \leq t \leq T$,

$$
|\delta_x^+ (\psi(t) - \tilde{\psi}(t))|_h^2 + |\delta_y^+ (\psi(t) - \tilde{\psi}(t))|_h^2 \leq C h^3, \quad 0 < h \leq h_0
$$

(31)

where $\psi(t) = \psi_{i,j}(t)$ is the pointwise interpolated exact solution and $\tilde{\psi}_{i,j}(t)$ is the solution of the semidiscrete scheme.

**Properties**

- Second order centered approximation (no upwinding).
- No need of boundary conditions on the vorticity and no uncontrolled pressure modes.
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Properties

- Second order centered approximation (no upwinding).
- No need of boundary conditions on the vorticity and no uncontrolled pressure modes.
Fourth order scheme for the Navier-Stokes equation

Centered fourth order scheme

The operators in space are just translated from the continuous ones on the discrete grid using:

- Fourth order Laplacian, fourth order Biharmonic

\[
\begin{align*}
\Delta \psi(x_i, y_j) &\approx \Delta_h \psi_{i,j} - \frac{h^2}{12} (\delta^4_x \psi_{i,j} + \delta^4_y \psi_{i,j}) \\
\Delta^2 \psi(x_i, y_j) &\approx \Delta^2_h \psi - \delta^4_x \left( I - \frac{h^2}{6} \delta^2_y \right) \psi_{i,j} + \delta^4_y \left( I - \frac{h^2}{6} \delta^2_x \right) \psi_{i,j} + 2\delta^2_x \delta^2_y \psi_{i,j}
\end{align*}
\]

- Fourth order convective term

\[
(\nabla^\perp \psi(x_i, y_j)) \cdot \nabla (\Delta \psi(x_i, y_j)) \approx -\psi_{y,i,j} \Delta_h \psi_{x,i,j} + \psi_{x,i,j} \Delta_h \psi_{y,i,j} - \frac{h^2}{12} \left( -\delta_x (\psi_{y,i,j} (\delta^4_x \psi_{i,j} + \delta^4_y \psi_{i,j})) \right) + \delta_y (\psi_{x,i,j} (\delta^4_x \psi_{i,j} + \delta^4_y \psi_{i,j}))
\]

Jean-Pierre CROISILLE - Univ. Metz, France

Hermitian Compact Schemes for the Navier-Stokes Equations
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\[
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\]
\[
\left. + \delta_y (\psi_{x,i,j} (\delta^4_x \psi_{i,j} + \delta^4_y \psi_{i,j})) \right)
\]
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\[
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\Delta^2 \psi(x_i, y_j) & \simeq \Delta^2_h \psi - \delta_x^4 \left( I - \frac{h^2}{6} \delta_y^2 \right) \psi_{i,j} + \delta_y^4 \left( I - \frac{h^2}{6} \delta_x^2 \right) \psi_{i,j} + 2 \delta_x^2 \delta_y^2 \psi_{i,j}
\end{align*}
\]

- Fourth order convective term

\[
(\nabla \perp \psi(x_i, y_j)) \cdot \nabla (\Delta \psi(x_i, y_j)) \simeq -\psi_{y,i,j} \Delta_h \psi_{x,i,j} + \psi_{x,i,j} \Delta_h \psi_{y,i,j} - \frac{h^2}{12} \left( -\delta_x \left( \psi_{y,i,j} \left( \delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j} \right) \right) \right) + \delta_y \left( \psi_{x,i,j} \left( \delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j} \right) \right)
\]
A high order IMEX time-scheme (Spalart-Moser-Rogers)

Algorithm: 3 biharmonic solving by time-step

\[
\begin{align*}
U &= \tilde{\Delta}_h \psi \\
D &= \nu \tilde{\Delta}^2_h (\psi) \\
C &= \tilde{C}_h (\psi),
\end{align*}
\] (33)

The scheme is

\[
\begin{align*}
U^1 &= \tilde{\Delta}_h \psi^n \\
U^2 &= U^1 + \Delta t \left( \gamma_1 (-C^1_h) + \alpha_1 D^1_h + \beta_1 D^2_h \right) + \frac{8}{15} \Delta t F^{n+4/15} \\
U^3 &= U^2 + \Delta t \left( \gamma_2 (-C^2_h) + \alpha_2 D^2_h + \beta_2 D^3_h \right) + \Delta t \left( \frac{2}{3} F^{n+1/3} - \frac{8}{15} F^{n+4/15} \right) \\
U^4 &= U^3 + \Delta t \left( \gamma_3 (-C^3_h) + \alpha_3 D^3_h + \beta_3 D^4_h \right) + \Delta t \left( \frac{1}{6} F^n + \frac{2}{3} F^{n+1/2} + \frac{1}{6} F^{n+1} - \frac{2}{3} F^{n+1/3} \right).
\end{align*}
\] (34)

The values of the parameters are

\[
\begin{align*}
\alpha_1 &= \frac{29}{96} & \alpha_2 &= \frac{-3}{40} & \alpha_3 &= \frac{1}{6} \\
\beta_1 &= \frac{160}{5} & \beta_2 &= \frac{-24}{3} & \beta_3 &= \frac{-1}{3} \\
\gamma_1 &= \frac{15}{17} & \gamma_2 &= \frac{12}{5} & \gamma_3 &= \frac{-4}{3} \\
\zeta_1 &= \frac{17}{15} & \zeta_2 &= \frac{-5}{3} & \zeta_3 &= \frac{-12}{5}.
\end{align*}
\] (35)
Assessing the fourth order accuracy

\( e = \) absolute error for \( \psi \), \( e_r = \), relative error for \( \psi_x \), \( e_x = \) absolute error for \( \psi_x \).

<table>
<thead>
<tr>
<th>mesh</th>
<th>9 × 9</th>
<th>Rate</th>
<th>17 × 17</th>
<th>Rate</th>
<th>33 × 33</th>
<th>Rate</th>
<th>65 × 65</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0.25 )</td>
<td>( e )</td>
<td>5.0867(-3)</td>
<td>4.06</td>
<td>3.0525(-4)</td>
<td>4.02</td>
<td>1.8835(-5)</td>
<td>4.00</td>
<td>1.1734(9)-6)</td>
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<tr>
<td></td>
<td>( e_r )</td>
<td>9.4936(-3)</td>
<td>4.00</td>
<td>5.7441(-4)</td>
<td>4.00</td>
<td>3.5460(-5)</td>
<td>4.00</td>
<td>2.2092(9)-6)</td>
</tr>
<tr>
<td></td>
<td>( e_x )</td>
<td>2.6390(-3)</td>
<td>3.89</td>
<td>1.7837(-4)</td>
<td>3.93</td>
<td>1.1670(-5)</td>
<td>3.98</td>
<td>7.3752(9)-7)</td>
</tr>
<tr>
<td>( t = 0.5 )</td>
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<td>3.2224(-3)</td>
<td>4.00</td>
<td>2.0085(-4)</td>
<td>4.00</td>
<td>1.2541(-5)</td>
<td>4.00</td>
<td>7.8361(9)-7)</td>
</tr>
<tr>
<td></td>
<td>( e_r )</td>
<td>7.7407(-3)</td>
<td>4.00</td>
<td>4.8536(-4)</td>
<td>4.00</td>
<td>3.0317(-5)</td>
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<td>1.8944(9)-6)</td>
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<tr>
<td></td>
<td>( e_x )</td>
<td>3.2285(-3)</td>
<td>4.02</td>
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<td>4.00</td>
<td>7.7745(9)-7)</td>
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<tr>
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<td>4.00</td>
<td>9.6887(-6)</td>
<td>4.00</td>
<td>6.0551(9)-7)</td>
</tr>
<tr>
<td></td>
<td>( e_r )</td>
<td>7.6730(-3)</td>
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<td>4.8119(-4)</td>
<td>4.00</td>
<td>3.0075(-5)</td>
<td>4.00</td>
<td>1.8796(9)-6)</td>
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<tr>
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<tr>
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<tr>
<td></td>
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<td>4.8103(-4)</td>
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<tr>
<td></td>
<td>( e_x )</td>
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<td>1.2255(-4)</td>
<td>4.00</td>
<td>7.6526(-6)</td>
<td>4.00</td>
<td>4.7826(9)-7)</td>
</tr>
</tbody>
</table>

Table 1: Compact scheme for Navier-Stokes with exact solution:

\( \psi = (1 - x^2)^3(1 - y^2)^3e^{-t} \) on \([-1, 1] \times [-1, 1]\). We represent \( e_\psi \) the \( l_2 \) error for the streamfunction and \( e_x \) the max error in the \( U^x \) velocity \( = -\partial_y \psi \). \( \Delta t = Ch^2 \).
Max $\psi$ behaviour at $Re = 10000$

Figure: Driven Cavity for $Re = 10000$ : Max streamfunction. Computations are done with $N = 65$, with $\Delta t = 1/90$. 

Jean-Pierre CROISILLE - Univ. Metz, France
Hermitian Compact Schemes for the Navier-Stokes Equations
Figure: Driven Cavity for $Re = 7500, 10000$ : Streamfunction Contours with the fourth-order scheme
Velocity in the middle of the cavity, $Re = 7500$, $Re = 10000$

Figure: Velocity components for the driven cavity problem. Left: $Re = 7500$, fourth-order scheme with $N = 65$ (solid line), Ghia-Ghia-Shin with $N = 257$ (circles). Right: $Re = 10000$ fourth-order scheme with $N = 65$ (solid line), Ghia-Ghia-Shin with $N = 257$ (circles).
Computing efficiency for NS (driven cavity)

<table>
<thead>
<tr>
<th>( \mathcal{N} = 65 ), ( \text{Re} = 1000 )</th>
<th>( \mathcal{N} = 129 ), ( \text{Re} = 1000 )</th>
<th>( \mathcal{N} = 256 ), ( \text{Re} = 5000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8000 it., ( \Delta t = 1/60 )</td>
<td>12000 it., ( \Delta t = 1/60 )</td>
<td>50000 it., ( \Delta t = 1/180 )</td>
</tr>
<tr>
<td>4 min (0.03 sec/it.)</td>
<td>23 min 30 sec. (0.11 sec/it.)</td>
<td>7 h 50 min. (0.56 sec/it.)</td>
</tr>
</tbody>
</table>

**Table:** Indicative CPU time for the driven cavity on a Laptop
Implementation so far

Outline

- Fourth order scheme with fast solver in $O(N^2 \ln_2(N))$. Fortran90 code.
- Driven cavity computations up to $Re = 10000$, beyond the first Hopf bifurcation.
- Numerical analysis
- Derivation and first implementation of the 3D NS equations in streamfunction formulation in a cube
- Design and tests of a cartesian embedded biharmonic scheme for irregular geometries
- Application to other models involving biharmonic equations (e.g. image processing).
Spectral analysis of fourth order problems. Application to the Stokes modes in a square/cube.

Still enhance the fast solver (also in 3D)

Other applications of fourth order problems solving: HJ (Hamilton-Jacobi), KS (Kuramoto-Sivashinsky), MEMS (Micro-Electro-Mechanical Systems), Image processing.

Driven cavity in a cube.

Irregular geometries on cartesian grids using embedded/immersed boundaries seem tractable.