High energy estimates of the resolvant

Joint work with Thierry Jecko

Sylvain Golénia

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Let *H* be a self-adjoint operator acting on a complex and separable Hilbert space $(\mathscr{H}, \|\cdot\|)$.

We consider *the Schrödinger equation*

$$\begin{cases} \mathrm{i}\partial_t f = Hf, \\ f(0) = f_0 \in \mathscr{H}, \end{cases}$$

where $\|f_0\| = 1$.

Let $f(t) = e^{-\mathrm{i}tH} f_0$ be the unique solution.

If f_0 is an eigenfunction associated to λ_0, we have : $Hf_0 = \lambda_0 f_0$ and

$$f(t) = e^{-itH} f_0 = e^{-it\lambda_0} f_0.$$

The solution is periodic.

If $f_0 \in \mathscr{H}_{pp}(H)^{\perp}$, where $\mathscr{H}_{pp}(H) := \overline{\text{span}\{f \text{ eigenfunction of } H\}}$.

Given K which is relatively compact with respect to H, we have :

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|Ke^{-itH} f_0\|^2 \, dt = 0.$$

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Aim : If $f_0 \in \mathscr{H}_{pp}(H)^{\perp}$, where $\mathscr{H}_{pp}(H) := \overline{\text{span}\{f \text{ fonction propre de } H\}}$.

Given K which is relatively compact with respect to H, we have :

$$\lim_{t \to \infty} K e^{-itH} f_0 = 0.$$

Example :

In $L^2(\mathbb{R}^n;\mathbb{C})$, set

$$H = -\Delta_{\mathbb{R}^n} + V(Q),$$

with $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ and where $V(Q)$ is the operator of multiplication by V.

If $\lim_{|x|\to\infty} V(x) = 0$ and if $f_0 \in \mathscr{H}_{pp}(H)^\perp$, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\mathbf{1}_{B(0,R)}(Q)e^{-itH}f_0\|^2\,dt = 0.$$

If $\lim_{|x|\to\infty} \langle x\rangle^{1+\varepsilon} V(x) = 0$, for some $\varepsilon > 0$, and if $f_0 \in \mathscr{H}_{pp}(H)^\perp$, then

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We measure the *evolution* with respect to the observable A, i.e. a self-adjoint operator in $\mathscr{H}$. Set

$$\mathcal{A}_{\mathrm{f}}(\mathrm{t}) := \langle \mathrm{e}^{-\mathrm{itH}}\mathrm{f}_0, \mathrm{A}\mathrm{e}^{-\mathrm{itH}}\mathrm{f}_0\rangle.$$

We want $\mathcal{A}_{\mathrm{f}}(\cdot)$ to be <u>increasing</u>.

We compute the *Heisenberg derivative* :

$$\mathcal{A}'_{\mathrm{f}}(\mathrm{t}) = \langle \mathrm{e}^{-\mathrm{itH}}\mathrm{f}_0, [\mathrm{H}, \mathrm{iA}]\mathrm{e}^{-\mathrm{itH}}\mathrm{f}_0\rangle.$$

We then suppose that $[\mathrm{H}, \mathrm{iA}]$ is “strictly positive” (the choice of A is crutial). Then there exists $\mathrm{c} > 0$ such that

$$\langle \mathrm{e}^{-\mathrm{itH}}\mathrm{f}_0, \mathrm{A}\mathrm{e}^{-\mathrm{itH}}\mathrm{f}_0\rangle \geq \mathrm{ct} + \langle \mathrm{f}_0, \mathrm{Af}_0\rangle$$

There is some *transport*, the particle *diffuses* along A.

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$$\mathcal{A}_{\mathrm{f}}'(\mathrm{t}) = \langle e^{-\mathrm{itH}} f_0, [H, iA] e^{-\mathrm{itH}} f_0 \rangle.$$

We then suppose that $[H, iA]$ is "strictly positive" (the choice of A is crutial). Then there exists $c > 0$ such that

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There is some *transport*, the particle *diffuses* along A.

Back to the example : Set

$$A := \frac{1}{2i}\left(P \cdot Q + Q \cdot P\right),$$

where $P := -i\nabla$ and Q is the operator of multiplication by x.

We have

$$[-\Delta, iA] = -2\Delta \geq 0.$$

$$E_{\mathcal{I}}(-\Delta)[-\Delta, iA]E_{\mathcal{I}}(-\Delta) = 2(-\Delta)E_{\mathcal{I}}(-\Delta) \geq 2\inf(\mathcal{I})E_{\mathcal{I}}(-\Delta).$$

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For $\mathcal{I} \Subset (0, \infty)$.

For the perturbation, we look for some compactness. We suppose

$$V(x) \to 0 \quad \text{et} \quad x \cdot \nabla V(x) \to 0,$$

for $|x| \to \infty$.

This gives

$$\begin{cases} [V, iA] = -Q \cdot \nabla V(Q) \in \mathcal{K}\big(\mathscr{H}^2, \mathscr{H}\big), \\ \varphi(-\Delta) - \varphi(H) \in \mathcal{K}\big(\mathscr{H}, \mathscr{H}^2\big), \text{ for } \varphi \in \mathcal{C}_c^\infty(\mathbb{R}). \end{cases}$$

We deduce that for all interval $\mathcal{I} \Subset (0, \infty)$, there exist $c > 0$ and K a compact operator, such that

$$E_{\mathcal{I}}(H)[H, iA]E_{\mathcal{I}}(H) \geq \underbrace{c}_{\text{diffusion}} E_{\mathcal{I}}(H) + \underbrace{K}_{\text{e.v.}}.$$

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Let $\mathcal{I}$ be a compact interval and A be a self-adjoint operator. We suppose that a *Mourre estimate* holds true (Mourre '81, Perry-Seagal-Simon '81, . . . Amrein-Boutet de Monvel-Georgescu '96) :

$$\mathrm{E}_{\mathcal{I}}(\mathrm{H})[\mathrm{H}, \mathrm{iA}]\mathrm{E}_{\mathcal{I}}(\mathrm{H}) \geq \underbrace{\mathrm{c}}_{\text{const. } >0} \mathrm{E}_{\mathcal{I}}(\mathrm{H}) + \underbrace{\mathrm{K}}_{\text{compact op.}},$$

in the form sense. We suppose $\mathrm{H} \in \mathcal{C}^2(\mathrm{A})$, i.e. that

$$t \mapsto e^{\mathrm{itA}}(\mathrm{H}+\mathrm{i})^{-1}e^{-\mathrm{itA}}$$

is strongly $\mathcal{C}^k$. Then

a. H has at most a finite number of eigenvalues in $\mathcal{I}$ (Virial Theorem).

b. Let $\mathcal{J} \Subset \mathcal{I}$ compact, such that H has no e.v. in $\mathcal{J}$, then, given K relatively compact with respect to H, we have :

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(LAP) $\sup_{\lambda \in \mathcal{J}, \varepsilon > 0} \left| \langle \mathrm{f}, (\mathrm{H} - \lambda - \mathrm{i}\varepsilon)^{-1}\mathrm{f}\rangle \right| \leq \mathrm{C}\|\langle \mathrm{A}\rangle^{\mathrm{s}}\mathrm{f}\|^2,$

for all $\mathrm{f} \in \mathcal{D}(\langle \mathrm{A}\rangle^{\mathrm{s}})$, $\mathrm{s} > 1/2$.

(Kato) $\displaystyle\int_{\mathbb{R}} \|\langle \mathrm{A}\rangle^{-\mathrm{s}} e^{-\mathrm{itH}}\mathrm{E}_{\mathcal{J}}(\mathrm{H})\mathrm{f}\|^2 \, \mathrm{dt} \leq \mathrm{C}\|\mathrm{f}\|^2.$

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The Mourre theory has been adapted and used by many authors to deal with thresholds (Fournais-Skibsted, Richard, Bony-Häfner, Bony-Faupin, . . .).
We follow the approach of Richard. First we need a different type of positivity.

Back to the example : For $c \in (0, 2)$ and setting $H := -\Delta + V(Q)$, we have

$$[H, iA] - c \times H = (2 - c) \times (-\Delta) - W(Q),$$

where

$$W(Q) := Q \cdot \nabla V(Q) + cV(Q).$$

Assuming also that $W(Q) \geq 0$, we get :

$$[H, iA] - c \times H > 0,$$

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In the abstract setting Richard supposes that there are a self-adjoint operator H and $\mathrm{c} \in (0,2)$ such that

$$[\mathrm{H}, \mathrm{iA}] - \mathrm{cH} > 0.$$

Under some additionnal regularity assumptions, he infers :

$$\sup_{\lambda\in[0,\infty),\varepsilon>0} |\langle \mathrm{f}, (\mathrm{H}-\lambda-\mathrm{i}\varepsilon)\mathrm{f}\rangle| \leq \|\mathrm{Bf}\|^2$$

for some weight B.

Pro :

- Goes till the threshold.
- No localisation in energy.
- The method does not feel the dimension.

Cons :

- Not the standard regularity asumptions.
- The weight B is not very natural in some cases.
- Cannot prove the continuity of the boundary value of the resolvant.
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We adapt the method to study properties of relativistic massive charged particles with spin-$1/2$. We follow the Dirac formalism. Set :

$$D_m := \alpha \cdot P + m\beta = -i \sum_{k=1}^{3} \alpha_k \partial_k + m\beta,$$

where $m > 0$ is the mass. Here we set $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta := \alpha_4$. The α_i, for $i \in \{1, 2, 3, 4\}$, are linearly independent self-adjoint linear maps, acting in $\mathbb{C}^4$, satisfying the anti-commutation relations :

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{i,j} \mathbf{1}_{\mathbb{C}^4}, \text{ where } i, j \in \{1, 2, 3, 4\}.$$

We add a Coulombic interaction. The Hamiltonian is given by

$$H_\gamma := D_m + \gamma V_c(Q), \text{ where } V_c := v_c \otimes Id_{\mathbb{C}^{2\nu}} \text{ and } v_c(x) := \sum_{k=1,\dots,n} \frac{z_i}{|x - a_i|},$$

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Theorem (Boussaid-G, '10)

There are $\kappa, \delta, C > 0$ such that

$$\sup_{|\lambda|\in[m,m+\delta],\, \varepsilon>0, |\gamma|\le\kappa} \|\langle Q\rangle^{-1}(H_\gamma - \lambda - i\varepsilon)^{-1}\langle Q\rangle^{-1}\| \le C.$$

In particular, H_γ has no eigenvalue in $\pm m$. Moreover, there is C' so that

$$\sup_{|\gamma|\le\kappa} \int_{\mathbb{R}} \|\langle Q\rangle^{-1} e^{-itH_\gamma} E_{\mathcal{I}}(H_\gamma) f\|^2 dt \le C' \|f\|^2,$$

where $\mathcal{I} = [-m-\delta, -m] \cup [m, m+\delta]$ and where $E_{\mathcal{I}}(H_\gamma)$ denotes the spectral measure of H_γ.

Few ideas :

1. Reduce to problem to a bounded perturbation.
2. Use a kind of Feshbach method with respect to the spin-up/down decomposition.
3. Reduce the analyse to a kind of non-selfadjoint Laplacian depending of a parameter.
4. Adapt the method of Richard to this setting (non-self-adjoint + parameter).
5. Conclude by perturbing the resolvant.

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In the present work, we start again with

$$[H, iA] - cH > 0.$$

We want to use the standard class of regularity

and prove that there is $\lambda_0 > 0$ such that

$$\sup_{\lambda \in [\lambda_0, \infty), \varepsilon > 0} |\langle f, (H - \lambda - i\varepsilon) f\rangle| \leq c \|\langle A \rangle f\|^2.$$

Moreover, we would like to cover a non-self-adjoint setting.

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and prove that there is $\lambda_0 > 0$ such that

$$\sup_{\lambda\in[\lambda_0,\infty),\varepsilon>0} |\langle f, (H-\lambda-i\varepsilon)f\rangle| \leq c\|\langle A\rangle f\|^2.$$

Moreover, we would like to cover a non-self-adjoint setting.

In the present work, we start again with

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Moreover, we would like to cover a non-self-adjoint setting.

Let $H^{\pm}$ be two closed and densely defined operators, acting in some Hilbert space $\mathscr{H}$ so that

$$\mathscr{D} := \mathcal{D}(H^{+}) = \mathcal{D}(H^{-}) \text{ and } (H^{+})^{*} = H^{-}.$$

We endow $\mathscr{D}$ with the graph norm of H^{+}. In particular, we have that $\mathcal{D}((H^{\pm})^{*}) = \mathscr{D}$.

Since $H^{\pm}$ are densely defined, share the same domain and are adjoint of the other, we have that $\Re(H^{\pm})$ and $\Im(H^{\pm})$ are closable operators on $\mathscr{D}$, indeed their adjoints are densely defined. We denote by $\Re(H^{\pm})$ and by $\Im(H^{\pm})$ the closure of these operators. It is possible that they are not self-adjoint, albeit there are symmetric. However, $\mathscr{D}$ is a core for them. Their domain is possibly bigger than $\mathscr{D}$.

We suppose that H^{+} is *dissipative*, i.e.

$$\langle f, \Im(H^{+})f\rangle \geq 0, \text{ for all } f \in \mathscr{D}.$$

This gives also that $\Im(H^{-}) \leq 0$. By the numerical range theorem, we infer that $\sigma(H^{\pm})$ is included in the half-plan containing $\pm i$.

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Consider now a non-negative self-adjoint operator S in $\mathscr{H}$ with domain $\mathscr{D}$. We set $\mathscr{G} := \mathcal{D}(S^{1/2})$. By duality and interpolation we have :

$$H^{\pm} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*).$$

Here we identified $\mathscr{H}$ with $\mathscr{H}^*$.

To perform this analysis, we consider an external operator, the conjugate operator. Let A be a self-adjoint operator in $\mathscr{H}$. We assume $S \in \mathcal{C}^1(A)$. Let $W_t := e^{itA}$ be the C_0-group associated to A in $\mathscr{H}$. We ask :

$$W_t \mathscr{D} \subset \mathscr{D}, \text{ for all } t \in \mathbb{R}.$$

By duality, we have W_t stabilizes $\mathscr{D}^*$ and by interpolation it also stablizes $\mathscr{G}$ and $\mathscr{G}^*$.

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Theorem

Let $H^\pm$ and A as above. Suppose that $H^\pm \in \mathcal{C}^2(A; \mathscr{D}, \mathscr{H})$, i.e., there is b s.t.

$$|\langle H^\mp f, Ag\rangle - \langle Af, H^\pm g\rangle| \leq b\|f\| \cdot \|\langle S\rangle g\|, \text{ for all } f, g \in \mathscr{D} \cap \mathcal{D}(A),$$

and

$$|\langle [H^\mp, A] f, Ag\rangle - \langle Af, [H^\pm, A] g\rangle| \leq b\|f\| \cdot \|\langle S\rangle g\|, \text{ for all } f, g \in \mathscr{D} \cap \mathcal{D}(A),$$

Assume that, for some $c_1 > 0$,

$$[\Re(H^\pm), iA] - c_1 \Re(H^\pm) \geq S \geq 0,$$
$$\pm \Im(H^\pm) \geq 0,$$

in the sense of forms on $\mathscr{G}$. Consider $\lambda_0 > 0$. Then, for all $s > 1/2$, there are $c > 0$ and $\mu_0 > 0$, so that

$$|\langle f, (H^\pm - \lambda \pm i\mu)^{-1} f\rangle| \leq c\|\langle|\langle S\rangle^{-1/2} A|\rangle^s f\|^2,$$

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Few remarks :

If S is only bounded from below, the theorem is still true if we replace λ_0 big enough.

If $\lambda_0 > 0$ we avoid the threshold effect.

If $c_0 = 0$ one can prove a LAP on $\mathbb{R}$ (see Mantoiu and al).

Note that by duality and interpolation there is $c_2 > 0$ such that

$$\left|\langle f, [[H^{\pm}, A], A] f\rangle\right| \leq c_2 \|\langle S\rangle^{1/2} f\|^2, \text{ for all } f \in \mathscr{G}.$$

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Example : In $L^2(\mathbb{R}^n;\mathbb{C})$, set

$$H = -\Delta_{\mathbb{R}^n} + V_1(Q) + iV_2(Q),$$

with

$$V_i \text{ and } Q^{1+\varepsilon}\cdot\nabla V_i(Q) \in L^\infty$$

for some $\varepsilon > 0$ and

$$V_2 \geq 0.$$

Given $s > 1/2$, there are $\lambda_0 > 0$ and $c > 0$ such that

$$|\langle f, (H-\lambda-i\mu)^{-1} f\rangle| \leq c\|\langle A\rangle^s f\|^2,$$

for all $\lambda \geq \lambda_0$ and $\mu > 0$, where $A := \frac{1}{2}(P\cdot Q + Q\cdot P)$.

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An other example : The bosonic field is described by the Hilbert space $\Gamma(\mathfrak{h})$, where $\mathfrak{h} := L^2(\mathbb{R}^3)$. Set $\mathfrak{h}^{0\otimes} = \mathbb{C}$ and $\mathfrak{h}^{n\otimes} = \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}$. Let S_n be the group of permutation of n elements. For each $\sigma \in S_n$, one defines the action on $\mathfrak{h}^{n\otimes}$ by $\sigma(f_{i_1} \otimes \ldots \otimes f_{i_n}) = f_{\sigma^{-1}(i_1)} \otimes \ldots \otimes f_{\sigma^{-1}(i_n)}$, where (f_i) is a basis of $\mathfrak{h}$. The action extends to $\mathfrak{h}^{n\otimes}$ by linearity to a unitary operator. The definition is independent of the choice of the basis. On $\mathfrak{h}^{n\otimes}$, we set

$$\Pi_n := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \text{ and } \Gamma_n(\mathfrak{h}) := \Pi_n(\mathfrak{h}^{n\otimes}). \tag{1}$$

Note that Π_n is an orthogonal projection. We call $\Gamma_n(\mathfrak{h})$ the n-particle bosonic space. The bosonic space is defined by

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \Gamma_n(\mathfrak{h}).$$

We denote by Ω the *vacuum*, the element $(1, 0, 0, \ldots)$.

Given a self-adjoint operator ω in $\mathfrak{h}$ and a finite dimensional Hilbert space $\mathscr{K}$. One defines the free Hamiltonian H_0 acting on the Hilbert space $\mathscr{H} := \mathscr{K} \otimes \Gamma(\mathfrak{h})$ by

$$H_0 := K \otimes 1_{\Gamma(\mathfrak{h})} + 1_{\mathscr{K}} \otimes d\Gamma(\omega), \tag{2}$$

with $\omega := |k|$.
Recall that for b self-adjoint in $\mathfrak{h}$ we define :

$$d\Gamma(b)|_{\Pi_n(\mathcal{D}(b)^{n\otimes})} := \sum_{j=1}^{n} 1 \otimes \ldots \otimes 1 \otimes \underbrace{b}_{j^{th}} \otimes 1 \otimes \ldots \otimes 1.$$

We recall also the definition of the *number operator* $N := 1_{\mathscr{K}} \otimes d\Gamma(1)$.

We now define the interaction. Let α be an element $\mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})$. This is a *form-factor*. We define $b(\alpha)$ on $\mathscr{H}$ by $b(\alpha) := \mathscr{K} \otimes h^{n\otimes} \to \mathscr{K} \otimes h^{(n-1)\otimes}$, where

$$b(\alpha)(\Psi \otimes \phi_1 \otimes \dots \phi_n) := \alpha^*(\Psi \otimes \phi_1) \otimes \phi_2 \otimes \dots \phi_n,$$

for $n \geq 1$ and by 0 otherwise. This operator is bounded and its norm is given by $\|\alpha\|_{\mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})}$. We define the *annihilation operator* on $\mathscr{K} \otimes \Gamma(\mathfrak{h})$ with domain $\mathscr{K} \otimes \Gamma_{fin}(\mathfrak{h})$ by

$$a(\alpha) := (N+1)^{1/2} b(\alpha) \big(1 \otimes \Pi\big),$$

where $\Pi := \sum_n \Pi_n$, see (1). As above, it is closable and its closure is denoted by $a(\alpha)$. Its adjoint is the *creation operator*. It acts as $a^*(\alpha) = b^*(\alpha)(N+1)^{1/2}$ on $\mathscr{H}$. Note that $b^*(\alpha)(\psi \otimes \phi_1 \otimes \dots \otimes \phi_n) = (\alpha\phi) \otimes \phi_1 \otimes \dots \otimes \phi_n$.
The *Field operator* is defined by

$$\phi(\alpha) := \frac{1}{\sqrt{2}} \big(a(\alpha) + a^*(\alpha)\big).$$

We consider its closure on $\mathscr{K} \otimes \mathcal{D}(N^{1/2})$.

Set

$$H_g := H_0 + g\phi(\alpha),$$

and $A := P \cdot Q + Q \cdot P$, where $\omega^{1/2}\alpha, \omega^{1/2}A\alpha$, and $\omega^{1/2}A^2\alpha$ are $\mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})$. Then given $g_max > 0$, there is $\lambda_0 > 0$we have

$$|\langle f, (H_g - \lambda - \mu)^{-1} f| \leq \|\langle d\Gamma(A)\rangle^s f\|^2$$

for all $\lambda \geq \lambda 0$ and $|g| \leq g_max$

Partial Proof : We define

$$H_\varepsilon^\pm := H^\pm \pm i\varepsilon[H^\pm, iA],$$

with the common domain $\mathscr{D}$ for $\varepsilon \geq 0$. By Kato-Rellich, since $H^\pm$ is closed and $[H^\pm, iA] \in \mathcal{B}(\mathscr{D}, \mathscr{H})$ imply that, for ε_0 small enough and $|\varepsilon| \leq \varepsilon_0$, $H_\varepsilon^\pm$ is closed on the same domain, namely $\mathscr{D}$.

Since $H^\pm \pm i$ is bijective, by writing

$$H_\varepsilon^\pm \pm i = \left(1 \pm i\varepsilon[H^\pm, iA]_\circ(H^\pm \pm i)^{-1}\right)(H^\pm \pm i),$$

as operators acting on $\mathscr{D}$, and using again $[H^\pm, iA] \in \mathcal{B}(\mathscr{D}, \mathscr{H})$, we see that $H_\varepsilon^\pm \pm i$ is bijective (and therefore closed) for ε small enough. Hence $(H_\varepsilon^\pm \pm i)^*$ is also bijective from $\mathcal{D}\big((H_\varepsilon^\pm)^*\big)$ onto $\mathscr{H}$.

Now since $(H_\varepsilon^\pm \pm i)^*$ is a bijective extension of the bijective map $H_\varepsilon^\mp \mp i$ (from $\mathscr{D}$ onto $\mathscr{H}$), $(H_\varepsilon^\pm \pm i)^* = H_\varepsilon^\mp \mp i$. Therefore, for $|\varepsilon| \leq \varepsilon_0$, $\mathcal{D}\big((H_\varepsilon^\pm)^*\big) = \mathscr{D}$ and $(H_\varepsilon^\pm)^* = H_\varepsilon^\mp$.

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Since $H^{\pm} \in \mathcal{C}^1(A; \mathscr{G}, \mathscr{G}^*)$, we obtain that

$$[H^{\pm}, A] = [\Re(H^{\pm}), A] + i[\Im(H^{\pm}), A] \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$$

and

$$H_\varepsilon^{\pm} - \lambda \pm i\mu \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*).$$

Now, take $f \in \mathscr{G}$. Take $\varepsilon, \lambda, \mu \geq 0$. We get :

$$\begin{aligned}
&- c_1\varepsilon \langle f, \Re(H_\varepsilon^{\pm} - \lambda \pm i\mu)f\rangle \pm \langle f, \Im(H_\varepsilon^{\pm} - \lambda \pm i\mu)f\rangle = \\
&= -c_1\varepsilon \langle f, (\Re(H^{\pm}) \mp \varepsilon[\Im(H^{\pm}), iA] - \lambda)\, f\rangle \pm \langle f, (\Im(H^{\pm}) \pm \mu \pm \varepsilon[\Re(H^{\pm}), iA])\, f\rangle \\
&= \varepsilon \langle f, ([\Re(H^{\pm}), iA] - c_1\Re(H^{\pm}))f\rangle + (c_1\lambda\varepsilon + \mu)\,\|f\|^2 \\
&\qquad \pm \left\langle f, \left(c_1\varepsilon^2[\Im(H^{\pm}), iA] + \Im(H^{\pm})\right) f\right\rangle \\
&\geq (c_1\lambda\varepsilon + \mu)\,\|f\|^2 + \varepsilon\|S^{1/2}f\|^2. \qquad (3)
\end{aligned}$$

Here we remove the condition on the commutator of the imaginary part by taking ε small enough.

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& = -c_1 \varepsilon \left\langle f, \left(\Re(H^{\pm}) \mp \varepsilon [\Im(H^{\pm}), iA] - \lambda\right) f \right\rangle \pm \left\langle f, \left(\Im(H^{\pm}) \pm \mu \pm \varepsilon [\Re(H^{\pm}), iA]\right) f \right\rangle \\
& = \varepsilon \left\langle f, \left([\Re(H^{\pm}), iA] - c_1 \Re(H^{\pm})\right) f \right\rangle + (c_1 \lambda \varepsilon + \mu) \|f\|^2 \\
& \qquad \pm \left\langle f, \left(c_1 \varepsilon^2 [\Im(H^{\pm}), iA] + \Im(H^{\pm})\right) f \right\rangle \\
& \geq (c_1 \lambda \varepsilon + \mu) \|f\|^2 + \varepsilon \|S^{1/2} f\|^2. \qquad (3)
\end{aligned}$$

Here we remove the condition on the commutator of the imaginary part by taking ε small enough.

We start with a crude bound. For $f \in \mathscr{G}$, we set :

$$\|f\|^2_{\mathscr{G}} := \|f\|^2 + \|S^{1/2}f\|^2 .$$

Using Cauchy-Schwartz, we obtain :

$$\sqrt{2}\max(c_1\varepsilon, 1)\,\|(H^{\pm}_{\varepsilon} - \lambda \pm i\mu)f\|_{\mathscr{G}*} \geq \min(c_1\lambda\varepsilon + \mu, \varepsilon)\|f\|_{\mathscr{G}} .$$

This shows that, as an operator in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, $H^{\pm}_{\varepsilon} - \lambda \pm i\mu$ is injective with closed range. As such, its adjoint belongs to $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and is an extension of $H^{\mp}_{\varepsilon} - \lambda \mp i\mu$. Thus it is actually $H^{\mp}_{\varepsilon} - \lambda \mp i\mu$, which is injective with closed range. All this implies the bijectivity of $H^{\pm}_{\varepsilon} - \lambda \pm i\mu$, viewed in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, and

$$G^{\pm}_{\varepsilon} := G^{\pm}_{\varepsilon}(\lambda, \mu) = (H^{\pm}_{\varepsilon} - \lambda \pm i\mu)^{-1} \text{ exists in } \mathcal{B}(\mathscr{G}^*, \mathscr{G}), \text{ for } \lambda \geq 0 \text{ and } \varepsilon, \mu > 0.$$

Moreover,

$$\|G^{\pm}_{\varepsilon}\|_{\mathcal{B}(\mathscr{G}^*, \mathscr{G})} \leq \sqrt{2}\max(c_1\varepsilon, 1)/\min(c_1\lambda\varepsilon + \mu, \varepsilon), \text{ for } \lambda \geq 0 \text{ and } \varepsilon, \mu > 0. \tag{4}$$

This bound seems not enough to lead the whole analysis.

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$$G_\varepsilon^\pm := G_\varepsilon^\pm(\lambda, \mu) = (H_\varepsilon^\pm - \lambda \pm i\mu)^{-1} \text{ exists in } \mathcal{B}(\mathscr{G}^*, \mathscr{G}), \text{ for } \lambda \geq 0 \text{ and } \varepsilon, \mu > 0.$$

Moreover,

$$\|G_\varepsilon^\pm\|_{\mathcal{B}(\mathscr{G}^*, \mathscr{G})} \leq \sqrt{2}\max(c_1\varepsilon, 1)/\min(c_1\lambda\varepsilon + \mu, \varepsilon), \text{ for } \lambda \geq 0 \text{ and } \varepsilon, \mu > 0. \tag{4}$$

This bound seems not enough to lead the whole analysis.

We start with a crude bound. For $f \in \mathscr{G}$, we set :

$$\|f\|_{\mathscr{G}}^2 := \|f\|^2 + \|S^{1/2}f\|^2 .$$

Using Cauchy-Schwartz, we obtain :

$$\sqrt{2}\max(c_1\varepsilon, 1)\,\|(H_\varepsilon^\pm - \lambda \pm i\mu)f\|_{\mathscr{G}^*} \geq \min(c_1\lambda\varepsilon + \mu, \varepsilon)\|f\|_{\mathscr{G}} .$$

This shows that, as an operator in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, $H_\varepsilon^\pm - \lambda \pm i\mu$ is injective with closed range. As such, its adjoint belongs to $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and is an extension of $H_\varepsilon^\mp - \lambda \mp i\mu$. Thus it is actually $H_\varepsilon^\mp - \lambda \mp i\mu$, which is injective with closed range. All this implies the bijectivity of $H_\varepsilon^\pm - \lambda \pm i\mu$, viewed in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, and

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This bound seems not enough to lead the whole analysis.

Then, we first restrict the domain of $G_\varepsilon^\pm$ to $\mathscr{H}$ and improve it.

Since (3) holds true on $\mathscr{D}$, the common domain of $H_\varepsilon^\pm$ and $(H_\varepsilon^\pm)^*$, we can apply the numerical range theorem. Thus the spectrum of $H_\varepsilon^\pm - \lambda \pm i\mu$ is contained in the half-plane delimited by the equation

$$\pm y \geq c_1 \varepsilon x + \mu.$$

Taking possibly ε_0 smaller (but positive), the distance from 0 to this half-plane is larger than $\mu/2$. Then $G_\varepsilon^\pm|_{\mathscr{H}} = (H_\varepsilon^\pm - \lambda \pm i\mu)^{-1}$ exists from $\mathscr{H}$ to $\mathscr{D}$. Thus

$$\|G_\varepsilon^\pm\|_{\mathcal{B}(\mathscr{H})} \leq 2/\mu, \text{ for } \mu > 0 \text{ and } \varepsilon \in [0, \varepsilon_0]. \quad (5)$$

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Take $\varepsilon, \mu > 0$. We fix $f \in \mathscr{H}$ and set :

$$F_\varepsilon^\pm := \left\langle f, G_\varepsilon^\pm f \right\rangle .$$

Notice that $\overline{F_\varepsilon^\pm} = F_\varepsilon^\mp$. Since $G_\varepsilon^\pm \mathscr{H} = \mathscr{D} \subset \mathscr{G}$ and using (4), we infer

$$\min\left(c_1 \lambda \varepsilon + \mu, \varepsilon\right) \left\| \langle S \rangle^{1/2} G_\varepsilon^\pm f \right\|^2 \le \sqrt{2} \max(c_1 \varepsilon, 1) \left| F_\varepsilon^\pm \right| . \tag{6}$$

Hence up to a smaller $\varepsilon_0 > 0$ and since $\mu > 0$, we obtain

$$\left\| \langle S \rangle^{1/2} G_\varepsilon^\pm f \right\|^2 \le \frac{1}{\varepsilon \min(c_1 \lambda, 1)} \left| F_\varepsilon^\pm \right| = \frac{c_0}{\varepsilon} \left| F_\varepsilon^\pm \right|, \text{ for all } \varepsilon \in (0, \varepsilon_0], \tag{7}$$

and where $c_0 := 1/\min(1, \lambda_0 c_1) \ge 1$. From (6), we derive

$$\left| F_\varepsilon^\pm \right| \le \left\| \langle S \rangle^{-1/2} f \right\| \left\| \langle S \rangle^{1/2} G_\varepsilon^\pm f \right\| \le \sqrt{\frac{c_0}{\varepsilon}} \left\| \langle S \rangle^{-1/2} f \right\| \sqrt{\left| F_\varepsilon^\pm \right|}$$

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$$\left| F_\varepsilon^\pm \right| \le \frac{c_0}{\varepsilon} \left\| \langle S \rangle^{-1/2} f \right\|^2, \text{ for all } \varepsilon \in (0, \varepsilon_0]. \tag{8}$$

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We now show that $G_\varepsilon^\pm \in \mathcal{C}^1(A)$. Recall that $G_\varepsilon^\pm$ is a bijection from $\mathscr{H}$ onto $\mathscr{D}$. Then by taking the adjoint, it is also a bijection from $\mathscr{D}^*$ onto $\mathscr{H}$. Remember now that e^{itA} stabilizes $\mathscr{G}$ and $\mathscr{G}^*$. By the resolvent equality in $\mathcal{B}(\mathscr{H})$, we have :

$$\frac{1}{t}[G_\varepsilon^\pm, e^{itA}] = -\frac{1}{t} \underbrace{G_\varepsilon^\pm}_{\mathscr{H} \longleftarrow \mathscr{G}^*} \underbrace{\left[H^\pm \pm i\varepsilon[H^\pm, iA], e^{itA}\right]}_{\mathscr{G}^* \longleftarrow \mathscr{G}} \underbrace{G_\varepsilon^\pm}_{\mathscr{G} \longleftarrow \mathscr{H}}$$

Let us now take the limit at 0. Since $H^\pm$ and $[H^\pm, iA]$ are in $\mathcal{C}^1(A; \mathscr{G}, \mathscr{G}^*)$, the r.h.s has a strong limit for all element in $\mathscr{H}$. Hence, $G_\varepsilon^\pm \in \mathcal{C}^1(A; \mathscr{H}, \mathscr{H})$ which actually means $G_\varepsilon^\pm \in \mathcal{C}^1(A)$.

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Easily, it follows that $G_\varepsilon^\pm \mathcal{D}(A) \subset \mathcal{D}(A) \cap \mathscr{D}$.

Considering commutators in the form sense, we have :

$$\begin{aligned} \frac{dF_\varepsilon^\pm}{d\varepsilon} &= \left\langle f, \frac{dG_\varepsilon^\pm}{d\varepsilon} f \right\rangle \qquad (9)\\ &= \pm i \left\langle G_\varepsilon^\mp f, [H^\pm, iA] G_\varepsilon^\pm f \right\rangle \\ &= \pm i \left\langle G_\varepsilon^\mp f, [H_\varepsilon^\pm - \lambda \pm i\mu, iA] G_\varepsilon^\pm f \right\rangle - \varepsilon \left\langle G_\varepsilon^\mp f, \big[[H^\pm, iA], iA\big] G_\varepsilon^\pm f \right\rangle \\ &= \pm \left\langle G_\varepsilon^\mp f, Af \right\rangle \mp \left\langle Af, G_\varepsilon^\pm f \right\rangle - \varepsilon \left\langle G_\varepsilon^\mp f, \big[[H^\pm, iA], iA\big] G_\varepsilon^\pm f \right\rangle , \qquad (10) \end{aligned}$$

for all $f \in \mathcal{D}(A)$.

Using the hypothesis on the second commutator, we infer :

$$\left\|\frac{dF_\varepsilon^\pm}{d\varepsilon}\right\| \leq \left\|\langle S\rangle^{-1/2}Af\right\|\left(\left\|\langle S\rangle^{1/2}G_\varepsilon^\pm f\right\| + \left\|\langle S\rangle^{1/2}G_\varepsilon^\mp f\right\|\right) + c_2\varepsilon\left\|\langle S\rangle^{1/2}G_\varepsilon^\pm f\right\|\cdot\left\|\langle S\rangle^{1/2}G_\varepsilon^\mp f\right\|. \tag{11}$$

Now we use four times (7) and the fact that $|F_\varepsilon^\pm| = |F_\varepsilon^\mp|$, and integrate to obtain, for all $f \in \mathcal{D}(A)$,

$$\left|F_{\varepsilon'}^\pm - F_\varepsilon^\pm\right| \leq \int_\varepsilon^{\varepsilon'} \left\{2s^{-1/2}\sqrt{c_0}\sqrt{|F_s^\pm|}\left\|\langle S\rangle^{-1/2}Af\right\| + c_2c_0\left|F_s^\pm\right|\right\} ds, \tag{12}$$

for $0 < \varepsilon \leq \varepsilon' \leq \varepsilon_0$.

Using the hypothesis on the second commutator, we infer :

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By the Gronwall lemma, we infer that there is $C_1 > 0$, independent of $c_0 \geq 1$, $\varepsilon \in (0, \varepsilon_0]$, $\lambda \geq \lambda_0$, and of $\mu > 0$, such that, for $f \in \mathcal{D}(A)$,

$$\begin{aligned}
\left|F_\varepsilon^\pm\right| &\leq e^{c_0 c_2(\varepsilon_0 - \varepsilon)} \left(\left|F_{\varepsilon_0}^\pm\right|^{1/2} + \int_\varepsilon^{\varepsilon_0} \left\{ \frac{1}{\sqrt{\eta}} e^{-\frac{1}{2} c_0 c_2 (\varepsilon_0 - \eta)} \right\} d\eta \ \left\| \langle S \rangle^{-1/2} A f \right\| \sqrt{c_0} \right)^2 \\
&\leq 2 e^{c_0 c_2 \varepsilon_0} \left(\left| F_{\varepsilon_0}^\pm \right| + C_1 c_0 \left\| \langle S \rangle^{-1/2} A f \right\|^2 \right) \\
&\leq 2 c_0 e^{c_0 c_2 \varepsilon_0} \left(\varepsilon_0^{-1} \left\| \langle S \rangle^{-1/2} f \right\|^2 + C_1 \left\| \langle S \rangle^{-1/2} A f \right\|^2 \right) . \qquad (13)
\end{aligned}$$

using (8).

We now plug this back into (12). Thus there exist $C_2, C_3 > 0$ with the same independence such that

$$\left|F^{\pm}_{\varepsilon'} - F^{\pm}_{\varepsilon}\right| \leq c_0^2 e^{c_0 c_2 \varepsilon_0} \int_{\varepsilon}^{\varepsilon'} \left\{ 2\frac{\sqrt{C_2}}{\sqrt{s}} + C_1 C_2 \right\} ds \, (\|f\|^2 + \|\langle S\rangle^{-1/2} A f\|^2)$$
$$= C_3 c_0^2 e^{c_0 c_2 \varepsilon_0} \big(\sqrt{\varepsilon'} - \sqrt{\varepsilon}\big)(\|f\|^2 + \|\langle S\rangle^{-1/2} A f\|^2).$$

Then, $\{F^{\pm}_{\varepsilon}\}_{\varepsilon \in (0, \varepsilon_0]}$ is a Cauchy sequence. We denote by $F^{\pm}_{0^+}$ the limit, as ε goes to 0. It remains to notice that $F^{\pm}_{0^+} = F^{\pm}_{0}$. Indeed, using the resolvent identity, (5), and (1),

$$\|G^{\pm}_0 - G^{\pm}_{\varepsilon}\|_{\mathcal{B}(\mathscr{H})} \leq \varepsilon \|G^{\pm}_{\varepsilon}\|_{\mathcal{B}(\mathscr{H})} \cdot \|[H^{\pm}, iA](H^{\pm} - \lambda \pm i\mu)^{-1}\|_{\mathcal{B}(\mathscr{H})} \leq \frac{c(\lambda)\varepsilon}{\mu^2},$$

with for some ε-independent $c(\lambda)$. This gives us the LAP. □

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