Beyond the Van Hove Timescale

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Joint work with Volker Bach and Matthias Westrich

Part I

Setting up the Problem

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But what lies beyond the Van Hove timescale?

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Our method does not at present allow us to deal with degenerate energy levels.

The Scalar Field

The Hilbert space for the free scalar photon field is the Bosonic Fock space

$$\mathcal{F} = \Gamma(L^2(\mathbb{R}^3)) = \mathbb{C} \oplus \Big(\bigoplus_{n=1}^{\infty} L^2(\mathbb{R}^3)^{\otimes_{\mathrm{s}} n} \Big).$$

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The free field energy, at zero temperature is the second quantized massless dispersion relation $k \to |k|$, i.e.,

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The choice of dimension equal to 3 is not of any significance.

The Free Hamiltonian

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We have

$$\sigma(H_0) = [e^{(1)}, \infty) \quad \text{and} \quad \sigma_{\mathrm{pp}}(H_0) = \sigma(H_{\mathrm{at}}).$$

The point spectrum is embedded into the continuous spectrum.

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for some $\Theta > 0$. The choice of $In(4/3)$ as the constraint of the real part of θ is for convenience.

Decay Assumptions

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We assume the existence of $C > 0$ and $\mu > 0$ such that

$$\forall |k| \leq 1, \theta \in R : \qquad |G_\theta(k)| \leq C|k|^{-\frac{1}{2}+\mu}$$

and

$$\sup_{\theta \in R} \int_{|k|\geq 1} |G_\theta(k)|^2 dk < \infty.$$

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It is an observation of Hasler and Herbst that μ may be chosen equal to zero if $L = 2$ and $G(k)$ does not have diagonal entries for any k. (As in the spin-boson model.)

The interacting Hamiltonian

The coupling G gives rise to an interaction term

$$\phi(G) = \int_{\mathbb{R}^3} \big(G(k)^* \otimes a(k) + G(k) \otimes a^*(k) \big) \, \mathrm{d}k.$$

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Here g is a (real) coupling constant.

First Attempt

Our initial intention was to find an expression for

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While we did succeed in this task, we were not able to set up an iteration scheme, as in the BFS renormalization group. (To many oscillatory integrals.)

Second Attempt

The starting point is an old idea, to express e^{-itH_g} as an inverse Laplace transform of the resolvent:

$$e^{-itH_g} = \text{s}-\lim_{T\to+\infty} \frac{1}{2\pi i} \int_{[-T,T]+ic} e^{-itz}(H_g - z)^{-1}\,\mathrm{d}z,$$

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To illustrate why a vacuum projection may help, consider the free case $g = 0$. Here $\langle 0|(H_0 - z)^{-1}|0\rangle = (H_{\mathrm{at}} - z)^{-1}$, which is an analytic function of z with simple poles at the eigenvalues $e^{(\ell)}$. $\ell = 1, 2, \ldots, L$.

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It is now an easy consequence of the residue theorem, to show that the inverse Laplace transform of $\langle 0|(H_0 - z)^{-1}|0\rangle$ is indeed $\mathrm{e}^{-itH_{\mathrm{at}}}$.

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The dilated Hamiltonian, computed first for real θ, becomes

$$H_g(\theta) := U_\theta^* H_g U_\theta = H_{\mathrm{at}} + e^{-\theta} H_{\mathrm{ph}} + gV(\theta),$$

where

$$V(\theta) = \int_{\mathbb{R}^3} \big(G_{\bar{\theta}}(k)^* \otimes a(k) + G_\theta(k) \otimes a^*(k) \big) \,\mathrm{d}k.$$

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Note that $H_g(\theta)$ is closed on $\mathcal{D}(H_{\mathrm{ph}})$ and not normal if $\mathrm{Im}\theta \neq 0$.

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Note that $U_\theta |0\rangle = |0\rangle$, such that we have the crucial identity

$$\langle 0|(H_g - z)^{-1}|0\rangle = \langle 0|(H_g(\theta) - z)^{-1}|0\rangle.$$

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Additionally, we need to get good estimates on $(H_g(\theta) - z)^{-1}$, which are not immediate since $H_g(\theta)$ is not a normal operator.

Part II

The Renormalization Group

Parameters

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- $0 < \underline{\rho} \leq \rho \leq \bar{\rho}$: a real scaling parameter.
- $0 < \xi < 1$: A measure of the rate of decay of Wick monomials.

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Kernels of Wick Monomials

A kernel of a Wick monomial of order (M, N) with $M, N \geq 0$ is a $\mathbb{C}$-valued function

$$(r, k^{(M,N)}, z, \theta, \rho, g) \to w_{M,N}(r, k^{(M,N)}, z; \theta, \rho, g),$$

where $k^{(M,N)} = (k^{(M)}, \tilde{k}^{(N)}) \in B_1^{3M} \times B_1^{3N}$ are momenta and $B_1 = \{k \in \mathbb{R}^3 \,|\, |k| \leq 1\}$.

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For almost every $k^{(M,N)}$, the function

$$(r, z, \theta, \rho, g) \to w_{M,N}(r, k^{(M,N)}, z; \theta, \rho, g)$$

is jointly continuous, admits a continuous partial r-derivative, and is holomorphic in the variables z, θ and g.

Norms of Wick Kernels

For $M + N \geq 1$, we introduce the norm

$$\|w_{M,N}\|_\mu^2 = \sup_{z,\theta,\rho,g} \int_{B_1^{M+N}} \sup_{0\leq r\leq 1} \left|w_{M,N}(r, k^{(M,N)}, z)\right|^2 \frac{dk^{(M,N)}}{\prod(k^{(M,N)})^{3+2\mu}},$$

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We consider only kernels for which

$$\|w_{M,N}\|_\mu < \infty \qquad \text{and} \qquad \|\partial_r w_{M,N}\|_\mu < \infty.$$

Wick Monomials

To a kernel $w_{M,N}$, $M+N \geq 1$, we associate the Wick monomial

$$W_{M,N}[z] = \int_{B_1^{M+N}} a^*(k^{(M)}) w_{M,N}(H_{\mathrm{ph}}, k^{(M,N)}, z) a(\tilde{k}^{(N)}) \frac{\mathrm{d}k^{(M,N)}}{\sqrt{\prod(k^{(M,N)})}},$$

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We in fact have

$$\|W_{M,N}[z]\|_{\mathrm{Op}} \leq M^{-M/2} N^{-N/2} \|w_{M,N}\|_{\mu} \leq \|w_{M,N}\|_{\mu}.$$

Wick Polynomials

Write $\mathcal{W}_{\geq 1}$ for sequences of kernels $w = \{w_{M,N}\}_{M+N\geq 1}$ with finite norm

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Wick polynomials are bounded operators on $\mathcal{H}_{\mathrm{red}}$ with

$$\|W[z]\|_{\mathrm{Op}} \leq \xi \|w\|_{\xi,\mu}.$$

Free Hamiltonian and Spectral Parameter

We decompose 'kernels' of order $(0,0)$ into a sum of two terms

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The operator $T[H_f, z]$ plays the role of a (complex dilated) field energy, and $E[z]$ plays the role of a spectral parameter.

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We decompose 'kernels' of order $(0,0)$ into a sum of two terms

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where $E[z] = -w_{0,0}(0,z)$.

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To $h \in \mathcal{W}_{\geq 0}$, we may now associate a bounded operator

$$H[z] = T[H_f, z] - E[z] + W[z].$$

Disc of Kernels

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Hamiltonians $H[z]$ associated to $h \in \mathcal{D}(\epsilon, \delta)$ are close to the free dilated field energy $e^{-\theta} H_{\mathrm{ph}} - z$, provided ϵ and δ are small.

Zeroes

Let $h \in \mathcal{D}(1/32, 1/32)$.

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Furthermore

$$|z[0]| \leq \delta \qquad \text{and} \qquad |z[r] - z[0] - e^{-\theta} r| \leq \frac{1}{2}(7\delta + 32\varepsilon) r.$$

Resolvent Estimate

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For $\kappa \in [0,1)$, let $C_\kappa = (1 + \overline{D}_\kappa)[0,\infty)$.

Then for $z \in \overline{D}_{1/2} \setminus (\overline{D}_{3\delta} + e^{-\theta} C_{4\epsilon/3})$, the operator $H[z]$ is invertible and

$$\|H[z]^{-1}\| \leq \frac{1}{\mathrm{dist}(z, e^{-\theta} C_{4\epsilon/3}) - 2\delta} \leq \frac{1}{\delta}.$$

Restricted Resolvent Estimate

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Let $\chi, \bar{\chi} \in C^\infty(\mathbb{R})$, $\chi, \bar{\chi} \geq 0$, $\chi^2 + \bar{\chi}^2 = 1$, $\mathrm{supp}(\chi) \subset (-\infty, 1]$ and $\mathrm{supp}(\bar{\chi}) \subset [3/4, \infty)$. Abbreviate $\chi_\rho(\lambda) = \chi(\lambda/\rho)$ and $\bar{\chi}_\rho(\lambda) = \bar{\chi}(\lambda/\rho)$.

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$$H_{\bar{\chi}_\rho}[z] = T[H_{\mathrm{ph}}, z] - E[z] + \bar{\chi}_\rho W[z] \bar{\chi}_\rho.$$

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Suppose $\delta < \underline{\rho}/60$ and $z \in z[0] + \overline{D}_{\widetilde{\rho}/2}$, where $\widetilde{\rho} = \rho/|E'[z[0]]|$. Then $H_{\bar{\chi}_\rho}[z]$ is invertible and (using $|E'[z[0]] - 1| \leq 8\delta$)

$$\|H_{\bar{\chi}_\rho}[z]^{-1}\| \leq \frac{1}{\frac{\rho}{20} - 3\delta}.$$

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is invertible, and if invertibility holds, we have

$$H^{-1} = \begin{pmatrix} F^{-1} & -F^{-1}B\overline{H}^{-1} \\ -\overline{H}^{-1}CF^{-1} & \overline{H}^{-1} + \overline{H}^{-1}CF^{-1}B\overline{H}^{-1} \end{pmatrix}.$$

Renormalization. First Step

Using a Hilbert space analogue of Schur's result and with smoothed out projections, we get the Feshbach operator

$$F_\rho[z] = T[H_{\mathrm{ph}}, z] - E[z] - \chi_\rho W[z] \bar{\chi}_\rho H_{\bar{\chi}_\rho}[z]^{-1} \bar{\chi}_\rho W[z] \chi_\rho .$$

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Then $H[z]$ is invertible if and only if $F_\rho[z]$ is invertible as an operator on $\mathbb{1}[H_{\mathrm{ph}} \leq \rho]\mathcal{H}_{\mathrm{red}}$.

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The rescaled Feshbach operator, acting on $\mathcal{H}_{\rm red}$, becomes

$$\frac{1}{\rho}\Gamma_\rho F_\rho[z]\Gamma_\rho^*,$$

where the factor $1/\rho$ is an energy rescaling chosen such that

$$\frac{1}{\rho}\Gamma_\rho H_{\rm ph}\Gamma_\rho^* = H_{\rm ph}.$$

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Jacob Schach Møller

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Composing the three constructions yields the one-step renormalized Hamiltonian

$$\mathcal{R}_\rho(h)[\zeta] = \frac{1}{\rho}\Gamma_\rho F_\rho[J^{-1}(\zeta)]\Gamma_\rho^*.$$

Recall that we started with a $h \in \mathcal{D}(\delta, \epsilon)$ and from that we constructed a renormalized Hamiltonian $\mathcal{R}_\rho(h)[\zeta]$, which is a bounded operator on $\mathcal{H}_{\mathrm{red}}$ for each $\zeta \in \overline{D}_{1/2}$.

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This is the hard part of the analysis ... so we skip it.

Contractivity of the RG Transformation

There exist (explicit) constants $0 < c_1, c_2, c_3 < 1$, depending only on the choice of χ, such that with:

$$0 < \underline{\rho} < \bar{\rho} = c_1^{1/\mu}, \qquad \xi = c_2\sqrt{\underline{\rho}},$$
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In the rest of Part II we fix

$$\delta_0 \leq c_3\underline{\rho} \qquad \text{and} \qquad \epsilon_0 \leq 1/32 - \delta_0.$$

Iterating the Renormalization Transform

Suppose we begin with $h_{(0)} \in \mathcal{D}(\delta_0, \epsilon_0)$. We may now iterate and get a sequence

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$$\frac{1}{2}\rho^n \leq \widetilde{\rho}_{(0)}\widetilde{\rho}_{(1)} \cdots \widetilde{\rho}_{(n-1)} \leq \frac{3}{2}\rho^n.$$

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Note that $e_{(\infty)}$ depends holomorphically on θ and g through both the $z_{(n)}[0]$'s and the $\widetilde{\rho}_{(n)}$'s.

Spectral Localization

From the point of view of the original disc $\overline{D}_{1/2}$, pertaining to $h_{(0)}$, the rescaled discs are concentric and intersects to $\{e_{(\infty)}\}$:

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Using isospectrality in the n'th annulus with $H_{(n)}[z]$ yields:

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where $\kappa = 3\epsilon_0 + 46\delta_0/\underline{\rho}$.

Improved Resolvent Estimate

As with Schur's theorem from Linear Algebra, one may reconstruct $H[z]^{-1}$ from $H_{(n)}[\zeta]^{-1}$, provided $z \in e_{(n)} + \overline{D}_{\frac{1}{2}\widetilde{\rho}_{(0)}\cdots\widetilde{\rho}_{(n-1)}}$ and not in the cone $e_{(\infty)} + e^{-\theta} C_\kappa$ from the previous slide.

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We find that for any $z \in \overline{D}_{1/2} \setminus (e_{(\infty)} + e^{-\theta}C_\kappa)$, we have

$$\|H[z]^{-1}\|_{\mathrm{Op}} \leq \frac{100}{\mathrm{dist}(z, e_{(\infty)} + e^{-\theta}C_\kappa)}.$$

Part III

Effective Dynamics

Initial Discs and Projections

Place discs around each unperturbed energy level with a radius

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$$(e^{(\ell-1)} + e^{-\theta}[0, \infty)) \cap (e^{(\ell)} + \overline{D}_{\rho_{(-1)}/2}) = \emptyset.$$

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We want to initialize the Renormalization Scheme using the projections

$$P^{(\ell)} = P^{(\ell)}_{\mathrm{at}} \otimes \chi_{\rho_0},$$

where $P^{(\ell)}_{\mathrm{at}}$ projects onto the ℓ'th atomic eigenspace.

Restriction on Coupling Strength

In order to do that, we need

$$H_{\overline{P}^{(\ell)}}[z] = H_{\rm at} + e^{-\theta} H_{\rm ph} - z + g\overline{P}^{(\ell)} V(\theta)\overline{P}^{(\ell)}$$

to be invertible for $|z - e^{(\ell)}| \leq \rho_{(-1)}/2$ as an operator on

$$\overline{P}^{(\ell)}_{\rm at}\mathcal{H} \oplus \mathbb{1}[H_{\rm ph} \geq 3\rho_{(-1)}/4]\mathcal{H}.$$

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In order to do that, we need

$$H_{\overline{P}^{(\ell)}}[z] = H_{\rm at} + e^{-\theta} H_{\rm ph} - z + g\overline{P}^{(\ell)} V(\theta)\overline{P}^{(\ell)}$$

to be invertible for $|z - e^{(\ell)}| \le \rho_{(-1)}/2$ as an operator on

$$\overline{P}^{(\ell)}_{\rm at}\mathcal{H} \oplus \mathbb{1}[H_{\rm ph} \ge 3\rho_{(-1)}/4]\mathcal{H}.$$

This is ok provided

$$|g| < g_0 = \frac{\Theta^{3/2}}{82\pi^{3/2}\sup_{\theta\in R}\sqrt{\int_{\mathbb{R}^3}(1+|k|^{-1})|G_\theta(k)|^2 \mathrm{d}k}}.$$

Feeding the Renormalization Scheme

For each ℓ we now feed the Hamiltonian

$$H_{(0)}[z] = \frac{1}{\rho_{(-1)}}\Gamma_{\rho_{(-1)}} F_{P^{(\ell)}}[e^{(\ell)} + \rho_{(-1)} z]\Gamma^*_{\rho_{(-1)}},$$

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Writing $w^{(\ell)}_{(0),0,0} = T^{(\ell)}_{(0)}[r,z] - E^{(\ell)}_{(0)}[z]$, one may verify that

$$\sup_{0\leq r\leq 1} |\partial_r T^{(\ell)}_{(0)}[r,z] - e^{-\theta}| \leq C|g|^2, \qquad |E^{(\ell)}_{(0)}[z] - z| \leq C|g|^2,$$
$$\|\{w_{(0),M,N}\}_{M+N\geq 1}\|_{\xi,\mu} \leq C|g|,$$

for some $C > 0$, independent of $z \in \overline{D}_{1/2}$, $\theta \in R_\Theta$ and $|g| < g_0$.

Resonances and Spectral Localization

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We furthermore find that

$$\sigma(H_g(\theta)) \subset S_\theta,$$

where

$$S_\theta = \bigcup_{\ell=1}^{L} (E^{(\ell)} + e^{-\theta} C_\kappa).$$

Final Resolvent Estimate

Employing the resolvent estimate for $H^{(\ell)}_{(0)}[z]^{-1}$, together with Neumann expansions when $|z - e^{(\ell)}| > \rho_{(-1)}$, we find that for any $z \in \mathbb{C} \setminus S_\theta$:

$$\|(H_g(\theta) - z)^{-1}\| \leq \frac{C}{\mathrm{dist}(z, S_\theta)},$$

for some $C > 0$.

The Integration Contour

Recall the formula for the dilated group (now semigroup)

$$e^{-itH_g(\theta)} = \text{s-}\lim_{T\to+\infty} \frac{1}{2\pi i} \int_{[-T,T]+ic} e^{-itz}(H_g(\theta) - z)^{-1}\,\mathrm{d}z.$$

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Let

$$f(r) = \max\{\mathrm{Im}E^{(\ell)} - \sin(\Theta/4)|\mathrm{Re}E^{(\ell)} - r| \; : \; |\ell = 1, 2, \cdots, L\}$$

and define a curve

$$\gamma_{(\infty)}(r) = r + if(r).$$

Ideally we would like to deform the integration contour to run along the curve. Note that S_θ sits below the curve (except for the resonances), provided $\delta_0 << \underline{\rho}$, i.e., for small enough g_0.

Decapitated Key Hole Path

The path just chosen hits the resonances and will give rise to logarithmically divergent integrals. To avoid that, for a given $n \in \mathbb{N}_0$, we replace the part of the curve inside $E^{(\ell)} + \overline{D}_{\rho_{(-1)}\widetilde{\rho}_{(0)}\cdots\widetilde{\rho}_{(n-1)}/2}$, with a another curve running along the boundary up just above the resonance, cutting across at a distance of $O(\rho^{n+1})$ and back down again. We call these curves, shaped like a series of decapitated keyholes, $\gamma_{(n)}$.

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The modified segments, we call $\gamma_{(n)}^{(\ell)}$. Then

$$\Big| e^{-itH_g(\theta)} - \frac{1}{2\pi i} \sum_{\ell=1}^{L} \int_{\gamma_{(n)}^{(\ell)}} e^{-itz} (H_g(\theta) - z)^{-1} \, \mathrm{d}z \Big| \\ \leq C e^{t \max \mathrm{Im} E^{(\ell)}} |\ln(\rho^n)| e^{-\sin(\Theta/4)\rho^n t/5}.$$

Passing to Renormalized Free Dynamics

We now use the Feshbach-Schur reconstruction to replace $(H_g(\theta) - z)^{-1}$ with $H_{(n)}^{(\ell)}[\zeta]^{-1}$ (up to reconstruction corrections), where z and ζ related though multiple affine linear rescalings.

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$$\begin{aligned}
&\Big|\int_{\gamma^{(\ell)}_{(n)}} e^{-itz}(H_g(\theta)-z)^{-1}\,\mathrm{d}z \\
&\quad -\int_\sigma A^{(\ell)}_{(n)} e^{-it(e^{(\ell)}+\rho_{(-1)}e^{(\ell)}_{(n)}+\rho_{(-1)}\widetilde{\rho}_{(0)}\cdots\widetilde{\rho}_{(n-1)}\zeta)} \\
&\quad \times \chi_{\rho_{(-1)}\rho^n}\Gamma_{\rho_{(-1)}\rho^n} H^{(\ell)}_{(n)}[\zeta]^{-1}\Gamma^*_{\rho_{(-1)}\rho^n}\chi_{\rho_{(-1)}\rho^n}\,\mathrm{d}\zeta\Big| \\
&\quad \le C e^{t\mathrm{Im}E^{(\ell)}}\Big(|\ln(\rho)|e^{-\sin(\Theta/4)\rho^n/5}+\frac{\xi\delta_0}{\underline{\rho}}e^{t\rho^{n+1}}\Big),
\end{aligned}$$

where $A^{(\ell)}_{(n)} = |E'_{(0)}[z_{(0)}[0]]\cdots E'_{(n-1)}[z_{(n-1)}[0]]|$ (for each ℓ).

Renormalized Free Hamiltonian

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The error in doing so is:

$$\Big|\int_\sigma A^{(\ell)}_{(n)} e^{-it(e^{(\ell)}+\rho_{(-1)}e^{(\ell)}_{(n)}+\rho_{(-1)}\widetilde{\rho}_{(0)}\cdots\widetilde{\rho}_{(n-1)}\zeta)}\chi_{\rho_{(-1)}\rho^n}\Gamma_{\rho_{(-1)}\rho^n}$$
$$\times\Big(H^{(\ell)}_{(n)}[\zeta]^{-1} - (T^{(\ell)}_{(n)}[H_{\mathrm{ph}},\zeta] - E^{(\ell)}_{(n)}[\zeta])^{-1}\Big)\Gamma^*_{\rho_{(-1)}\rho^n}\chi_{\rho_{(-1)}\rho^n}\,\mathrm{d}\zeta\Big|$$
$$\leq Ce^{t\mathrm{Im}E^{(\ell)}}\frac{\xi\delta_0}{\underline{\rho}}e^{ct\rho^{n+1}}$$

Coup de Grace

Now that we are down to a function of H_{ph}, we may sandwich with a vacuum projection to arrive at $(T[0,\zeta] - E[\zeta])^{-1}$, which has a simply pole at the (rather an) approximate resonance position:

$$\frac{1}{2\pi i}\int_\sigma A^{(\ell)}_{(n)} e^{-it(e^{(\ell)}+\rho_{(-1)}e^{(\ell)}_{(n)}+\rho_{(-1)}\widetilde{\rho}_{(0)}\cdots\widetilde{\rho}_{(n-1)}\zeta)}$$

$$\times (T^{(\ell)}_{(n)}[0,\zeta] - E^{(\ell)}_{(n)}[\zeta])^{-1}\,\mathrm{d}\zeta$$

$$= A^{(\ell)}_{(n)} e^{-it(e^{(\ell)}+\rho_{(-1)}e^{(\ell)}_{(n)}+\rho_{(-1)}\widetilde{\rho}_{(0)}\cdots\widetilde{\rho}_{(n-1)}z_{(n)}[0])}$$

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&= A^{(\ell)}_{(n)} e^{-itE^{(\ell)}_{(n)}}.
\end{aligned}$$

where $E^{(\ell)}_{(n)} = e^{(\ell)} + \rho_{(-1)}e^{(\ell)}_{(n+1)}$ and

$$|E^{(\ell)} - E^{(\ell)}_{(n)}| \leq C\rho^{n+1}\frac{\delta_0}{\underline{\rho}} \qquad \text{and} \qquad |A^{(\ell)}_{(n)} - 1| \leq 16\delta_0.$$

Summing Up

Finally we have, for any n,

$$\Big\| \langle 0|e^{-itH_g}|0\rangle - \mathrm{diag}(A^{(1)}_{(n)}e^{-itE^{(1)}_{(n)}}, \dots, A^{(L)}_{(n)}e^{-itE^{(L)}_{(n)}}) \Big\| \\ \leq Ce^{t\max \mathrm{Im} E^{(\ell)}}\big(|\ln(\rho^n)|e^{-ct\rho^n} + \frac{\xi\delta_0}{\underline{\rho}}e^{ct\rho^{n+1}}\big).$$

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One my exploit this estimate, together with analyticity of the $E^{(\ell)}$'s as functions of g, to argue that

$$\mathrm{Im} E^{(\ell)} \leq 0,$$

for all $|g| \leq g_1$, for a $g_1 > 0$ sufficiently small.

Time Scales

Each non-real $E^{(\ell)}$ defines a timescale through its imaginary part

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where $a_{2k_\ell} < 0$.

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Levels with longer time scales give oscillatory contributions that should be factored out, whereas levels with shorter time scales die out.

Part IV

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To the Beaches!!