On the Preparation of States in Quantum Mechanics

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Introduction

How to prepare a QM system in a prescribed state?

Question

S quantum mechanical system. $\mathcal{A}_S$: C^*-algebra generated by all observables on S, ω_S positive linear functional on $\mathcal{A}_S$. How can we prepare the system S in the state ω_S?

Three main ways

Via

- Duplication of S and selection;
- Adiabatic evolution;
- Weak coupling of S with an environment E.

Duplication and Selection

Create n copies of S. Choose a selector (apparatus) to select only copies of S in ω_S. Example: send particles with spin 1/2 in a Stern and Gehrlach setup where $(\nabla B \cdot \vec{e}_z) \neq 0$. Select the upper or the lower beam. Throw out the rest.

Adiabatic evolution

- $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ = GNS-representation of $(\mathcal{A}_S, \omega)$,

$$\omega(a_S) = \langle \Omega_\omega, \pi_\omega(a_S)\Omega_\omega \rangle, \qquad \forall a_S \in \mathcal{A}_S.$$

  Build an experiment with unitary time evolution $U(t,t')$ on $\mathcal{H}_\omega$

$$\frac{d}{dt}U(t,t') = -iH(t/\tau)U(t,t'), \qquad U(t,t) = \mathbb{1}, \qquad \forall t, t' \in \mathbb{R}.$$

  $\tau > 0$: adiabatic time parameter. $H(t)^* = H(t)$, $H(0)\Omega_\omega = E(0)\Omega_\omega$.

- Setting $s = t/\tau$, $s' = t'/\tau$, $U_\tau(s,s') = U(\tau s, \tau s')$,

$$\frac{d}{ds}U_\tau(s,s') = -i\tau H(s)U_\tau(s,s').$$

Known results (see e.g Avron-Elgart[99], Teufel[01])

Set $\Omega(0) := \Omega_\omega$. If $E(s)$ is a non-degenerate e.v. of $H(s)$ for all $s \in [0,1]$, denote by $P(s) := |\Omega(s)\rangle\langle\Omega(s)|$ the projection onto 1D eigenspace corresponding to $E(s)$. Under some regularity assumptions on $H(s)$ and $P(s)$,

$$\sup_{s\in[0,1]} \|U_\tau(s,0)\Omega_\omega - \Omega(s)\| \underset{\tau\to\infty}{\longrightarrow} 0.$$

Weak coupling with a dispersive environment

- $\overline{S} = S \vee E$, E environment. $\mathcal{A}_{\overline{S}}$: C^*-algebra generated by observables on $\overline{S}$, $\mathcal{A}_S \subset \mathcal{A}_{\overline{S}}$: C^*-subalgebra generated by observables on S only.
- $\alpha_{t,s}$: family of *automorphisms on $\mathcal{A}_{\overline{S}}$ (time evolution), $\alpha_{t,s}\alpha_{s,s'} = \alpha_{t,s'}$.
- $\mathcal{S}_0$ (dense) subset of the set of states on $\overline{S}$, and ω_S the state to be reached. Asymptotic preparation in ω_S if

$$\lim_{t\to\infty} \omega(\alpha_{t,0}(a_S)) = \omega_S(a_S), \qquad \forall a_S \in \mathcal{A}_S, \tag{1}$$

for all $\omega \in \mathcal{S}_0$.

Drawback of each technique

- Duplication and selection: need to create many copies of S.
- Adiabatic evolution: initial state ω must be known, take a long time (asymptotic limit)
- Weak coupling with environment: take a long time (asymptotic limit)

The spin-boson Model

Description

Toy model for an atom (n-level system) coupled to the quantized e.m. field (or phonon field).

Hilbert space and Hamiltonian

- Hilbert space: $\mathcal{H} := \mathcal{H}_S \otimes \mathcal{F}_+(L^2(\mathbb{R}^3))$, $\dim(\mathcal{H}_S) = n$.
- Hamiltonian:

$$H(t) := H_0 + \lambda(t) H_I,$$

$$H_0 = \sum_{i=1}^{n} E_i P_i \otimes \mathbb{1}_E + \mathbb{1}_S \otimes \int_{\mathbb{R}^3} d^3k\, \omega(k) a^*(k) a(k) := H_S + H_E,$$

$$H_I := G \otimes (a(\phi) + a^*(\phi)).$$

- $\omega(k) = |k|$ for all $k \in \mathbb{R}^3$,
  $\lambda(t)$ is a positive monoton decreasing function,
  $E_1 < ... < E_n$, $\dim(\mathrm{Ran} P_i) = 1$, $P_i^* = P_i$, $\sum_{i=1}^n P_i = \mathbb{1}_S$,
  $\phi \in L^2(\mathbb{R}^3)$, $G^* = G$.

Assumptions

(A1) Initial state

$$\Psi = \varphi \otimes \Omega,$$

Ω vacuum Fock state, $\varphi \in \mathcal{H}_S$ unit ray.

Remark: Ω can be replaced by any following states: a finite particle state, a coherent state, the KMS state at temperature $T > 0$. φ can be replaced by a density matrix. One could also consider entangled states.

(A2) Decay of correlations

The form factor ϕ satisfies: $\phi, \phi/\omega^{1/2} \in L^2(\mathbb{R}^3)$. Setting

$$f(t) := \langle \Omega | \Phi(e^{i\omega t}\phi)\Phi(\phi)|\Omega\rangle = \int d^3k \, |\phi(k)|^2 e^{-it\omega(k)}, \qquad t \geq 0,$$

we assume that there exists a constant $\alpha > 2$ such that

$$|f(t)| \propto \frac{1}{(1+t)^{\alpha}}.$$

Assumptions

(A3) Fermi-Golden rules

For all $i \geq 2$,

$$\sum_{j=1}^{i-1} \int d^3k \; |G_{ij}|^2 |\phi(k)|^2 \delta(E_j - E_i + \omega(k)) > 0.$$

(A4) Evolution of coupling

There exists a constant

$$-1/2 < \gamma < 0,$$

such that

$$\lambda(t) = (\lambda(0)^{1/\gamma} + t)^{\gamma}.$$

Remarks

(A2) Time decay can be deduced from infrared behavior of $\phi(\vec{k})$. $\phi(\vec{k}) \sim |k|^{\mu}$ with $\mu > -1/2$.
(A3) Excited states of the atom turn into resonances.
(A4) Sort of adiabatic condition.

Main result

Theorem

Suppose that Assumptions (A1)-(A4) are satisfied. Then, there exists a constant $\lambda_c > 0$ such that, for any $0 < \lambda(0) < \lambda_c$,

$$\langle \Psi(t) | (O \otimes \mathbb{1}_E) \Psi(t) \rangle \underset{t \to \infty}{\longrightarrow} \mathrm{Tr}(P_1 O) \tag{2}$$

for all initial states Ψ as given in (A1) and for all observables $O \in \mathcal{B}(\mathcal{H}_S)$. P_1 is the projection onto the 1D-eigenspace of H_S corresponding to the eigenvalue E_1. $\Psi(t)$ is the time evolved state of Ψ under the dynamics generated by the family of Hamiltonians $H(t)$.

Strategy

- Mainly inspired by De Roeck-Kupiainen[13], but also De Roeck, Froehlich, Pizzo[10] : brute force method based on Dyson series expansion and resuming of perturbations.

Step 1: Analysis of reduced dynamics on the Van Hove time scale

- $U(t,s)$: unitary operators generated by the time dependent Hamiltonian $H(t)$. For any $O \in \mathcal{B}(\mathcal{H}_S)$, define $\langle O \rangle^{(t,s)} \in \mathcal{B}(\mathcal{H}_S)$ by

$$\langle \varphi | \langle O \rangle^{(t,s)} \psi \rangle := \langle \varphi \otimes \Omega | U^*(t,s)(O \otimes \mathbb{1}) U(t,s)(\psi \otimes \Omega) \rangle, \qquad \forall \varphi, \psi \in \mathcal{H}_S.$$

- On time scales $t - s \sim \lambda^{-2}(s)$, for weak coupling, reduced dynamics is close to a jump process (e.g. Davies[74], and Fermi golden rules). We find out the Lindblad operator generating this jump process. By perturbation theory for isolated non-degenerate eigenvalues, we can decompose

$$\langle \;\cdot\; \rangle^{(s+\tau\lambda^{-2}(s),s)} = R(s) + R'(s),$$

where $R(s) : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is a one-dimensional projection and $R'(s) : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is a perturbation.

Strategy

Step 2: Reduced dynamics at arbitrary large times

- Main idea: discretize time on the van Hove time scale. Set $t_i = t_{i-1} + \tau\lambda^{-2}(t_{i-1})$, $t_0 = 0$. Photons are scattered at infinity once emitted. Therefore, we expect the field memory effects to disappear for very large times:

$$\langle \ \cdot \ \rangle^{(t_N,0)} = R(0) \ldots R(t_{N-1}) + \underset{N\to\infty}{o}(1).$$

- Rigorous proof with Dyson series and cluster expansion. Brute force: nothing else than reorganizing sums and products of operators in a clever way. We will find a polymer set $\mathbb{G}_N^c$ (constructed from Feynman graphs) such that

$$\langle \Psi(t_N) | (O \otimes \mathbb{1}_E) \Psi(t_N) \rangle = \sum_{q=1}^{N} \frac{1}{q!} \sum_{\substack{\mathcal{G}_1,\ldots,\mathcal{G}_q \in \mathbb{G}_N^c \\ \mathrm{dist}(\mathcal{G}_i,\mathcal{G}_j)\geq 1,\ \exists i_0,\ \mathbf{I}_{N+1}\in\mathcal{V}(\mathcal{G}_{i_0})}} p(\mathcal{G}_1) \ldots p(\mathcal{G}_q)$$

  with scalar weights $p(\mathcal{G}) \in \mathbb{C}$. Possible to do so because the $R(t_i)$'s are 1D projections!

Strategy

Step 3: Take the limit $N \to \infty$

- Find an upper bound for the weights $p(\mathcal{G})$. Show absolute convergence of the cluster expansion using classical criteria [Cammarota,82] [Kotecky-Preiss,86] [Ueltschi,04].
- Rewrite $\langle \Psi(t_N) | (O \otimes \mathbb{1}_E) \Psi(t_N) \rangle$ with an exponential, and divide by $1 = \langle \Psi(t_N) | (\mathbb{1}_S \otimes \mathbb{1}_E) \Psi(t_N) \rangle$. Many terms cancel.
- Take the limit $N \to \infty$.

Analysis of the reduced dynamics on the Van Hove time scale

Strong convergence of the Dyson series

Lemma 1 [Well-known]

Let $t, s \in \mathbb{R}$. The Dyson series

$$U(t,s) = \sum_{k=0}^{\infty} (-i)^k \int_s^t du_k ... \int_s^{u_2} du_1 \lambda(u_1)...\lambda(u_k)\ e^{-itH_0} H_I(u_k)...H_I(u_1) e^{isH_0}, \quad (3)$$

converges strongly on $\mathcal{H}_S \otimes F(L^2(\mathbb{R}^3))$. $F(L^2(\mathbb{R}^3)) \subset \mathcal{F}_+(L^2(\mathbb{R}^3))$: dense subspace of finite particles vectors,
$H_I(u) = e^{iuH_0} H_I e^{-iuH_0}$.

Lemma 2 [Wick's theorem]

Ω quasi-free state: for $\Phi(f) = a(f) + a^*(f)$,

$$\langle \Omega | \Phi(\phi_{v_1})...\Phi(\phi_{v_{2k}})\Omega\rangle = \sum_{\text{pairings } \pi} \prod_{\substack{(i,j)\in\pi \\ i<j}} \langle \Omega | \Phi(\phi_{v_i})\Phi(\phi_{v_j})\Omega\rangle.$$

(=0 for 2k+1 terms)

Reorganization of the sum using Wick's theorem

- Plug the Dyson series into the r.h.s of

$$\langle\varphi|\langle O\rangle^{(t,s)}\psi\rangle := \langle\varphi\otimes\Omega|U^*(t,s)(O\otimes\mathbb{1})U(t,s)(\psi\otimes\Omega)\rangle$$

  and use Lemma 2.

- Example: term of order λ^2.

$$i^2\int_s^t du_1\int_s^t dv_1\ \lambda(u_1)\lambda(v_1)\langle e^{isH_S}\varphi\otimes\Omega|\Big(-H_I(u_1)O(t)H_I(v_1)$$

$$+O(t)H_I(u_1)H_I(v_1)\chi(u_1>v_1)+H_I(u_1)H_I(v_1)O(t)\chi(u_1<v_1)\Big)(e^{isH_S}\varphi\otimes\Omega)\rangle$$

$$=i^2\int_s^t du_1\int_s^t dv_1\ \lambda(u_1)\lambda(v_1)\langle e^{isH_S}\varphi|\Big(-G(u_1)O(t)G(v_1)$$

$$+O(t)G(u_1)G(v_1)\chi(u_1>v_1)+G(u_1)G(v_1)O(t)\chi(u_1<v_1)\Big)f(u_1-v_1)(e^{isH_S}\varphi)\rangle,$$

  where $f(u_1-v_1)=\langle\Omega|\Phi(\phi e^{iu_1\omega})\Phi(\phi e^{iv_1\omega})\Omega\rangle$.

- Introduce index $r\in\{0,1\}$ (write $u^{(r)}$) to take care whether $G(u)$ appears on the left or on the right of O. Define $\mathbf{M}(iG(u^{(0)}))O=iG(u)O$ and $\mathbf{M}(iG(u^{(1)}))O=-iOG(u)$.

Reorganization of the sum using Wick's theorem

- Taking also the signs into account, integrating only over $\chi(u_1 < v_1)$, get

$$\int_s^t du_1 \int_s^t dv_1 \; \lambda(u_1)\lambda(v_1)\chi(u_1 < v_1) \sum_{r_1, r_1' \in \{0,1\}} f(u_1^{(r_1)}, v_1^{(r_1')}) \langle e^{isH_S}\varphi| \Big(\mathbf{M}(iG(u_1^{(r_1)}))\mathbf{M}(iG(v_1^{(r_1')}))O(t)\Big)(e^{isH_S}\varphi)\rangle.$$

- Set $\mathbf{F}(u_1^{(r_1)}, v_1^{(r_1')}) = f(u_1^{(r_1)}, v_1^{(r_1')})\mathbf{M}(iG(u_1^{(r_1)}))\mathbf{M}(iG(v_1^{(r_1')}))$, and do this for all order of λ.

Lemma 3

$$e^{isH_S}\langle O\rangle^{(t,s)}e^{-isH_S} = \sum_{k=0}^{\infty} \int_{[s,t]^{2k}} d\mu(\underline{w}_k)\lambda(\underline{w}) \sum_{\underline{r}\in\{0,1\}^{2k}} \mathcal{T}_S \left[\prod_{i=1}^{k} \mathbf{F}(u_i^{(r_i)}, v_i^{(r_i')})\right][O(t)] \tag{4}$$

$w^{(r,r')} := (u^{(r)}, v^{(r')})$, $\underline{w}_k^{(\underline{r})} = (w_1^{(r_1,r_1')}, ..., w_k^{(r_k,r_k')})$,
$\lambda(\underline{w}_k) = \lambda(u_1)...\lambda(u_k)\lambda(v_1)...\lambda(v_k)$,
$d\mu(\underline{w}_k) = du_1 \; ... \; du_k \; dv_1 \; ... \; dv_k \; \chi(s < u_1 < ... < u_k < t) \prod_{i=1}^{k} \chi(u_i < v_i)$.

Comparison with a piecewise constant coupling

- $\langle O\rangle_0^{(t,s)}$: defined as $\langle O\rangle^{(t,s)}$, but with $\lambda(u)=\lambda(s)$ for all $u\in[s,t]$.

Lemma 4

If the Assumption (A4) is satisfied, then, for all $\epsilon>0$, there exists $\lambda_c(\epsilon,\tau)>0$ such that, for any $0<\lambda(0)<\lambda_c(\epsilon,\tau)$,

$$||\langle O\rangle^{(s+\tau\lambda^{-2}(s),s)}-\langle O\rangle_0^{(s+\tau\lambda^{-2}(s),s)}||\leq\epsilon\|O\|$$

for all $s\geq 0$.

- Sketch of the proof: Use the Dyson series (4). Get the bound

$$\frac{||\langle O\rangle^{(t,s)}-\langle O\rangle_0^{(t,s)}||}{\|O\|}\leq\sum_{k=1}^{\infty}\int_{[s,t]^{2k}}d\mu(\underline{w}_k)\ 4^k\left(\lambda^{2k}(s)-\lambda^{2k}(t)\right)\prod_{i=1}^{k}|f(v_i-u_i)|||G||^2$$

Integrate over the v_i's, then over the k-dimensional simplex, for all k. Get bound $e^{4\tau||f||_{L^1}||G||^2}-e^{4\tau\lambda^{-2}(s)||f||_{L^1}\ ||G||^2\lambda^2(t)}$ for $t-s=\tau\lambda^{-2}(s)$. Use (A4).

Comparison with a semigroup

- Set $\mathcal{L}_S := [H_S, \cdot]$, $\mathcal{L}_S : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$.

Lemma 5 Davies [74], De Roeck-Kupiainen[13]

There is a Linbladian $iM : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$, a constant $C > 0$ independent of λ and τ, such that

$$||\langle O \rangle_0^{(t,s)} - e^{i(t-s)\mathcal{L}_S + i(t-s)\lambda^2(s)M} O|| \leq C\|O\|\lambda^2(s) e^{C\lambda^2(s)(t-s)} |\ln(\lambda(s))|,$$

for all $t, s \geq 0$.

- Spectral properties of $\mathcal{L}_S$: 0 is n-fold degenerated eigenvalue, other eigenvalues: $E_i - E_j \in \mathbb{R}$, $i \neq j$.
- Spectral properties of iM (easy to prove by direct calculation): $[iM, \mathcal{L}_S] = 0$, 0 is a **non-degenerate** eigenvalue of iM, all other eigenvalues have a strictly negative real part. The projection $|\mathbb{1}_S\rangle\langle P_1|$ acting on $B(\mathcal{H}_S)$ satisfies $M|\mathbb{1}_S\rangle\langle P_1| = |\mathbb{1}_S\rangle\langle P_1|M = 0$. (Notation: $\langle A|B\rangle = \mathrm{Tr}(A^* B)$).

Spectral properties of the reduced evolution

Lemma 6

Let $0 < \epsilon_0 < 1$. There are positive constants $\tau(\epsilon_0)$ and $\lambda_c(\epsilon_0, \tau) > 0$ such that, for any $\tau > \tau(\epsilon_0)$ and for any $\lambda(0) < \lambda_c(\epsilon_0, \tau)$,

$$\langle \ \cdot \ \rangle^{(s+\tau\lambda^{-2}(s),s)} = R(s) + R'(s) \tag{5}$$

for all $s \geq 0$.

- $||R'(s)O|| \leq \epsilon_0 \|O\|$ and $R(s) = |\mathbb{1}_S\rangle\langle\Pi(s)|$ projects onto subspace spanned by the eigenvector $\mathbb{1}_S$ of $\langle \ \cdot \ \rangle^{(s+\tau\lambda^{-2}(s),s)}$.
- $[R(s), \langle \ \cdot \ \rangle^{(s+\tau\lambda^{-2}(s),s)}]$ and $R(s)R'(s) = R'(s)R(s) = 0$.
- $\Pi(s) \underset{s\to\infty}{\longrightarrow} P_1$.

- Sketch of the proof: perturbation theory for isolated non-degenerate eigenvalue. Use that

$$|\mathbb{1}_S\rangle\langle P_1| = -\frac{1}{2i\pi}\int_\Gamma \frac{1}{e^{i\tau\lambda(s)^{-2}\mathcal{L}_S+i\tau M} - z}dz,$$

Γ contains only e.v. 1.

Graphical representation and cluster expansion

A taste of cluster expansions

- Let $\mathbb{X}$ be a first-countable set. $x \in \mathbb{X}$ is called a polymer. Let $\xi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a symmetric function with the property that

$$|1 + \xi(x, y)| \leq 1, \qquad \forall x, y \in \mathbb{X}.$$

  For hardcore polymer models, ξ is used to encode an adjacency relation $\sim$, i.e. a symmetric and irreflexive binary relation. Then $\xi(x, y) = -1$ if $x \sim y$ and 0 otherwise.

- In statistical physics, look at the partition function

$$Z := 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{x_1, \dots, x_n \in \mathbb{X}} p(x_1) \dots p(x_n) \prod_{1 \leq i < j \leq n} (1 + \xi(x_i, x_j)) \,.$$

  The weights $p(x_i) \in \mathbb{C}$. Legitimate question: does the series absolutely converge?

- If it does, then consider graphs $g = (\mathcal{V}(g), \mathcal{L}(g))$, with $\mathcal{V}(g) \subseteq \mathbb{N}_n = \{1, \dots, n\}$.
  $\mathbf{G}(A)$: graphs g with $\mathcal{V}(g) = A \subseteq \mathbb{N}_n$
  $\mathbf{C}(A)$: connected graphs on A.

- Use that

$$\prod_{1\le i<j\le n}(1+\xi(x_i,x_j)) = \sum_{g\in\mathbf{G}(\mathbb{N}_n)}\prod_{(i,j)\in\mathcal{L}(g)}\xi(x_i,x_j).$$

- Then

$$Z = 1+\sum_{n\ge 1}\frac{1}{n!}\sum_{x_1,\dots,x_n\in\mathbb{X}} p(x_1)\dots p(x_n)\sum_{g\in\mathbf{G}(\mathbb{N}_n)}\prod_{(i,j)\in\mathcal{L}(g)}\xi(x_i,x_j)$$

$$= 1+\sum_{n\ge 1}\frac{1}{n!}\sum_{k=1}^{n}\frac{1}{k!}\sum_{A_1\cup\dots\cup A_k=\mathbb{N}_n}\ \sum_{x_1,\dots,x_n\in\mathbb{X}} p(x_1)\dots p(x_n)$$

$$\prod_{l=1}^{k}\Big(\sum_{g\in\mathbf{C}(A_l)}\prod_{(i,j)\in\mathcal{L}(g)}\xi(x_i,x_j)\Big)$$

- Write $p(x_{A_l}) := \prod_{m\in A_l} p(x_m)$. Count the number of partitions of N_n into k disjoint unions of m_l elements, get

$$Z = 1+\sum_{n\ge 1}\frac{1}{n!}\sum_{k=1}^{n}\frac{1}{k!}\sum_{m_1+\dots+m_k=n}\frac{n!}{m_1!\dots m_k!}\prod_{l=1}^{k}\Big(\sum_{x_{\mathbb{N}_{m_l}}} p(x_{\mathbb{N}_{m_l}})$$

$$\sum_{g\in\mathbf{C}(\mathbb{N}_{m_l})}\prod_{(i,j)\in\mathcal{L}(g)}\xi(x_i,x_j)\Big)$$

- Exchange sum over k and over n, get

$$Z = 1 + \sum_{k\geq 1} \frac{1}{k!} \prod_{l=1}^{k} \Big(\sum_{m_l \geq 1} \frac{1}{m_l!} \sum_{x_{\mathbb{N}_{m_l}}} p(x_{\mathbb{N}_{m_l}}) \sum_{g \in \mathbf{C}(\mathbb{N}_{m_l})} \prod_{(i,j)\in\mathcal{L}(g)} \xi(x_i, x_j) \Big)$$
$$= \exp\left(\sum_{m\geq 1} \frac{1}{m!} \sum_{x_1,\dots,x_m \in \mathbb{X}} p(x_1)\dots p(x_m) \sum_{g\in\mathbf{C}(\mathbb{N}_m)} \prod_{(i,j)\in\mathcal{L}(g)} \xi(x_i, x_j) \right).$$

- We have rewritten the partition function as the exponential of the "truncated" weights. Very useful, since we may write 1 with a similar cluster expansion (with a smaller set $\mathbb{X}$ or different weights p). $\Rightarrow$ Dividing by 1, many terms may disappear in the argument of the exponential!

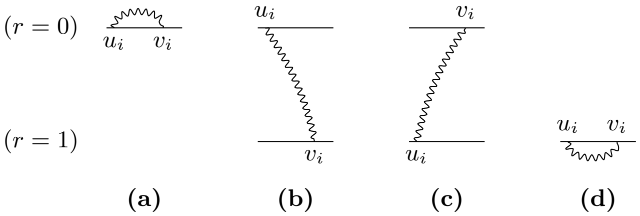
- How could we get such an expansion for our present problem? Product of operators in the Dyson series is non-commutative! Answer: because the projections $R(s)$ are 1D.

Dyson series and Feynman rules

- Back to the Dyson series

$$\langle O\rangle^{(t,0)} = \sum_{k=0}^{\infty} \int_{[0,t]^{2k}} d\mu(\underline{w}_k)\lambda(\underline{w}) \sum_{\underline{r}\in\{0,1\}^{2k}} \mathcal{T}_S \left[\prod_{i=1}^{k} \mathbf{F}(u_i^{(r_i)}, v_i^{(r'_i)})\right] [O(t)]$$

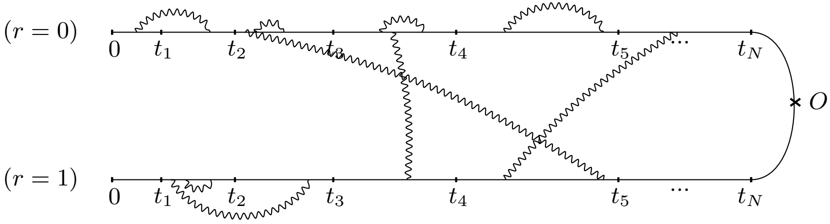
- Represent the contribution of each pairing $\underline{w}^{(\underline{r})}$ to the Dyson series via Feynman rules



- The contribution of **(a)**-**(d)** is just given by the corresponding $\mathbf{F}(u_i^{(r_i)}, v_i^{(r'_i)})$. For, e.g., **(a)** $= \lambda(u_i)\lambda(v_i)f(u_i - v_i)\mathbf{L}(iG(u_i))\mathbf{L}(iG(v_i))$.

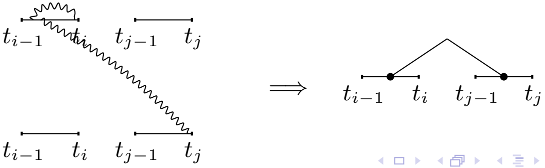
Dyson series and Feynman rules

- To each pairing corresponds a unique Feynman graph. Example:



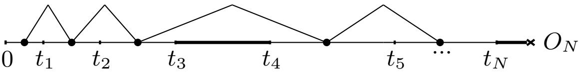
  where $t_{i+1} = t_i + \tau\lambda(t_i)^{-2}$, $t_0 = 0$.

- Idea: we want to reorganize the terms in the Dyson series such that the operators $\langle\cdot\rangle^{(t_{i+1},t_i)}$ appear explicitly. To do so: To each pairing $\underline{w}^{(\underline{r})}$, associate a graph $\mathcal{G}$ on the vertex set $\{I_1, ..., I_{N+1}\}$, where $I_i = [t_{i-1}, t_i[$, $i = 1, ..., N+1$.



Resuming: building graphs

- Last Feynman graph gives for instance



- Set of graphs obtained this way: $\mathbb{F}_N$. $\mathcal{G}$ in $\mathbb{F}_N$ is of the form $\mathcal{G} = (\{I_1, ..., I_{N+1}\}, \mathcal{L}(\mathcal{G}))$.
- $\underline{w}^{(\underline{r})}$ is **compatible** with the graph $\mathcal{G}$ if it generates $\mathcal{G}$ following the construction above. For a given $\mathcal{G}$, resum over all pairings compatible with $\mathcal{G}$.

$$\langle O \rangle^{(t_N,0)} = \sum_{\mathcal{G} \in \mathbb{F}_N} \langle O \rangle_{\mathcal{G}}^{(t_N,0)},$$

$$\langle O \rangle_{\mathcal{G}}^{(t_N,0)} := \sum_{k=0}^{\infty} \sum_{\underline{r} \in \{0,1\}^{2k}} \int_{[0,t]^{2k}} d\mu(\underline{w}_k) \lambda(\underline{w}) \ \mathbb{1}_{\mathcal{G}}(\underline{w}^{(\underline{r})}) \mathcal{T}_S \left[\mathbf{F}(\underline{w}^{(\underline{r})}) \right] [O_{t_N}] .$$

Resuming: isolated vertices

- We now fix a graph $\mathcal{G} \in \mathbb{F}_N$. Suppose that the vertex I_{i_0}, $i_0 \neq N+1$, is isolated. $\mathbb{1}_{\mathcal{G}}(\underline{w}^{(\underline{r})})$ selects the pairings $\underline{w}^{(\underline{r})}$ with property that $u_i^{(r_i)} \in I_{i_0}$ implies $v_i^{(r_i')} \in I_{i_0}$ (and vice-versa). We sum all these pairs.

$$\langle O \rangle_{\mathcal{G}}^{(t_N,0)} = \sum_{k=0}^{\infty} \sum_{\underline{r}\in\{0,1\}^{2k}} \sum_{m=0}^{k} \int_{[0,t_N]^{2k}} d\mu(\underline{w}_k)\lambda(\underline{w})\; \mathbb{1}_{\mathcal{G},\mathbf{I}_{i_0}}^{(m)}(\underline{w}^{(\underline{r})})\mathcal{T}_S\left[\prod_{i=1}^{k}\mathbf{F}(\underline{w}^{(\underline{r})})\right][O_{t_N}]$$

Splitting the integrals, we get that

$$\langle O \rangle_{\mathcal{G}}^{(t_N,0)} = \sum_{k=0}^{\infty} \sum_{\underline{\tilde{r}}\in\{0,1\}^{2k}} \int_{([0,t_N]\setminus I_{i_0})^{2k}} d\mu(\underline{\tilde{w}}_k)\lambda(\underline{\tilde{w}})\; \mathbb{1}_{\mathcal{G}\setminus\mathbf{I}_{i_0}}(\underline{\tilde{w}}^{(\underline{\tilde{r}})})$$

$$\mathcal{T}_S\left(\mathbf{F}(\underline{\tilde{w}}^{(\underline{\tilde{r}})}) \sum_{m=0}^{\infty} \sum_{\underline{\hat{r}}\in\{0,1\}^{2m}} \int_{I_{i_0}^{2m}} d\mu(\underline{\hat{w}}_m)\lambda(\underline{\hat{w}})\mathcal{T}_S\left[\mathbf{F}(\underline{\hat{w}}_m^{(\underline{\hat{r}})})\right]\right)[O_{t_N}].$$

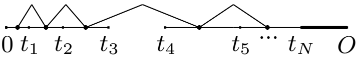
We recognize the operator $e^{it_{i_0-1}\mathcal{L}_S}\langle\cdot\rangle^{(t_{i_0},t_{i_0-1})}e^{-it_{i_0}\mathcal{L}_S}$ on the second line!

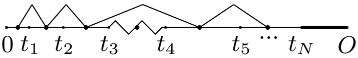
- Do this for all m isolated vertices of $\mathcal{G}$, and use the decomposition $\langle\cdot\rangle^{(t_i,t_{i-1})} = R_{t_i} + R'_{t_i}$. Get 2^m terms of the form

$$\langle O\rangle_{\mathcal{G}}^{(t_N,0)} = \sum_{k=0}^{\infty} \sum_{\underline{r}\in\{0,1\}^{2k}} \int_{([0,t_N]\setminus(I_{i_0}\cup\ldots\cup I_{i_m}))^{2k}} d\mu_k(\underline{w})\lambda(\underline{w})\ \mathbb{1}_{\mathcal{G}\setminus(\mathbf{I}_{i_0}\cup\ldots\cup\mathbf{I}_{i_m})}(\underline{w}^{(\underline{r})})$$

$$\mathcal{T}_S\left(\left(\prod_{l=0}^{m-1} e^{-it_{i_l}\mathcal{L}_S}\right)\mathbf{F}(\underline{w}^{(\underline{r})})\left(\prod_{l=0}^{m-1} e^{it_{i_l-1}\mathcal{L}_S}\right) R_{t_{i_0}}\ \ldots\ R'_{t_{i_{m-1}}}\right)[O_{t_N}].$$

- Outline the difference between getting an R or an R' by constructing 2^m new graphs from $\mathcal{G}$. Let the isolated interval blank if R, mark it with a snake if R'.





- Consider the set $\mathbb{G}_N$ of graphs $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{L}(\mathcal{G}))$, where $\mathcal{V}(\mathcal{G}) \subset \bigcup_{j=1}^{N+1} I_j$ and the lines $\mathcal{L}(\mathcal{G})$ join the center of intervals in $\mathcal{V}(\mathcal{G})$. The vertex I_{N+1} carries the operator $\mathbf{L}(O)$. Set $R'_{t_{N+1}} = \mathbf{L}(O)$. Then

$$\langle O\rangle^{(t_N,0)} = \sum_{\mathcal{G}\in\mathbb{G}_N, I_{N+1}\in\mathcal{V}(\mathcal{G})} \langle O\rangle_{\mathcal{G}}^{(t_N,0)}$$

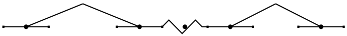
Decomposition into connected graphs

where

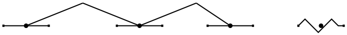
$$\langle O\rangle_{\mathcal{G}}^{(t_N,0)} := \sum_{k=0}^{\infty} \sum_{\underline{r}\in\{0,1\}^{2k}} \int_{[0,t_N]^{2k}} d\mu_k(\underline{w})\lambda(\underline{w})\ \mathbb{1}_{(\mathcal{B}(\mathcal{G}),\mathcal{L}(\mathcal{G}))}(\underline{w}^{(\underline{r})})$$

$$\mathcal{T}_S \left(\left(\prod_{l,\mathbf{I}_l\notin\mathcal{B}(\mathcal{G})} e^{it_{l-1}\mathcal{L}_S} e^{-it_l\mathcal{L}_S} \right) \mathbf{F}(\underline{w}^{(\underline{r})}) \prod_{i,\mathbf{I}_i\in\mathcal{R}'(\mathcal{G})} R'_{t_i} \prod_{j,\mathbf{I}_j\notin\mathcal{V}(\mathcal{G})} R_{t_j} \right) [\mathbb{1}_S].$$

- $\mathcal{G}\in\mathbb{G}_N$ is connected if any two vertices of $\mathcal{G}$ can be joined by a succession of edges or adjacent lines



A connected graph



A disconnected graph

- Decompose each graph into a disjoint union of non-adjacent connected subgraphs, $\mathcal{G}=\mathcal{G}_1\cup ....\cup\mathcal{G}_q$. Denote the set of connected graphs by $\mathbb{G}_N^c$.

Decomposition into connected graphs

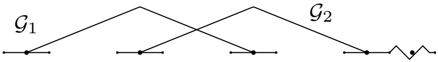
Lemma 7

Let $\varphi \in \mathcal{H}_S$. There is a function $p_\varphi : \mathbb{G}_N^c \to \mathbb{C}$ such that

$$\langle \varphi | \langle O \rangle_{\mathcal{G}}^{(t_N,0)} \varphi \rangle = p_\varphi(\mathcal{G}_1) \dots p_\varphi(\mathcal{G}_q)$$

for all $\mathcal{G} = \mathcal{G}_1 \cup \dots . \cup \mathcal{G}_q$, with $\mathcal{G}_i \in \mathbb{G}_N^c$ and $\text{dist}(\mathcal{G}_i, \mathcal{G}_j) \geq 1$, $i \neq j$.

- Example:



Decomposition into connected graphs

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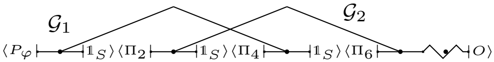

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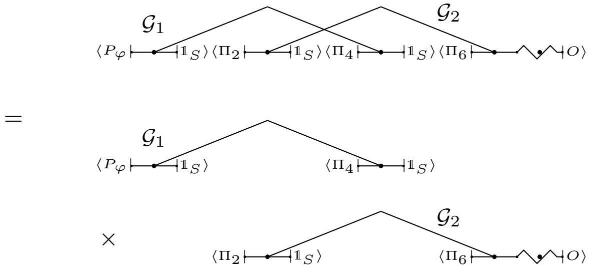


The cluster expansion

Lemma 8

$$\langle \Psi(t_N) | (O \otimes 1) \Psi(t_N) \rangle = \sum_{q=1}^{N} \frac{1}{q!} \sum_{\substack{\mathcal{G}_1,\ldots,\mathcal{G}_q \in \mathbb{G}_N^c \\ \mathrm{dist}(\mathcal{G}_i,\mathcal{G}_j)\geq 1,\ \exists i_0,\ \mathbf{I}_{N+1}\in\mathcal{V}(\mathcal{G}_{i_0})}} p_\varphi(\mathcal{G}_1) \ldots p_\varphi(\mathcal{G}_q), \quad (6)$$

$$\langle \Psi(t_N) | \Psi(t_N) \rangle = 1 + \sum_{q=1}^{N} \frac{1}{q!} \sum_{\substack{\mathcal{G}_1,\ldots,\mathcal{G}_q \in \mathbb{G}_N^c \\ \mathrm{dist}(\mathcal{G}_i,\mathcal{G}_j)\geq 1,\ \mathbf{I}_{N+1}\notin\mathcal{V}(\mathcal{G}_i)}} p_\varphi(\mathcal{G}_1) \ldots p_\varphi(\mathcal{G}_q). \quad (7)$$

Remark:

- The weights depend on initial state φ (and on O) but the bounds we derive afterwards are uniform in φ.
- (6) and (7) look pretty much the same. The only difference is that a vertex carrying the operator O must belong to one (and only one) of the connected graphs in (6).

The large time limit: convergence of the cluster expansion

The Kotecky-Preiss convergence criterion

- We come back to cluster expansions on a general first-countable set $\mathbb{X}$. What criterion can ensure absolute convergence of the cluster expansion?

Lemma 9, Kotecky-Preiss[86]

Assume that there exists a non negative function $a : \mathbb{X} \to \mathbb{R}_+$ such that

$$\sum_{x' \in \mathbb{X}} |p(x')||\xi(x,x')|e^{a(x')} \leq a(x) \qquad \forall x \in \mathbb{X}. \tag{8}$$

We also assume that $\sum_{x \in \mathbb{X}} |p(x)|e^{a(x)} < \infty$. Then the cluster expansion converges absolutely,

$$Z = \exp\Big(\sum_{n \geq 1} \frac{1}{n!} \sum_{x_1, \ldots, x_n \in \mathbb{X}} p(x_1)...p(x_n)\varphi^T(x_1, ..., x_n)\Big),$$

and, for all $x_1 \in \mathbb{X}$,

$$1 + \sum_{n \geq 2} \frac{1}{(n-1)!} \sum_{x_2, \ldots, x_n \in \mathbb{X}} |p(x_2)|...|p(x_n)| \; |\varphi^T(x_1, ..., x_n)| \leq e^{a(x_1)}.$$

KP criterion for our problem

- $\mathbb{X} = \mathbb{G}_N^c$, $\xi : \mathbb{G}_N^c \times \mathbb{G}_N^c \to \{-1, 0\}$, defined by

$$\xi(\mathcal{G}, \mathcal{G}') := \begin{cases} -1 & \text{if } \operatorname{dist}(\mathcal{G}, \mathcal{G}') = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the weights $p_\varphi(\mathcal{G})$.

Lemma 10

Let $\mathcal{G}$ in $\mathbb{G}_N^c$. Then

$$|p_\varphi(\mathcal{G})| \leq e^{4\tau ||f||_{L^1} \; ||G||^2 |\mathcal{B}(\mathcal{G})|} \Big(\prod_{\mathscr{L} \in \mathcal{L}(\mathcal{G})} \eta(\mathscr{L}) \Big) \epsilon_0^{|\mathcal{R}'(\mathcal{G})|}.$$

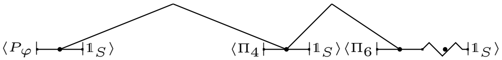
where

$$\eta((I_i, I_j)) := 4||G||^2 \int_{t_{i-1}}^{t_i} du \int_{t_{j-1}}^{t_j} dv |f(v-u)| \lambda(u) \lambda(v),$$

$$|\mathcal{B}(\mathcal{G})| := \text{number of vertices linked to others via edges},$$

$$|\mathcal{R}'(\mathcal{G})| := \text{number of vertices carrying perturbation } R'.$$

- Sketch of the proof: Take for instance



- Use that for a pairing $\underline{w}_k^{(r)}$ contributing to this graph, its contribution is bounded by

$$\left\| \langle P_\varphi | \mathcal{T}_S \left(\mathbf{F}(\underline{w}_k^{(r)}) e^{it_1 \mathcal{L}_S} R_{t_4} e^{-it_4 \mathcal{L}_S} e^{it_5 \mathcal{L}_S} R_{t_6} e^{-it_6 \mathcal{L}_S} e^{it_7 \mathcal{L}_S} R'_{t_8} e^{-it_8 \mathcal{L}_S} \right) [\mathbb{1}_S] \right\|.$$

For any $O \in \mathcal{B}(\mathcal{H}_S)$,

$$\|e^{\pm it\mathcal{L}_S} O\| \leq \|O\|,$$
$$\|R_t O\| \leq \|(R_t - |\mathbb{1}_S\rangle\langle P_1|)O\|| + \|\mathbb{1}_S \langle P_1 | O\rangle\| \leq 2\|O\|,$$
$$\|\mathbf{F}(\underline{w}_k^{(r)}) O\| \leq \|O\| \|G\|^{2k} \prod_{i=1}^{k} |f(u_i - v_i)| \lambda(u_i) \lambda(v_i).$$

- Get the upper bound $2^2 \epsilon_0 \|G\|^{2k} \prod_{i=1}^{k} |f(u_i - v_i)| \lambda(u_i)\lambda(v_i)$.
- Sort out two pairs corresponding to the two edges, and integrate them independently from the rest $\Rightarrow$ gives the upper bound

$$2^2 \epsilon_0 \ \eta((I_1, I_5)) \ \eta((I_5, I_7)) e^{3 \times 4\tau \|G\|^2}.$$

Summability and decay of the interaction

Lemma 11

Let $a : \mathbb{G}_N^c \to \mathbb{R}_+$ defined by $a(\mathcal{G}) = |\mathcal{B}(\mathcal{G})| + |\mathcal{M}\mathcal{R}'\mathcal{G}|$. Then

$$\sum_{\mathcal{G}' \in \mathbb{G}_N^c,\ \mathrm{dist}(\mathcal{G}', \mathcal{G}) = 0} |p_\varphi(\mathcal{G}')| \ e^{a(\mathcal{G}')} \leq a(\mathcal{G}).$$

- $|\mathcal{M}\mathcal{R}'\mathcal{G}|$ is the number of adjacent blocks of intervals carrying an operator R'.
- The function a is defined independently of N and φ.
- Main ingredient of the proof: the summability of the weights $\eta(\mathscr{L})$ of the edges. Because of the decay of the 2pt-correlation function, $f(t) \propto (1+t)^\alpha$, $\alpha < -2$,

$$\sum_{j \geq 0, j \neq i} [\lambda(t_{i-1})\lambda(t_{j-1})]^{-1} \ \eta(\mathbf{I}_i, \mathbf{I}_j) \leq C(\alpha)$$

for all $i \in \mathbb{N}$. $C(\alpha)$ depends on α but neither on N nor i.

- Sketch of the proof: we have that

$$\sum_{\mathcal{G}' \in \mathbb{G}_N^c,\ \mathrm{dist}(\mathcal{G}',\mathcal{G})=0} |p_\varphi(\mathcal{G}')|\ e^{a(\mathcal{G}')} \le |\mathrm{Adj}(\mathcal{G})| \sup_{I_i \in \mathrm{Adj}(\mathcal{G})} \sum_{\mathcal{G}' \in \mathbb{G}_N^c, I_i \in \mathcal{G}'} |p_\varphi(\mathcal{G}')|\ e^{a(\mathcal{G}')}.$$

- $|\mathrm{Adj}(\mathcal{G})| \le 2a(\mathcal{G})$.
- Case $|\mathcal{B}(\mathcal{G}')| = 0$.

$$\sum_{\mathcal{G}' \in \mathbb{G}_N^c, I_i \in \mathcal{G}', |\mathcal{B}(\mathcal{G}')|=0} |p_\varphi(\mathcal{G}')|\ e^{a(\mathcal{G}')} \le e\epsilon_0 \Big(1 + \sum_{n_1+n_2 \ge 1} \epsilon_0^{n_1+n_2}\Big) \sim \frac{\epsilon_0}{1-\epsilon_0}.$$

- Case $|\mathcal{B}(\mathcal{G}')| = |\mathcal{V}(\mathcal{G}')| = 2$. Then

$$\begin{aligned}\sum_{\mathcal{G}' \in \mathbb{G}_N^c, I_i \in \mathcal{G}', |\mathcal{B}(\mathcal{G}')|=|\mathcal{V}(\mathcal{G}')|=2} |p_\varphi(\mathcal{G}')| e^{a(\mathcal{G}')} &\le e^{8\tau \|f\| \|G\|^2 + 2} \sum_{j, j \neq i} \eta((I_i, I_j)) \\ &\le C(\alpha) [e^{4\tau \|f\| \|G\|^2 + 1} \lambda(0)]^2\end{aligned}$$

- General case for the graphs $\mathcal{G}'$ needs some more work, but idea essentially the same.

The large time limit

- We write
$$\frac{\langle \Psi(t_N) | (O \otimes 1) \Psi(t_N) \rangle}{1}$$
as the exponential of the truncated weights. We get that
$$\langle \Psi(t_N) | (O \otimes \mathbb{1}_E) \Psi(t_N) \rangle = \sum_{\mathcal{G} \in \mathbb{G}_N^c,\ \mathbf{I}_{N+1} \in \mathcal{V}(\mathcal{G})} p_\varphi(\mathcal{G})\ \omega(\mathcal{G}), \tag{9}$$
where $|\omega(\mathcal{G})| \leq e^{a(\mathcal{G})}$ for all $\mathcal{G}$. The scalar $\omega(\mathcal{G})$ does not depend on O.
- Dominant contribution to (9) : $p_\varphi(\{I_{N+1}\})$. Let us assume for the moment that $\sum_{\mathcal{G} \in \mathbb{G}_N^c,\ \mathbf{I}_{N+1}, \mathbf{I}_N \in \mathcal{V}(\mathcal{G})} p_\varphi(\mathcal{G})\ \omega(\mathcal{G}) \to 0$ when $N \to \infty$. Then we have, taking $O = \mathbb{1}_S$,
$$\langle \Psi(t_N) | (\mathbb{1}_S \otimes \mathbb{1}_E) \Psi(t_N) \rangle = p_\varphi(\{I_{N+1}\}) \omega(\{I_{N+1}\}) + \underset{N \to \infty}{o}(1) = 1!$$
Since $p_\varphi(\{I_{N+1}\}) = \langle \Pi(t_N) | \mathbb{1}_S \rangle = 1$, we get that $\omega(\{I_{N+1}\}) \to 1$ when $N \to \infty$. For general O,
$$p_\varphi(\{I_{N+1}\}) = \langle \Pi(t_N) | O \rangle \to \mathrm{Tr}(P_1 O).$$

The large time limit

Lemma 12

$$\sum_{\mathcal{G}\in\mathbb{G}_N^c,\ \mathbf{I}_{N+1},\mathbf{I}_N\in\mathcal{V}(\mathcal{G})} p_\varphi(\mathcal{G})\ \omega(\mathcal{G}) \underset{N\to\infty}{\longrightarrow} 0.$$

- Sketch of the proof:

  The graphs must contain the vertex I_N and must be connected. E.g., let us sum over all $\mathcal{G}$ with $|\mathcal{B}(\mathcal{G})| = 0$. For such graphs,

$$p_\varphi(\mathcal{G}) = \langle \Pi(t_{N-|\mathcal{V}(\mathcal{G})|})|R'_{t_{N-|\mathcal{V}(\mathcal{G})|+2}} \dots R'_{t_N} O\rangle.$$

  Consequently,

$$|p_\varphi(\mathcal{G})| \leq \begin{cases} ||O||\epsilon_0^{|\mathcal{V}(\mathcal{G})|-2} \|R_{t_{N-|\mathcal{V}(\mathcal{G})|+1}} R'_{t_{N-|\mathcal{V}(\mathcal{G})|+2}}\| & |\mathcal{V}(\mathcal{G})| \leq N, \\ ||O||\epsilon_0^N & |\mathcal{V}(\mathcal{G})| = N+1, \end{cases} \tag{10}$$

  and

$$\sum_{\substack{\mathcal{G}\in\mathbb{G}_N^c,\ \mathbf{I}_{N+1},\mathbf{I}_N\in\mathcal{V}(\mathcal{G}) \\ |\mathcal{B}(\mathcal{G})|=0}} |p(\mathcal{G})| e^{a(\mathcal{G})} \leq e||O||\Big(\sum_{k=2}^{N} \epsilon_0^{k-2}\ \|R_{t_{N-k+1}} R'_{t_{N-k+2}}\| + \epsilon_0^N\Big).$$

- $R_{t_i} R'_{t_{i+1}} = (R_{t_i} - R_{t_{i+1}}) R'_{t_{i+1}} + R_{t_{i+1}} (\langle \cdot \rangle^{(t_{i+1}, t_i)} - R_{t_{i+1}}) = (R_{t_i} - R_{t_{i+1}}) R'_{t_{i+1}}$. The sequence $(\|R_{t_i} - R_{t_{i+1}}\|)_i$ converges to zero.
- We also get that

$$\sum_{k=2}^{2N} \epsilon_0^{k-2} \, \|R_{t_{2N-k+1}} R'_{t_{2N-k+2}}\| \leq \max_{2 \leq j \leq N} \|R_{t_{2N-j+1}} R'_{t_{2N-j+2}}\| \frac{1}{1 - \epsilon_0} + \epsilon_0^N \sum_{k=2}^{N} \epsilon_0^{k-2} \, \|R_{t_{N-k+1}} R'_{t_{N-k+2}}\|. \underset{N \to \infty}{\longrightarrow} 0.$$

- The general case works similarly.