Radiative corrections to the binding energy for a spin $1/2$ charged particle

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Joint works with
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• “Quantitative estimates on the binding energy for Hydrogen in non-relativistic QED. II. The spin case.”, arXiv :1306 :4464 (2013)
J.-M. Barbaroux, S. W.

• “Contribution of the Spin-Zeeman term to the binding energy for hydrogen in non-relativistic QED”, Annals of the University of Bucharest (2013)
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• “On the ground state energy of the translation invariant Pauli-Fierz model. II.”, Documenta Mathematica (2012).
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• “Non-analyticity of the ground state energy of the Hamiltonian for Hydrogen atom in non-relativistic QED”, Journal of Physics A : Mathematical and Theoretical (2010)
J.-M. Barbaroux., S. W.

• “Quantitative estimates on the binding energy for Hydrogen in non-relativistic QED”, Annales Henri Poincaré (2010).
J.-M. Barbaroux., Thomas Chen, Vitali Vougalter, S.W.

• “On the ground state energy of the translation invariant Pauli-Fierz model”, Proc. Amer. Math. Soc., 136 (3), 1057-1064 (2008).
J.-M. Barbaroux., Thomas Chen, Vitali Vougalter, S.W.

• “Quantitative estimates on the enhanced binding for the Pauli-Fierz operator”, J. Math. Phys., vol. 46, no12 (2005).
J.-M. Barbaroux., Helmut Linde, S. W.

• “Binding conditions for atomic N-electron systems in non-relativistic QED”, Ann. Henri Poincaré. 4 (6), 1101 - 1136 (2003).
J.-M. Barbaroux., Thomas Chen, S. W.

1. Pauli-Fierz Operator
2. Ground state - Binding energy
3. Unitary transform - Self-Energy - Change of units
4. Preliminary results
5. Increase of the binding energy - spin case

We study the Hamiltonian in NRQED for an atom.

- System with 1 electron, described as quantum, *non relativistic*, pointwise particle with charge $-e$ and spin $\frac{1}{2}$
- The electron interacts with the quantized magnetic field
- One static pointwise nucleus, with positive charge - The electron interacts with the field of the nucleus via the Coulomb potential.
- One also study the free case ("self-energy" operator).

Hamiltonian

One electron interacting with a pointwise nucleus of charge e, $(Z = 1)$.

$$H_{\mathrm{p}} = -\Delta + V$$

Coulomb Potential : $V(x) = -\frac{e^2}{|x|}$
Electron mass $m = 1/2$; Planck constant $\hbar = 1$; velocity of light $c = 1$.
Fine structure constant : $\alpha = e^2 \approx 1/137$

Binding energy

Energy necessary to remove the electron to spatial infinity :

$$\Sigma(0) - \Sigma(V) = \inf \sigma(-\Delta) - \inf \sigma(-\Delta - \alpha/|x|).$$

- ▷ Coulomb uncertainty principle : $\int_{\mathbb{R}^3} \frac{1}{|x|} |\psi(x)|^2 \mathrm{d}x \leq \|\nabla\psi\| \, \|\psi\|$
- ▷

$$\langle \psi, (-\Delta + V)\psi \rangle \geq \|\nabla\psi\|^2 - \alpha \|\nabla\psi\| \, \|\psi\| = \underbrace{\left(\|\nabla\psi\|^2 - \frac{e^2}{2} \|\psi\| \right)^2}_{\geq 0} - \frac{\alpha^2}{4} \|\psi\|^2$$

- ▷ $\Sigma(V) = \inf \sigma(-\Delta + V) = -\frac{\alpha^2}{4}$.

Binding energy

$$\Sigma(0) - \Sigma(V) = \inf \sigma(-\Delta) - \inf \sigma(-\Delta + V) = \frac{\alpha^2}{4} \, .$$

Hamiltonian (Coulomb gauge) $N = 1$ electron

Coulomb potential case : Pauli-Fierz operator

$$H_{\text{PF}} = \underbrace{\left(-i\nabla_x \otimes \mathbb{I}_f + \sqrt{\alpha}\mathbb{A}(x)\right)^2}_{\text{kinetic energy}} - \underbrace{\frac{\alpha Z}{|x|} \otimes \mathbb{I}_f}_{\text{Coulomb electrostatic potential}}$$

$$+ (q-1) \underbrace{\sqrt{\alpha}\sigma \cdot \mathbb{B}(x)}_{\text{Zeeman term}} + \underbrace{\mathbb{I}_{el} \otimes H_f}_{\text{radiation field energy operator}}$$

Free case : "self-energy" operator

$$T_{\text{self.en.}} = \underbrace{(-i\nabla_x + \sqrt{\alpha}\mathbb{A}(x))^2}_{\text{kinetic energy}} + (q-1) \underbrace{\sqrt{\alpha}\sigma \cdot \mathbb{B}(x)}_{\text{Zeeman term}} + \underbrace{H_f}_{\text{Field energy}}$$

System of units : Electron mass $m = 1/2$; Planck const. $\hbar = 1$; speed of light $c = 1$; fine structure constant : $\alpha = e^2 \approx 1/137$

Hilbert space

$$\mathfrak{H} = \mathfrak{H}_{\text{part}} \otimes \mathfrak{F}_s$$

- $\mathfrak{H}_{\text{part}} = L^2(\mathbb{R}^3, \mathbb{C}^q)$: Hilbert space for $N = 1$ electron. $\mathbb{R}^3$ is configuration space, $\mathbb{C}^q$ for spin
  $q = 1$: “spinless” particule ; $q = 2$: electron (with spin)
- $\mathfrak{F}_s$: Bosonic Fock space

$$\mathfrak{F}_s = \Omega_f \mathbb{C} \oplus \underbrace{\bigoplus_{n=1}^{\infty} \bigotimes_s^n \left(\underbrace{L^2(\mathbb{R}^3, \mathbb{C}^2)}_{\text{one photon space (momentum variable} \times 2 \text{ polarizations transv.)}} \right)}_{\mathfrak{F}_s^{(n)} \ n\text{-photon space}}$$

- Vacuum : Ω_f.

- creation/annihilation operators : $a^*_\lambda(k)$, $a_\lambda(k)$. Fulfils C.C.R : $[a_\lambda(k), a^*_{\lambda'}(k)] = \delta_{\lambda,\lambda'}\delta(k-k')$, $[a^\sharp_\lambda(k), a^\sharp_{\lambda'}(k)] = 0$, $a_\lambda(k)\Omega_f = 0$
- Field energy :

$$H_f = \sum_{\lambda=1,2} \int \omega(k) a^*_\lambda(k) a_\lambda(k) \mathrm{d}k, \quad \omega(k) = |k|$$

$H_f = \bigoplus_n H_f{}^{(n)}$, $H_f\Omega_f = 0$,
$(H_f^{(n)}\Phi^{(n)})(k_1, k_2, \cdots, k_n) = \sum_{j=1}^n |k_j| \Phi^{(n)}(k_1, k_2, \cdots, k_n)$

- Photon number operator :

$$N_f = \sum_{\lambda=1,2} \int a^*_\lambda(k) a_\lambda(k) \mathrm{d}k$$

i.e., $(N_f\Phi)^{(n)}(k_1, \cdots, k_n) = n\Phi^{(n)}(k_1, \cdots, k_n)$.

- Magnetic vector potential : (Coulomb gauge)

$$\mathbb{A}(x) = \mathbb{A}^-(x) + \mathbb{A}^+(x)$$
$$= \sum_{\lambda=1,2} \int \frac{\chi_\Lambda(|k|)}{2\pi|k|^{\frac{1}{2}}} \epsilon_\lambda(k) \mathrm{e}^{ik.x} a_\lambda(k) \mathrm{d}k + \mathrm{h.c.}$$

- Polarization vectors : $\epsilon_\lambda(k)$, $\epsilon_1(k) \cdot \epsilon_2(k) = 0$, $k \cdot \epsilon_\lambda(k) = 0$.
- UV (Ultraviolet) cutoff : $\chi_\Lambda(|k|)$
- Coupling between electron and quantized magnetic field $\sqrt{\alpha}\sigma \cdot \mathbb{B}$, with $\mathbb{B} = \mathrm{Curl}\, \mathbb{A}$,

$$\mathbb{B}(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi_\Lambda(|k|)}{2\pi|k|^{1/2}} k \times i\varepsilon_\lambda(k) e^{ikx} a_\lambda(k) \mathrm{d}k + \mathrm{h.c.}\,,$$

and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, σ_i are 2×2 Pauli matrices.

$$H_{\mathrm{PF}} = (P - \sqrt{\alpha}\mathbb{A}(x))^2 + V + (q-1)\sqrt{\alpha}\sigma \cdot \mathbb{B}(x) + H_f, \quad V = -\frac{\alpha}{|x|}, \quad P = i\nabla_x$$

And also

$$H_{\mathrm{PF}} = H_{\mathrm{p}} + H_f + H_I(\alpha)$$

where

$$\begin{aligned} H_{\mathrm{p}} &= (-\Delta + V) \otimes \mathbb{I}_f \\ H_f &= \text{field energy operator} \\ H_I(\alpha) &= \text{interaction} \\ &= -2\mathrm{Re}\,\sqrt{\alpha}P.\mathbb{A}(x) + \alpha\mathbb{A}(x)^2 + \sqrt{\alpha}\sigma \cdot \mathbb{B}(x) \end{aligned}$$

Hamiltonian

- The Hamiltonian H_{PF} is self-adjoint, with domain $\mathcal{D}(H_{\mathrm{part.}} + H_f)$
- Stability of the first kind :

$$\inf \sigma(H_{\mathrm{PF}}) > -\infty$$

- Stability of the second kind : N electrons and M nuclei with charge Z_k $(k = 1, \ldots, M)$

$$\inf \sigma(H_{\mathrm{PF}}) \geq -C(\Lambda, \max\{Z_k\})\,(M + N)$$

Ground state

$\inf \sigma(H_{\mathrm{PF}})$ is an eigenvalue of multiplicity q .

Hamiltonian

$$H_{\text{PF}} = T + V \otimes \mathbb{I}_f \quad \text{sur} \quad \mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathfrak{F}$$

$$T = (-i\nabla_x \otimes \mathbb{I}_f + \sqrt{\alpha}\mathbb{A}(x))^2 + \mathbb{I}_{el} \otimes H_f - c_{\text{n.o.}}\alpha$$

Binding energy :

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) = \inf \sigma(T) - \inf \sigma(T + V)$$

Remark :

$$\begin{aligned} H_{\text{PF}} &= T + V \\ &= -\Delta_x + V + H_f + \underbrace{\left(-2\text{Re}\sqrt{\alpha}\mathbb{A}(x) \cdot i\nabla_x + \alpha\mathbb{A}(x)^2 - c_{\text{n.o.}}\alpha\right)}_{:=H_I(\alpha)} \end{aligned}$$

$$H_{\mathrm{PF}} = -\Delta_x + V + H_f + H_I(\alpha)$$

1. The free particle binds a larger quantity of (low-energetic) photons than the confined particle.
2. The binding energy should increase :

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) > \Sigma(0) - \Sigma(V) = \frac{\alpha^2}{4}$$

$$\mathcal{U} = \mathrm{e}^{iP_f \cdot x}$$

The e^- momentum variable is shifted by $P_f = \sum_\lambda \int k\, a_\lambda^*(k) a_\lambda(k) \mathrm{d}k$.
The photon "position" is shifted by x.

- $\mathcal{U}(i\nabla_x)\mathcal{U}^* = i\nabla_x - P_f$
  ($i\nabla_x$ acquires the meaning of the total momentum, i.e., momentum of particle + field).
- $\mathcal{U}\mathbb{A}(x)\mathcal{U}^* = \mathbb{A}(0)$ and $\mathcal{U}\mathbb{B}(x)\mathcal{U}^* = \mathbb{B}(0)$.
- $\mathcal{U}T\mathcal{U}^* = (P - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 + (q-1)\sigma.\mathbb{B}(0) + H_f - c_{\text{n.o.}}\alpha,$
  $P := i\nabla_x$.
- $\mathcal{U}(T+V)\mathcal{U}^* = \mathcal{U}\, H_{\text{PF}}\, \mathcal{U}^* =$
  $\underbrace{(P - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 + (q-1)\sigma.\mathbb{B}(0) + H_f}_{T} - \frac{\alpha}{|x|} - c_{\text{n.o.}}\alpha$

Equivalent Hamiltonian

$$H := \mathcal{U}(T+V)\mathcal{U}^*$$

$$= \underbrace{\left(P - P_f - \sqrt{\alpha}\mathbb{A}(0)\right)^2 + (q-1)\sigma.\mathbb{B}(0) + H_f}_{T} - \frac{\alpha}{|x|} - c_{n.o}\alpha$$

$$= \underbrace{(P^2 - \frac{\alpha}{|x|})}_{\text{Schrödinger operator}} + \underbrace{(P_f + \sqrt{\alpha}\mathbb{A}(0))^2 + (q-1)\sigma.\mathbb{B}(0) + H_f - c_{n.o}\alpha}_{\text{Self-Energy with total momentum } 0,\ T(0)}$$

$$-2\mathrm{Re}\, P.(P_f + \sqrt{\alpha}\mathbb{A}(0))$$

Self-energy at fixed total momentum :

The operator $T = -\Delta + H_f + H_I(\alpha)$ commutes with the total momentum P_{tot},

$$P_{\text{tot}} = (i\nabla_x \otimes \mathbb{I}_f) + (\mathbb{I}_{\text{el}} \otimes P_f), \quad P_f = \sum_\lambda \int k\, a_\lambda^*(k) a_\lambda(k) \mathrm{d}k$$

$$T \simeq \int_{\mathbb{R}^3}^{\oplus} T(p)\mathrm{d}p \quad T(p) \text{ acting on } \mathfrak{H}_0 \simeq \mathbb{C}^q \otimes \mathfrak{F}$$

Theorem ([F'74], [CF'07], [C'08])

$$\inf \sigma(T(0)) = \inf \sigma(T) = \Sigma_\alpha(0)$$

$\Sigma_\alpha(0)$ *is an eigenvalue of* $T(0)$: $T(0)\Psi_0^{GS} = \Sigma_\alpha(0)\Psi_0^{GS}$.

$$H_{\mathrm{PF}} = (i\nabla_x - \sqrt{\alpha}\mathbb{A}(x))^2 + (q-1)\sigma.\mathbb{B}(x) + H_f - \frac{\alpha}{|x|} - c_{n.o}\alpha$$

Change of variables (change of units)

$$\begin{cases} X &= \alpha x \\ K &= \frac{1}{\alpha^2}k \end{cases}$$

For $\mathcal{W}$ the associated unitary transform,

$$\mathcal{W}\, H_{\mathrm{PF}}\, \mathcal{W}^* = \alpha^2 H_{\mathrm{BFS}}$$

$$H_{\mathrm{BFS}} = (i\nabla_X - \alpha^{\frac{3}{2}}\mathbb{A}(\alpha X))^2 + (q-1)\alpha^{\frac{3}{2}}\sigma.\mathbb{B}(\alpha X) + H_f - \frac{1}{|X|} - c_{no}\alpha.$$

UV cutoff ~ 1 in "BFS" units implies UV cutoff $\sim \alpha^2$ in atomic units.

Increase of the binding energy - some rigorous results

- [HiSp'01] (large α, dipol approximation)
- [HVV'02] (small α)
- [HS'03] ($q = 2$, αZ fixed)
- [CVV'03] : $\Sigma_\alpha(0) - \Sigma_\alpha(V) > \alpha^2/4$ (α small, $q = 1$ or 2).
- [HHSp'05] : $\Sigma_\alpha(0) - \Sigma_\alpha(V)$ up to the order α^3, with error $\mathcal{O}(\alpha^{\frac{7}{2}} \log \alpha)$ (scalar boson, $q = 1$)
- [BFP'06] : Expansion in α for $\inf \sigma(H_\alpha)$ and its associated ground state, for arbitrary N

$$\inf \sigma(H_{\mathrm{BFS}}) = \epsilon_0 + \sum_{k=1}^{2N} \epsilon_k(\alpha)\alpha^{k/2} + o(\alpha^N)$$

"The quantity $\inf \sigma(H_{\mathrm{BFS}})$ is not analytic nor even smooth at $\alpha = 0$, but its derivatives of sufficiently high order diverge as $\alpha \to 0$"

- [GH'09, HH'11] : Analyticity in α of the ground state energy $\inf \sigma(H_{\mathrm{BFS}})$ ($q = 1$).

Theorem [BCVV'10]

$$\Sigma_\alpha(0) - \Sigma_\alpha(V)$$

$$= \underbrace{\frac{\alpha^2}{4}}_{\Sigma(0)-\Sigma(V)} + \underbrace{e^{(3)}\alpha^3}_{\text{increase of bind. energy}} + e^{(4)}\alpha^4 + e^{(5)} \underbrace{\alpha^5 \log\alpha}_{\text{infrared divergence}} + o(\alpha^5\log\alpha)$$

$$e^{(3)} = \frac{2}{3\pi}\int_0^\infty \frac{\chi_\Lambda^2(t)}{1+t}\,\mathrm{d}t > 0$$

$$e^{(4)} = \frac{1}{6}\langle \mathbb{A}^-(0)(H_f + P_f^2)^{(-1)}\mathbb{A}^+(0)\cdot\mathbb{A}^-(0)\Omega_f, (H_f + P_f^2)^{-1}\Omega_f\rangle$$

$$+\frac{1}{12}\sum_{i=1}^3 \|(P_f^2 + H_f)^{-\frac{1}{2}} P_f^i (P_f^2 + H_f)^{-1}\mathbb{A}^+(0)\cdot\mathbb{A}^+(0)\Omega_f\|^2 - \frac{1}{2}\|\mathbb{A}^-(0).(H_f + P_f^2)^{-1}\mathbb{A}^+(0)\Omega_f\|^2$$

$$+4a_0^2\|(-\Delta - \frac{1}{|x|} + \frac{1}{4})^{-\frac{1}{2}} Q^\perp \Delta f_1\|^2, \quad a_0 = \int \frac{k_1^2+k_2^2}{4\pi^2|k|^2}\frac{1}{|k|^2+|k|}\chi_\Lambda(|k|)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

$$e^{(5)} = \frac{4}{\pi}\|(-\Delta - \frac{1}{|x|} + \frac{1}{4})^{\frac{1}{2}}\nabla f_1\|^2 \neq 0$$

Self-energy (Translationnaly invariant case) [BCVV'08]

$$\Phi_2 := -(H_f + P_f^2)^{-1}\mathbb{A}^+(0)\cdot\mathbb{A}^+(0)\Omega_f\ ,$$
$$\Phi_3 := -(H_f + P_f^2)^{-1}P_f\cdot\mathbb{A}^+(0)\Phi_2\ ,$$
$$\Phi_1 := -(H_f + P_f^2)^{-1}P_f\cdot\mathbb{A}^-(0)\Phi_2\ .$$

$$\Theta_0^{\rm trial} := \Omega_f + \alpha\Phi_2 + 2\alpha^{\frac{3}{2}}\Phi_1 + 2\alpha^{\frac{3}{2}}\Phi_3$$

Then

$$\Sigma_\alpha(0) = \inf\sigma(T) = \inf\sigma(T(0))$$
$$= \langle\Theta_0^{\rm trial}, T(0)\Theta_0^{\rm trial}\rangle + \mathcal{O}(\alpha^4)$$
$$= -\alpha^2\|\Phi_2\|_*^2 + \alpha^3(2\|\mathbb{A}^-(0)\Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2) + \mathcal{O}(\alpha^4)\ ,$$

where $\langle\phi,\ \psi\rangle_* = \langle\phi,\ (H_f + P_f^2)\psi\rangle$.
For the true GS $\Psi_0^{\rm GS}$ of $T(0)$, $\Psi_0^{\rm GS} = \Theta_0^{\rm trial} + R_0$, with $\|R_0\| = \mathcal{O}(\alpha)$, $\|R_0\|_* = \mathcal{O}(\alpha^2)$.

Ground state energy of $T(0)$ - improved estimate [BV'11]

We have

$$\Sigma_\alpha(0) \;=\; d^{(2)}\alpha^2 \;+\; d^{(3)}\alpha^3 \;+ d^{(4)}\alpha^4 \;+\; \mathcal{O}(\alpha^5)\;,$$

with

$$d^{(2)} \;:=\; -\|\Phi_2\|_*^2, \quad d^{(3)} \;:=\; 2\|A^-\Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2$$

$$d^{(4)} \;:=\; -\left(\frac{2\|A^-\Phi_2\|^2 - 4\|\Phi_1\|_*^2 - 4\|\Phi_3\|_*^2}{\|\Phi_2\|_*}\right)^2$$

$$+8\Re\langle\Phi_1, A^- \cdot A^-\Phi_3\rangle + 8\|A^-\Phi_1\|^2 + 8\|A^-\Phi_3\|^2 - 16\|\tilde{\Phi}_2\|_*^2 - 16\|\Phi_4\|_*^2 + \|\Phi_2\|^2\|\Phi_2\|_*^2\;,$$

$$\Phi_2 := -(H_f + P_f^2)^{-1}A^+ \cdot A^+\Omega_f, \quad \Phi_3 := -(H_f + P_f^2)^{-1}P_f \cdot A^+\Phi_2\;, \quad \Phi_1 := -(H_f + P_f^2)^{-1}P_f \cdot A^-\Phi_2\;,$$

$$\tilde{\Phi}_2 := -P_{\Phi_2}{}^{\perp}(H_f + P_f^2)^{-1}\left(P_f \cdot A^+\Phi_1 + P_f \cdot A^-\Phi_3 + \frac{1}{2}A^+ \cdot A^-\Phi_2\right)$$

$$\Phi_4 := -(H_f + P_f^2)^{-1}\left(P_f \cdot A^+\Phi_3 + \frac{1}{4}A^+ \cdot A^+\Phi_2\right)\;,$$

where $P_{\Phi_2}{}^{\perp}$ is the orthogonal projection onto $\{\varphi \in \mathfrak{F} \mid \langle\varphi,\, \Phi_2\rangle_* = 0\}$.

Corollary

The ground state energy $\Sigma_\alpha(V)$ of H fulfils

$$\Sigma_\alpha(V) = \tilde{d}^{(2)}\alpha^2 + \tilde{d}^{(3)}\alpha^3 + \tilde{d}^{(4)}\alpha^4 + \tilde{d}^{(5)}\alpha^5 \log\alpha + o(\alpha^5 \log\alpha)\,.$$

- Remaining difficulties from spinless case
  - Standard Kato's perturbation theory is useless
  - α-dependence in V (Coulomb potential) and in the interaction term $H_I(\alpha)$
  - The magnetic potential $\mathbb{A}(x)$ contains a frequency space singularity $1/|k|^{\frac{1}{2}}$: Infrared divergence problem

- Additional difficulties
  - Additional term $\sqrt{\alpha}\sigma.\mathbb{B}(x)$ with a priori "high order"
  - Twice degenerate ground state for T and H
  - Weaker (but optimal) photon number estimate on ground states for T and H.

$$\langle \Psi^{\mathrm{GS}}, N_f \, \Psi^{\mathrm{GS}} \rangle \leq c\alpha \, .$$

Theorem [BV'12] - Self-energy

$$T = (i\nabla_x - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 + \sqrt{\alpha}\sigma\cdot\mathbb{B}(0) + H_f - c_{\text{n.o.}}\alpha$$

$$\Sigma_\alpha(0) = \inf\sigma(T) = \inf\sigma(T(0))$$
$$= -\alpha\|\Gamma_1\|_*^2 + -\alpha^2(2\|A^-\Gamma_1\|^2 - \|\Gamma_2\|_*^2 + \|\Gamma_1\|_*^2\|\Gamma_1\|^2) + \mathcal{O}(\alpha^3)$$

Ground state of $T(0)$:

$$\Psi_0^{\text{GS}} = \Omega_f + \sqrt{\alpha}\Gamma_1^{(a,b)} + \alpha\Gamma_2^{(a,b)} + R$$

$$\Gamma_1^{(a,b)} = -(H_f + P_f^2)^{-1}\sigma.\mathbb{B}^+(0)\Omega_f \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\Gamma_2^{(a,b)} = -(H_f + P_f^2)^{-1}[\sigma.\mathbb{B}^+(0)\Gamma_1^{(a,b)} + 2\mathbb{A}^+(0).P_f\Gamma_1^{(a,b)} + \mathbb{A}(0)^+.\mathbb{A}(0)^+\Omega_f \begin{pmatrix} a \\ b \end{pmatrix}]$$

Remark : Photon number estimate $\langle\Psi_0^{\text{GS}}, N_f\Psi_0^{\text{GS}}\rangle = \mathcal{O}(\alpha)$ (instead of $\mathcal{O}(\alpha^2)$ in the spinless case).

Theorem [BV'13 (ArXiv)]

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) = \inf \sigma(T(0)) - \inf \sigma(H)$$
$$= \frac{1}{4}\alpha^2 + (e^{(3)} + e^{(3),\text{Zeeman}})\alpha^3 + \mathcal{O}(\alpha^{3+\frac{1}{3}})$$

with

$$e^{(3)} = \frac{2}{3\pi}\int_0^\infty \frac{\chi_\Lambda^2(t)}{1+t}\mathrm{d}t > 0$$
$$e^{(3),\text{Zeeman}} = \frac{2}{3\pi}\int_0^\infty \frac{t^2\chi_\Lambda^2(t)}{(1+t)^3}\mathrm{d}t > 0$$

- No α term in the binding energy (as expected)
- An additional α^3 term due to the Zeeman term in H compared to the spinless particle case

The general strategy follows the one of [BCVV'10] for the spinless case :

Iterative procedure

Its implementation is however very different.

Step 0 : The trial state $\Psi^{\text{trial}} = \Psi_0^{\text{GS}} f_\alpha$ yields :

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) \geq \alpha^2/4$$

- Let f_α be the GS of $(-\Delta - \alpha/|x|)$, with e.v. $-e_0 = -\alpha^2/4$.
- Let Ψ_0^{GS} be the G.S. of the operator $T(0) = \left(-P_f - \sqrt{\alpha}\mathbb{A}(0)\right)^2 + \sqrt{\alpha}\mathbb{B}(0) + H_f$
- Normalized trial state : $\Theta^{\text{trial}} = f_\alpha(x)\Psi_0^{\text{GS}} \in L^2(\mathbb{R}^3) \otimes \mathfrak{F}$

$$\Sigma_\alpha(V) \leq \langle \Theta^{\text{trial}}, \underbrace{H}_{\mathcal{U}(T+V)\mathcal{U}^*} \Theta^{\text{trial}} \rangle$$

$$= \langle \Theta^{\text{trial}}, \left(: (P - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 : + \sqrt{\alpha}\mathbb{B}(0) + H_f - \frac{\alpha}{|x|} \right) \Theta^{\text{trial}} \rangle$$

$$= \underbrace{\langle (P^2 - \frac{\alpha}{|x|}) f_\alpha \Psi_0^{\text{GS}}, f_\alpha \Psi_0^{\text{GS}} \rangle}_{-\alpha^2/4} + \underbrace{\langle f_\alpha \Psi_0^{\text{GS}}, T(0) f_\alpha \Psi_0^{\text{GS}} \rangle}_{\Sigma_\alpha(0)}$$

$$- \underbrace{2\text{Re}\,\langle P \cdot (P_f + \sqrt{\alpha}\mathbb{A}(0)) f_\alpha \Psi_0^{\text{GS}}, f_\alpha \Psi_0^{\text{GS}} \rangle}_{0 \text{ by sym. } \langle \partial/\partial x_i f_\alpha, f_\alpha \rangle = 0}$$

To derive sharp upper and lower bound, we "perturb" around $\Psi_0^{\mathrm{GS}} f_\alpha$.

Step 1 : Lower bound on $\Sigma_\alpha(0) - \Sigma_\alpha(V)$ by choosing a "good" trial function :

$$\Psi^{\mathrm{trial}} = \Psi_0^{\mathrm{GS}} f_\alpha + \alpha^{\frac{1}{2}} 2P \cdot P_f (H_f + P_f^2)^{-1} \Gamma_1 f_\alpha + \alpha^{\frac{1}{2}} 2(H_f + P_f^2)^{-1} P \cdot A^+ \Omega_f \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Psi_0^{\mathrm{GS}} = \Omega_f + \sqrt{\alpha}\Gamma_1^{(1,0)} + \alpha\Gamma_2^{(1,0)} + R$$

$$\Gamma_1^{(1,0)} = -(H_f + P_f^2)^{-1} \sigma.\mathbb{B}^+(0)\Omega_f \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Gamma_2^{(1,0)} = -(H_f + P_f^2)^{-1} [\sigma.\mathbb{B}^+(0)\Gamma_1^{(1,0)} + 2\mathbb{A}^+(0).P_f\Gamma_1^{(1,0)} + \mathbb{A}(0)^+.\mathbb{A}(0)^+\Omega_f \begin{pmatrix} 1 \\ 0 \end{pmatrix}]$$

Step 2 : Photon number bound estimates :

$$\langle \Psi^{\mathrm{GS}}, N_f \Psi^{\mathrm{GS}} \rangle = \cancel{\mathcal{O}(\alpha^2)} \quad \mathcal{O}(\alpha)$$

Proposition

Let $K = (i\nabla - \sqrt{\alpha}A(x))^2 + \sqrt{\alpha}\sigma \cdot B(x) + H_f - \frac{\alpha}{|x|}$ *, be the Pauli-Fierz operator defined without normal ordering. Let* $\Psi^{GS} \in T(0)$ *be a ground state of K,*

$$K\,\Psi^{GS} = E\,\Psi^{GS} \,,$$

normalized by

$$\|\,\Psi^{GS}\,\| = 1 \,.$$

Let

$$N_f := \sum_{\lambda=1,2} \int a_\lambda^*(k)\, a_\lambda(k)\, \mathrm{d}k$$

denote the photon number operator. Then, there exists a constant c independent of α*, such that for any sufficiently small* $\alpha > 0$*, the estimate*

$$\langle \Psi^{GS} \,,\, N_f\, \Psi^{GS} \rangle \leq c\,\alpha \qquad (1)$$

is satisfied.

Proof. The strategy of the proof of Proposition 1 is similar to the proof of the photon number bound in [BCVV, 2010]. However, due to the occurrence of the spin-Zeeman term $\sqrt{\alpha}\sigma.B$, the photon number is one order larger in powers of the fine structure constant.
If we denote by $\tilde{H} \;:=\; : K :$ the operator K with normal ordering, we have

$$
\begin{aligned}
&\|(i\nabla - \sqrt{\alpha}A(x))\Psi^{\mathrm{GS}}\|^2 \\
&= \langle \Psi^{\mathrm{GS}}, K\Psi^{\mathrm{GS}}\rangle + \langle \Psi^{\mathrm{GS}}, \frac{\alpha}{|x|}\Psi^{\mathrm{GS}}\rangle - \langle \Psi^{\mathrm{GS}}, H_f\Psi^{\mathrm{GS}}\rangle \\
&\quad - 2\Re\sqrt{\alpha}\langle \Psi^{\mathrm{GS}}, \sigma \cdot B^{-}(x)\Psi^{\mathrm{GS}}\rangle \\
&\le \langle \Psi^{\mathrm{GS}}, \tilde{H}\Psi^{\mathrm{GS}}\rangle + c_{\mathrm{n.o.}}\alpha + \langle \Psi^{\mathrm{GS}}, \frac{\alpha}{|x|}\Psi^{\mathrm{GS}}\rangle.
\end{aligned}
$$

Since

$$\Re\langle\phi,\,(i\nabla-\sqrt{\alpha}A(x))^2-\alpha/|x|+H_f+2\sqrt{\alpha}\sigma\cdot B^-(x)\phi\rangle=\langle\phi,\,\tilde{H}\phi\rangle+c_{\mathrm{n.o.}}\alpha\|\phi\|^2\,,$$

and since from [GLL] we have, for α small enough

$$-\langle\Psi^{\mathrm{GS}},\,H_f\Psi^{\mathrm{GS}}\rangle-2\Re\sqrt{\alpha}\langle\Psi^{\mathrm{GS}},\,\sigma\cdot B^-(x)\Psi^{\mathrm{GS}}\rangle\leq 0\,.$$

Now we have, with $P := i\nabla$, and the inequality $-2\Re P.A^-(x) \geq -P^2 - (A^-(x))^2$,

$$
\begin{aligned}
0 &\geq \langle \Psi^{\mathrm{GS}}, \tilde{H}\, \Psi^{\mathrm{GS}} \rangle \\
&= \Re\Big\langle \Psi^{\mathrm{GS}}, \Big[-\Delta - 4\sqrt{\alpha} P \cdot A^-(x) + \alpha : A(x) :^2 \\
&+ 2\sqrt{\alpha}\sigma \cdot B^-(x) + H_f - \frac{\alpha}{|x|} \Big] \Psi^{\mathrm{GS}} \Big\rangle \\
&\geq -2\langle \Psi^{\mathrm{GS}}, \frac{\alpha}{|x|} \Psi^{\mathrm{GS}} \rangle + \langle \Psi^{\mathrm{GS}}, (1 - 8\sqrt{\alpha})(-\Delta)\Psi^{\mathrm{GS}} \rangle \\
&+ 8\sqrt{\alpha}\langle \Psi^{\mathrm{GS}}, -\Delta \Psi^{\mathrm{GS}} \rangle - 2\sqrt{\alpha}\langle \Psi^{\mathrm{GS}}, P^2 \Psi^{\mathrm{GS}} \rangle - 2\sqrt{\alpha}\langle \Psi^{\mathrm{GS}}, (A^-(x))^2 \Psi^{\mathrm{GS}} \rangle \\
&+ \frac{1}{4}\langle \Psi^{\mathrm{GS}}, H_f \Psi^{\mathrm{GS}} \rangle + \frac{1}{2}\langle \Psi^{\mathrm{GS}}, H_f \Psi^{\mathrm{GS}} \rangle + 2\langle \Psi^{\mathrm{GS}}, \alpha : (A^-(x))^2 : \Psi^{\mathrm{GS}} \rangle \\
&+ \frac{1}{4}\langle \Psi^{\mathrm{GS}}, H_f \Psi^{\mathrm{GS}} \rangle + 2\sqrt{\alpha}\Re\langle \Psi^{\mathrm{GS}}, \sigma \cdot B^-(x) \Psi^{\mathrm{GS}} \rangle + \langle \Psi^{\mathrm{GS}}, \frac{\alpha}{|x|} \Psi^{\mathrm{GS}} \rangle .
\end{aligned}
$$

In the right hand side, the sum of the first and the second term is bounded below by $-c\alpha^2$ for some constant $c > 0$, the sum of the third and fourth term is positive, and the sum of the fifth and sixth term is positive using [GLL]. Again using [GLL], we obtain that the sum of the seventh and eight term is larger than $-c\alpha^2$ and the sum of the ninth and tenth term is positive. Therefore, we obtain

$$\langle \Psi^{\mathrm{GS}}, \frac{\alpha}{|x|} \Psi^{\mathrm{GS}} \rangle \le c\alpha^2 \,.$$

This implies

$$\| (i\nabla - \sqrt{\alpha} A(x)) \Psi^{\mathrm{GS}} \|^2 \le c\alpha \,.$$

We set

$$v := i\nabla - \sqrt{\alpha} A(x) \,.$$

Using

$$[a_\lambda(k), H_f] = |k|, \quad [a_\lambda(k), v] = \frac{\epsilon_\lambda(k)}{2\pi |k|^{\frac{1}{2}}} \zeta(|k|) \mathrm{e}^{ik.x}$$

Applying the pull-through formula yields

$$a_\lambda(k)E\Psi^{\mathrm{GS}} = a_\lambda(k)K\Psi^{\mathrm{GS}}$$
$$= \Big[(H_f + |k|)a_\lambda(k) - \frac{1}{|x|}a_\lambda(k) + [a_\lambda(k), v]v + v[a_\lambda(k), v] + v^2 a_\lambda(k)$$
$$+ \sqrt{\alpha}\sigma\cdot B(x)a_\lambda(k) - i\sqrt{\alpha}\frac{\zeta(|k|)\,\sigma\cdot(\epsilon_\lambda(k)\wedge k)}{|k|^{\frac{1}{2}}2\pi}\,\mathrm{e}^{ik.x}\Big]\Psi^{\mathrm{GS}}\ .$$

Thus

$$a_\lambda(k)\,\Psi^{\mathrm{GS}}$$
$$= -\frac{\sqrt{\alpha}\zeta(|k|)}{2\pi|k|^{\frac{1}{2}}}\,\frac{2}{K+|k|-E}\Bigg(\Big(i\nabla-\sqrt{\alpha}A(x)\Big)\cdot\epsilon_\lambda(k)\mathrm{e}^{ik.x}$$
$$+ i\sigma\cdot\frac{\epsilon_\lambda(k)\wedge k}{2\pi}\mathrm{e}^{ik.x}\Bigg)\,\Psi^{\mathrm{GS}}\ .$$

We obtain

$$\| a_\lambda(k)\Psi^{\mathrm{GS}} \| \le c \, \frac{\sqrt{\alpha}\,\zeta(|k|)}{|k|^{\frac{3}{2}}} \, \| \left(i\nabla - \sqrt{\alpha}A(x) \right) \Psi^{\mathrm{GS}} \| + c \, \frac{\sqrt{\alpha}\,\zeta(|k|)}{|k|^{\frac{1}{2}}} \, \|\Psi^{\mathrm{GS}} \|$$
$$\le c \left(\frac{\alpha}{|k|^{\frac{3}{2}}} + \frac{\sqrt{\alpha}}{|k|^{\frac{1}{2}}} \right) \zeta(|k|) \|\Psi^{\mathrm{GS}}\| \,,$$

This a priori bound exhibits the L^2-critical singularity in frequency space. It does not take into consideration the exponential localization of the ground state due to the confining Coulomb potential.

To account for the latter, following the proof of [BCVV, 2010], we use two results from the work of Griesemer, Lieb, and Loss, [GLL], which provides the bound

$$\left\| a_\lambda(k)\Psi^{\mathrm{GS}} \right\| < \frac{c\sqrt{\alpha}\,\zeta(|k|)}{|k|^{\frac{1}{2}}} \left\| \, |x|\Psi^{\mathrm{GS}} \right\| .$$

Moreover,

$$\left\| \exp[\beta|x|]\Psi^{\mathrm{GS}} \right\|^2 \leq c \left[1 + \frac{1}{\Sigma_0 - E - \beta^2} \right] \| \Psi^{\mathrm{GS}} \|^2 ,$$

for any

$$\beta^2 < \Sigma_0 - E = O(\alpha^2) .$$

For the 1-electron case, Σ_0 is the infimum of the self-energy operator, and E is the ground state energy of $\tilde{H}$.

Choosing $\beta = O(\alpha)$ yields,

$$\begin{aligned} \| \, |x| \Psi^{\mathrm{GS}} \, \| &\leq \| |x|^4 \Psi^{\mathrm{GS}} \|^{\frac{1}{4}} \, \| \Psi^{\mathrm{GS}} \|^{\frac{3}{4}} \leq \frac{(4!)^{\frac{1}{4}}}{\beta} \left\| \exp[\beta |x|] \Psi^{\mathrm{GS}} \right\|^{\frac{1}{4}} \| \Psi^{\mathrm{GS}} \|^{\frac{3}{4}} \\ &\leq \frac{c}{\beta} \left[1 + \frac{1}{\Sigma_0 - E - \beta^2} \right]^{\frac{1}{8}} \| \, \Psi^{\mathrm{GS}} \, \| \\ &\leq c_1 \alpha^{-\frac{5}{4}} \, . \end{aligned}$$

Thus,

$$\left\| a_\lambda(k) \Psi^{\mathrm{GS}} \right\| < \frac{c \alpha^{-\frac{3}{4}} \zeta(|k|)}{|k|^{\frac{1}{2}}} \, .$$

We see that binding to the Coulomb potential weakens the infrared singularity by a factor $|k|$, but at the expense of a large constant factor α^{-2}.

We arrive at

$$\begin{aligned} \langle\, \Psi^{\mathrm{GS}}\,,\, N_f\, \Psi^{\mathrm{GS}}\,\rangle &= \int \left\| a_\lambda(k)\Psi^{\mathrm{GS}} \right\|^2 \mathrm{d}k \\ &\leq \int_{|k|<\delta} \frac{c\,\alpha^{-\frac{3}{2}}}{|k|}\mathrm{d}k + \int_{\delta\leq|k|\leq\Lambda} \left(\frac{c\,\alpha^2}{|k|^3} + \frac{c\,\alpha}{|k|} \right) \mathrm{d}k \\ &\leq c\alpha^{-\frac{3}{2}}\delta^2 + c\alpha^2 \log\delta + c\alpha \\ &\leq c\alpha\,, \end{aligned}$$

for $\delta = \alpha^{\frac{5}{4}}$. This proves the result.

Step 3 : A priori estimates on states “orthogonal” to f_α : For φ s.t.

$\langle \varphi^{(n)}(k_1, \lambda_1, \ldots k_n, \lambda_n; \ . \ ), f_\alpha(.) \begin{pmatrix} a \\ b \end{pmatrix} \rangle = 0$, (a.e. k_i, all λ_i, all a and b)

$$\langle \varphi, H\varphi \rangle \geq (\Sigma_\alpha(0) - \frac{\alpha^2}{4})\|\varphi\|^2 + \underbrace{\frac{3}{32}\alpha^2\|\varphi\|^2}_{\frac{1}{2}\text{ dist. between first 2 levels of } H_{\text{part}}} + \cancel{\nu\|H_f^{\frac{1}{2}}\varphi\|^2}$$

Step 4 : We prove $\Sigma_\alpha(0) - \Sigma_\alpha(V) \leq c\alpha^2$. More precisely.
Given a ground state Ψ^{GS} of H such that $\|\Pi_0 \Psi^{\mathrm{GS}}\| = 1$, we decompose it into a part parallel to f_α and a part orthogonal to f_α, where f_α is the ground state of the Schrödinger operator $-\Delta - \alpha/|x|$. Namely, we define $\phi \in \mathbb{C}^2 \otimes \mathfrak{F}$ and $G \in \mathfrak{H}$ by

$$\Psi^{\mathrm{GS}} = f_\alpha \phi + G \,,$$

with $G \in \mathfrak{H}$ orthogonal to the ground state f_α in the sense of Definition given in this talk.
Next we define a splitting for the state ϕ.
Given $\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \in \mathbb{C}^2$, let

$$\Gamma_1^{(\tilde{a},\tilde{b})} := -(H_f + P_f^2)^{-1} \sigma.B^+ \Omega_f \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} .$$

With this definition, $\Gamma_1^{(1,0)}$ equals the one photon component Γ_1 of an approximate ground state of $T(0)$ as defined by [BV, 2012].

Then we consider the following decomposition for ϕ :
Let a, b and γ_0 be defined by

$$\Pi_0\phi = \gamma_0 \begin{pmatrix} a \\ b \end{pmatrix} \Omega_f, \quad \text{with} \quad |a|^2 + |b|^2 = 1 \,,$$

and let γ_1 and R_1 de defined by

$$\Pi_1\phi = (\sqrt{\alpha}\gamma_1\Gamma_1^{(a,b)} + R_1), \quad \langle R_1,\, \Gamma_1^{(a,b)}\rangle_* = 0 \,.$$

Here the bilinear form $\langle .,.\rangle_*$ acts on $(\mathbb{C}^2 \otimes \mathfrak{F})^2$.
For G we define the following decomposition :

$$\Pi_0 G =: g \,,$$

and for

$$\Gamma_1(g) := -(H_f + P_f^2)^{-1}\sigma.B^+ g \,,$$

similarly we split $\Pi_1 G$ as

$$\Pi_1 G =: \sqrt{\alpha}\beta_1\Gamma_1(g) + L_1$$

where β_1 and L_1 are uniquely defined by the condition

$$\langle \Gamma_1(g),\, L_1\rangle_* = 0 \,.$$

For

$$\Lambda := \sup_{\zeta(r)\neq 0} |r|$$

where $\zeta(.)$ is the ultraviolet cutoff, we define

$$M(\Pi_1 G) = \begin{cases} \|PL_1\|^2 & \text{if} \quad |\beta_1| < 8(1+\Lambda) \\ \|(P-P_f)\Pi_1 G\|^2 & \text{if} \quad |\beta_1| \geq 8(1+\Lambda) \end{cases},$$

Proposition

i) For some $c > 0$ *we have* $\Sigma_0 - \Sigma \leq c\alpha^2$ *.*
ii) For Ψ^{GS} *a ground state of* H *such that* $\|\Pi_0 \Psi^{GS}\| = 1$ *we have*

$$\|H_f^{\frac{1}{2}} \Pi_{\geq 2} \Psi^{GS}\| = \mathcal{O}(\alpha)\,,$$
$$\|\nabla g\| = \mathcal{O}(\alpha)\,,$$
$$\|R_1\|_* = \mathcal{O}(\alpha), \quad \|L_1\|_* = \mathcal{O}(\alpha)\,,$$

$$(\gamma_1 - \gamma_0) = \mathcal{O}(\alpha^{\frac{1}{2}})\,,$$
$$(\beta_1 - 1)\|g\| = \mathcal{O}(\alpha^{\frac{1}{2}})\,,$$
$$\|(P - P_f)\Pi_{\geq 2}\Psi^{GS}\| = \mathcal{O}(\alpha)\,.$$

and

$$M(\Pi_1 G) = \mathcal{O}(\alpha^2)\,.$$

Step 5 : Upper bound up to α^2 with error $\mathcal{O}(\alpha^{\frac{8}{3}})$. This yields refined norm estimates and photon number estimates on remainder.

Proposition

[Refined photon number bound] We have

$$\|\Pi_{\geq 2}\Psi^{GS}\|,\ \|R_1\|,\ \|L_1\| = \mathcal{O}(\alpha^{\frac{5}{6}})\,.$$

The assumption $\|\Pi_0\Psi^{\mathrm{GS}}\| = 1$ imply that $|\gamma_0|$ is bounded by 1. Therefore, from Proposition above we obtain that $|\gamma_1|$ is bounded. Similarly, we have $\beta_1\|g\|$ bounded.

For $\mathcal{R} := \Psi^{\mathrm{GS}} - \sqrt{\alpha}\gamma_1\Gamma_1^{(a,b)}f_\alpha - \sqrt{\alpha}\beta_1\Gamma_1(g)$, we get

$$\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\mathcal{R}\|^2\mathrm{d}k \le 3\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\Psi^{\mathrm{GS}}\|^2\mathrm{d}k$$

$$+ 3\alpha|\gamma_1|^2\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\Gamma_1^{(a,b)}\|^2\mathrm{d}k + 3\alpha|\beta_1|^2\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\Gamma_1(g)\|^2\mathrm{d}k\mathrm{d}x$$

Moreover explicit computations shows that

$$\sum_{\lambda=1,2}\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\Gamma_1^{(a,b)}\|^2\mathrm{d}k \le c\alpha^{\frac{2}{3}} ,$$

and similarly

$$\sum_{\lambda=1,2}\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\Gamma_1(g)\|^2\mathrm{d}k\,\mathrm{d}x \le c\|g\|^2\alpha^{\frac{2}{3}} .$$

Therefore boundedness of $|\gamma_1|$ and $|\beta_1|\,\|g\|$ yields

$$\int_{|k|\leq\alpha^{\frac{1}{3}}} \|a_\lambda(k)\mathcal{R}\|^2 \mathrm{d}k \leq 3\left(\int_{|k|\leq\alpha^{\frac{1}{3}}} \|a_\lambda(k)\Psi^{\mathrm{GS}}\|^2 \mathrm{d}k\right) + c\alpha^{\frac{5}{3}}\,.$$

Thus, using the bounds given in the proof of the photon number estimate, we obtain

$$\begin{aligned}&\int_{|k|\leq\alpha^{\frac{1}{3}}} \|a_\lambda(k)\mathcal{R}\|^2 \mathrm{d}k\\ &\leq 3\left(\int_{|k|\leq\alpha^{\frac{7}{4}}} \frac{c\alpha^{-\frac{3}{2}}}{|k|}\,\mathrm{d}k + \int_{\alpha^{\frac{7}{4}}\leq|k|\leq\alpha^{\frac{1}{3}}} \frac{\alpha^2}{|k|^3} + \frac{\alpha}{|k|}\,\mathrm{d}k\right) + c\alpha^{\frac{5}{3}} \leq c\alpha^{\frac{5}{3}}\,.\end{aligned}$$

This implies

$$\begin{aligned}\langle \mathcal{R}, N_f \mathcal{R}\rangle &= \sum_{\lambda=1,2}\int_{|k|\le\alpha^{\frac{1}{3}}} \|a_\lambda(k)\mathcal{R}\|^2 \mathrm{d}k + \sum_{\lambda=1,2}\int_{|k|>\alpha^{\frac{1}{3}}} \|a_\lambda(k)\mathcal{R}\|^2 \mathrm{d}k \\ &\le c\alpha^{\frac{5}{3}} + \sum_{\lambda=1,2}\int_{|k|>\alpha^{\frac{1}{3}}} |k|\,\alpha^{-\frac{1}{3}}\|a_\lambda(k)\mathcal{R}\|^2 \mathrm{d}k \\ &\le c\alpha^{\frac{5}{3}} + \alpha^{-\frac{1}{3}}\|H_f^{\frac{1}{2}}\mathcal{R}\|^2 \le c\alpha^{\frac{5}{3}}\,,\end{aligned}$$

where in the last inequality we used

$$\begin{aligned}\|H_f^{\frac{1}{2}}\mathcal{R}\|^2 &= \|H_f^{\frac{1}{2}}\Pi_{\ge 2}\Psi^{\mathrm{GS}}\|^2 + \|H_f^{\frac{1}{2}} f_\alpha R_1\|^2 + \|\,H_f^{\frac{1}{2}} L_1\|^2 \\ &\le \|H_f^{\frac{1}{2}}\Pi_{\ge 2}\Psi^{\mathrm{GS}}\|^2 + \|R_1\|_*^2 + \|\,L_1\|_*^2\,,\end{aligned}$$

and the estimates of Proposition.
The identity

$$\langle \mathcal{R}, N_f \mathcal{R}\rangle = \langle \Pi_{\ge 2}\Psi^{\mathrm{GS}}, N_f\,\Pi_{\ge 2}\Psi^{\mathrm{GS}}\rangle + \langle f_\alpha R_1, N_f\, f_\alpha R_1\rangle + \langle L_1, N_f\, L_1\rangle\,,$$

conclude the proof.

Step 6 : Upper bound up to α^2 with error $\mathcal{O}(\alpha^{\frac{8}{3}})$. This yields again refined norm estimates and photon number estimates on remainder.

Step 7 : Upper bound up to α^3 with error $\mathcal{O}(\alpha^{3+\frac{1}{3}})$.

THANK YOU !