The Bernstein-Gel’fand-Gel’fand complex and Kasparov theory for SL(3, C)

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Abstract
In the case of the group SL(3, C), we describe how the Bernstein-Gel’fand-Gel’fand complex can be used to construct an element of Kasparov’s equivariant K-homology. The resulting construction is a model for the γ-element. The key to the construction is the introduction of a lattice of operator ideals associated to the natural fibrations of the complete flag variety for SL(3, C).

1 Introduction
Kasparov’s analytic K-homology is a blend of topology and analysis: a typical source for the construction of a K-homology cycle is an elliptic differential complex. If the elliptic complex is equivariant with respect to the action of a group G, and if moreover the group action satisfies some additional conformality property with respect to some Hermitian metric, then one can construct an element of equivariant K-homology (see [Kas84] for a classic example of this). Unfortunately, if G is a semisimple Lie group of rank greater than one, it seems probable that non-trivial examples of such complexes cannot exist. This paper describes a means of constructing an equivariant K-homology class from the Bernstein-Gel’fand-Gel’fand complex for SL(3, C), a differential complex which is neither elliptic nor conformal, but which satisfies some weaker (‘directional’) form of these conditions.

The motivation for this construction comes from the Baum-Connes conjecture. Although an understanding of the conjecture is not essential to this paper, it is useful for perspective. The Baum-Connes conjecture (with coefficients) states that the analytic assembly map of [BCH94],

\[ \mu_{G,A} : K^G_j(EG; A) \to K_j(A \rtimes_r G). \]

is an isomorphism, for any coefficient G-C*-algebra A. See [Hig98] for an overview of the conjecture and its many consequences. One of the major outstanding cases of the conjecture is the case of a simple Lie group of real-rank greater than one, such as the group G = SL(3, C).

Let us recall the method of proof for the real-rank one simple groups. If G is a semisimple Lie group and K its maximal compact algebra, then the conjecture for G with coefficient algebra A = C_0(G/K) is straightforward to
prove. On the other hand, the conjecture with trivial coefficients $A = \mathbb{C}$ implies the conjecture with any other coefficients. It follows that if the algebras $C_0(G/K)$ and $\mathbb{C}$ are $KK^G$-equivalent then the Baum-Connes conjecture holds for all closed subgroups of $G$. Kasparov demonstrated the existence of elements $\alpha \in KK^G(C_0(G/K), \mathbb{C})$ (the ‘Dirac element’) and $\beta \in KK^G(\mathbb{C}, C_0(G/K))$ (the ‘dual-Dirac element’) for which $\alpha \beta = 1 \in KK^G(C_0(G/K), C_0(G/K))$. The reverse composition, $\beta \alpha$ is a canonical idempotent in $KK^G(\mathbb{C}, \mathbb{C})$, called $\gamma_G$, or just $\gamma$.

It has been proven that $\gamma_G = 1$ for $SO_0(n, 1)$ in [Kas84], and for $SU(n, 1)$ in [JK95]. Importantly, though, the $KK^G$-cycles representing $\gamma$ which enable these proofs are built using the compact homogeneous space $G/B$, where $B$ is the maximal parabolic subgroup, rather than $G/K$. The following theorem summarizes Kasparov’s method for identifying such a model of $\gamma_G$.

**Theorem 1.1.** Let $\iota : \mathbb{C} \to C(G/B)$ be the inclusion of the constant functions. Suppose $\theta \in KK^G(C(G/B), \mathbb{C})$ is such that $\text{Res}^G_{K}(\iota^* \theta) = 1 \in KK^K(\mathbb{C}, \mathbb{C})$. Then $\iota^* \theta = \gamma_G$.

It was observed by N. Higson that, in each of the rank-one cases, the construction of such a $\theta$ was made by using some close variant of the Bernstein-Gel’fand-Gel’fand (BGG) complex for $G$. For complex semisimple $G$, the BGG complex is described as follows.

**Theorem 1.2.** ([BGG75, Theorem 10.1])

There exists a complex consisting of (smooth section spaces of) direct sums of $G$-homogeneous line bundles over $G/B$, and $G$-equivariant differential operators between them, which resolves the trivial representation of $G$.

Although the action of $G$ on this elliptic complex is not conformal, it is separately conformal on each line-bundle summand. In juxtaposition with Theorem 1.1 this suggests that the $\gamma$-element for a general complex semisimple group might be constructible from the BGG resolution. The purpose of this paper is to demonstrate that this is indeed possible for the group $SL(3, \mathbb{C})$.

We note from the outset that it is known that $\gamma_G \neq 1$ for any group $G$ which has Kazhdan’s property $T$. Therefore, a direct translation of Kasparov’s method cannot prove the Baum-Connes conjecture for simple Lie groups of rank greater than one—some subtle variation of Kasparov’s argument would be required. Nevertheless, it is hoped that a construction of this kind will be useful for further study of the Baum-Connes conjecture.

It is instructive to keep in mind the much easier case of the rank two semisimple group $G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Let us briefly sketch the construction of $\gamma$ for this group. First, recall that the $\gamma$-element for $SL(2, \mathbb{C})$ is obtained by taking the Dolbeault operator for $\mathbb{C}P^1$,

$$ L^2 \Omega^{0,0} \mathbb{C}P^1 \xrightarrow{j} L^2 \Omega^{0,1} \mathbb{C}P^1, $$

which is precisely the BGG complex for $SL(2, \mathbb{C})$, and replacing the operator $j$ by its phase (in the sense of polar decomposition of unbounded operators). For

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\[ ^1 \text{Bernstein, Gel’fand and Gel’fand’s formulation of this result is purely algebraic. See CSS01 Appendix A for a discussion of the geometric interpretation.} \]
$G = \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$, the BGG resolution is again the Dolbeault complex for $G/B \cong \mathbb{CP}^1 \times \mathbb{CP}^1$, but decomposed into its ‘directional’ components as follows:

\[
\begin{array}{ccc}
L^2\Omega^{0,1}\mathbb{C}P^1 \otimes L^2\Omega^{0,0}\mathbb{C}P^1 & \oplus & L^2\Omega^{0,1}\mathbb{C}P^1 \otimes L^2\Omega^{0,1}\mathbb{C}P^1 \\
\uparrow & & \uparrow \\
L^2\Omega^{0,0}\mathbb{C}P^1 \otimes L^2\Omega^{0,0}\mathbb{C}P^1 & \oplus & L^2\Omega^{0,0}\mathbb{C}P^1 \otimes L^2\Omega^{0,1}\mathbb{C}P^1 \\
\downarrow & & \downarrow \\
L^2\Omega^{0,0}\mathbb{C}P^1 \otimes L^2\Omega^{0,0}\mathbb{C}P^1 & \oplus & L^2\Omega^{0,1}\mathbb{C}P^1 \otimes L^2\Omega^{0,1}\mathbb{C}P^1 \\
\downarrow & & \downarrow \\
\end{array}
\]

If one replaces each of these four differential operators by its phase, one obtains the starting data for taking the Kasparov product of two copies of the gamma element of $\text{SL}(2, \mathbb{C})$. One must now use an application of the Kasparov Technical Theorem to transform this into a genuine Fredholm module. Note that this final step involves in a crucial way the lattice of operator ideals

\[
\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{B} & & \mathcal{B} \otimes \mathcal{K} \\
\mathcal{K} \otimes \mathcal{B} & & \mathcal{K} \otimes \mathcal{K} \\
\end{array}
\]

where $\mathcal{B}$ and $\mathcal{K}$ are the algebras of bounded and compact operators, respectively, on $L^2\Omega^\bullet \mathbb{C}P^1$. We will not go into the details of this example further, but leave the reader to ponder the analogy with the structures introduced here for $\text{SL}(3, \mathbb{C})$.

The structure of this paper is as follows. In Section 2 we set up notation and recall some basic facts concerning homogeneous line bundles over the flag manifold $G/B$ for $G = \text{SL}(3, \mathbb{C})$. Section 3 contains the crucial definitions of a lattice of $C^\ast$-ideals associated to the fibrations of $G/B$. We describe some of the basic properties of these ideals. In Section 4 we study the bounded operators which will take the place of the differential operators of the BGG resolution. In Section 5 we combine the results of the previous sections to give the construction of the $\gamma$-element for $\text{SL}(3, \mathbb{C})$.

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2 Notation and Preliminaries

2.1 Lie groups

Throughout this paper $G$ will denote the group $\text{SL}(3, \mathbb{C})$, and we use the following notation for various subgroups: $K$ is the maximal compact subgroup $\text{SU}(3)$; $B$ is the minimal parabolic subgroup of invertible upper triangular matrices; $N$ and $Z$ are the nilpotent subgroups of upper and lower triangular unipotent matrices, respectively; $M$ is the group of diagonal matrices with entries of modulus

2In this paper, we do not use the BGG resolution itself for the construction, but build the corresponding $KK_G$-cycle directly using noncommutative harmonic analysis. The BGG resolution serves only as very strong guidance for the construction.
one; $A$ is the group of diagonal matrices with positive real entries; $H = MA$ is the Cartan subgroup. The corresponding Lie algebras are $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{j}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{h}$.

We set

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_\rho = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{t}_C \cong \mathfrak{g}.$$ 

We use $X'$ to denote the conjugate transpose of an element $X \in \mathfrak{t}_C$.

The dual of a complex vector space $V$ will be denoted by $V^\dagger$. By extending the inclusions of $\mathfrak{m}$ and $\mathfrak{a}$ in $\mathfrak{h}$ to $\mathbb{C}$-linear identifications $\mathfrak{m}_C, \mathfrak{a}_C \cong \mathfrak{h}$, we will identify infinitesimal characters of $\mathfrak{m}$ and $\mathfrak{a}$ with elements of $\mathfrak{h}^\dagger$. Characters of $\mathfrak{h}$ will be denoted by $\chi = \chi_\mathfrak{M} \oplus \chi_\mathfrak{A}$ where $\chi_\mathfrak{M}$ and $\chi_\mathfrak{A}$ are the restrictions of $\chi$ to $\mathfrak{M}$ and $\mathfrak{A}$, respectively. Infinitesimal characters of $\mathfrak{b}$ will be identified with their restrictions to $\mathfrak{h}$.

The set of roots of $K$ will be denoted by $\Delta$. We let $\alpha_1, \alpha_2 \in \mathfrak{h}^\dagger$ be the weights of $X_1, X_2 \in \mathfrak{t}_C$, respectively, and fix these as the simple roots. Also put $\rho = \alpha_1 + \alpha_2$. The weight lattice of $K$ is denoted by $\Lambda_W$. We use $\delta_j$ ($j = 1, 2, 3$) to denote the characters

$$\delta_j : \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto t_j$$

of $M$. Thus, $\delta_1$ and $-\delta_3$ are the fundamental weights for $K$.

The Weyl group of $G$ is denoted $W$.

### 2.2 Homogeneous bundles over the flag manifold

Throughout this paper, $X$ will denote the homogeneous space $X = G/B = K/M$.

For each weight $\mu$ of $K$, we let $E_\mu$ denote the $G$-homogeneous bundle over $X$ induced from $\mu \oplus \rho$. Thus, sections of $E_\mu$ are identified with functions $s : G \to \mathbb{C}$ which satisfy the $B$-equivariance property

$$s(xb) = e^{-(\mu \oplus \rho)}(b)s(x) \quad \text{for all } x \in G, b \in B. \quad (2.1)$$

Equivalently,

$$B_L s = (\mu \oplus \rho)(B) s \quad \text{for all } B \in \mathfrak{b}, \quad (2.2)$$

where $B_L$ denotes the left-invariant differential operator on $G$ determined by $B$. The space of continuous sections of $E_\mu$ will be denoted $C(X; E_\mu)$. The group $G$ acts on $C(X; E_\mu)$ by pull-back:

$$(g \cdot s)(x) = s(g^{-1}x). \quad (2.3)$$

Thanks to the Iwasawa decomposition $G = KAN$,

$$C(X; E_\mu) \cong \{ s \in C(K) \mid s(km) = e^{-\mu(m)s(k)} \quad \forall k \in K, m \in M \}
= \{ s \in C(K) \mid M_L s = -\mu(M)s \quad \forall M \in \mathfrak{m} \},$$

The completion of $C(X; E_\mu)$ with respect to the inner product

$$\langle s_1, s_2 \rangle = \int_K \overline{s_1(k)} s_2(k) dk$$

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will be denoted \( L^2(\mathcal{X}; E_\mu) \). The pull-back action (2.3) of \( G \) on \( C(\mathcal{X}; E_\mu) \) extends to a unitary representation of \( G \) on \( L^2(\mathcal{X}; E_\mu) \), which we denote by \( U_\mu \).

The product of two sections \( s \in C(\mathcal{X}; E_\mu) \) and \( t \in C(\mathcal{X}; E_\nu) \) belongs to \( C(\mathcal{X}; E_{\mu+\nu}) \), and multiplication by \( s \) extends to a bounded linear map

\[
s : L^2(\mathcal{X}; E_\nu) \rightarrow L^2(\mathcal{X}; E_{\mu+\nu}); \quad t \mapsto st,
\]

whose norm is the \( L^\infty \)-norm of \( s \).

We will also occasionally need to refer to the ‘nilpotent’ (or ‘noncompact’) picture of \( E_\mu \). Using the almost everywhere defined decomposition \( G = \text{ZMAN} \), sections of \( E_\mu \) are determined on an open dense subset of \( \mathcal{X} \) by their restriction to \( Z \). This restriction yields a trivializing chart for \( E_\mu \) over \( Z \rightarrow \mathcal{X} \), and taking \( G \)-translates of this chart yields a trivializing atlas.

### 2.3 The Peter-Weyl transform

Let \( K \) denote the set of (equivalence classes of) irreducible unitary representations of \( K \). For each representation \( \sigma \in K \), let \( V^\sigma \) denote its representation space, and \( |\sigma| \) its dimension. Let \( \sigma^\dagger \) be the contragredient representation, acting on \( V^{\sigma^\dagger} \). The pairing of \( V^{\sigma^\dagger} \) with \( V^\sigma \) will be denoted by \( (\cdot, \cdot) \).

For any weight \( \mu \in \Lambda_W \), the Peter-Weyl isomorphism

\[
\bigoplus_{\sigma \in K} V^{\sigma^\dagger} \otimes V^\sigma \rightarrow L^2(K)
\]

\[
\xi^\dagger \otimes \xi \mapsto \{ k \mapsto |\sigma|^\frac{1}{2}(\xi^\dagger(k)\xi) \},
\]

restricts to an isomorphism

\[
L^2(\mathcal{X}; E_{-\mu}) \cong \bigoplus_{\sigma \in K} V^{\sigma^\dagger} \otimes (V^\sigma)_\mu,
\]

where \((V^\sigma)_\mu\) is the \( \mu \)-weight space of \( \sigma \). We refer to this isomorphism as the Peter-Weyl transform. Under the Peter-Weyl transform, the multiplication operators (2.4) are described in terms of Clebsch-Gordan-type rules for tensor products of \( SU(3) \)-representations, while the group representation \( U_{-\mu} \) of \( K \) becomes \( \bigoplus_{\sigma \in K} (\sigma^\dagger \otimes 1) \). The full representation \( U_{-\mu}(G) \) is harder to describe—see Section 3.3

### 2.4 K-invariant differential operators

Any element \( T \) of the complexified universal enveloping algebra \( U(\mathfrak{k}) \) defines a left-invariant differential operator on \( C^\infty(K) \). If \( T \) is a homogeneous element of weight \( \beta \), then \( T \) maps smooth sections of \( E_{-\mu} \) to sections of \( E_{-(\mu+\beta)} \), for any \( \mu \in \Lambda_W \). Thus \( T \) defines an unbounded operator between the corresponding \( L^2 \)-section spaces. Here, let us use the space of \( K \)-finite vectors in \( L^2(\mathcal{X}; E_{-\mu}) \) as the initial domain of definition. Under the Peter-Weyl transform, \( T \) acts as \( \bigoplus_{\sigma \in K} 1 \otimes \sigma(T) \).

**Remark 2.1.** It is important to note that while \( T \) maps \( M \)-equivariant functions of \( K \) to \( M \)-equivariant functions, it does not, in general, map \( B \)-equivariant functions of \( G \) to \( B \)-equivariant functions. Thus \( T \) is a well-defined operator on \( L^2(\mathcal{X}; E_{\mu \nu}) \) only if defined using the compact picture.
The conjugate transpose map $X \to X' \overset{def}{=} \overline{X^T}$, for $(X \in \mathfrak{k}_C \cong \mathfrak{g})$, extends to an algebra anti-automorphism of $\mathfrak{u}(\mathfrak{k}_C)$. For any $T \in \mathfrak{u}(\mathfrak{k}_C)$, the operators $T$ and $T'$ are formally adjoint, and the operators $T'T$ on $L^2(\mathfrak{X}; E_{-\mu})$, and

$$\begin{pmatrix} 0 & T' \\ T & 0 \end{pmatrix}$$

on $L^2(\mathfrak{X}; E_{-\mu} \oplus E_{-(\mu+\beta)})$ are essentially self-adjoint. We use $|T|$ to denote the absolute value of $T$, and $\text{Ph}(T)$ or $T'$ to denote the phase, as defined by the polar decomposition, $T = T|T|$.

3 $C^*$-algebras associated to the fibrations of $\mathfrak{X}$

3.1 Spectral decompositions of $L^2(\mathfrak{X}; E_\mu)$

Associated to the simple roots $\alpha_1, \alpha_2$ are the parabolic subgroups

$$P_1 = \begin{cases} \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \end{cases}, \quad P_2 = \begin{cases} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \end{cases} \subseteq G,$$

and the corresponding homogeneous spaces $\mathcal{X}_i = G/P_i$ ($i = 1, 2$). Let $M_i = P_i \cap K$, so that also $\mathcal{X}_i = K/M_i$. Note that $M_i = K_i \times K_i'$, where

$$K_1 = \begin{pmatrix} \text{SU}(2) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_1' = \{ \begin{pmatrix} \omega.I & 0 \\ 0 & 0 & \omega^{-2} \end{pmatrix} \mid |\omega| = 1 \},$$

and

$$K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{SU}(2) \end{pmatrix}, \quad K_2' = \{ \begin{pmatrix} \omega^{-2} & 0 & 0 \\ 0 & 0 & \omega.I \end{pmatrix} \mid |\omega| = 1 \}.$$

Suppose now that $\varpi$ is a representation of $K$ on a Hilbert space $H$. Fix $i = 1$ or 2. One may decompose $H$ first into isotypical subspaces for the restriction of $\varpi$ to $M_i$, and then further into isotypical components for the restriction to $\mathcal{X}_i$. The isotypical subspaces of $H$ for $M$ are, of course, the weight spaces of $H$, so this double decomposition yields a decomposition of each weight space of $H$ into $M_i$-types. In fact, for a fixed weight $\mu \in \Lambda_W$, the action of $K_i'$ on the $\mu$-weight space $H_\mu$ is fixed, so the decomposition of $H_\mu$ can be equivalently given in terms $K_i$-types. With $i = 1$ or 2, we use $\pi_k$ throughout to denote the irreducible representation of $K_i \cong \text{SU}(2)$ with highest weight $k \in \mathbb{N}$.

**Definition 3.1.** We use $P_k^{(H_{-\mu})}$, or more concisely $P_k^{(i)}$, to denote the orthogonal projection of $H_\mu$ onto the component of $K_i$-type $\pi_k$.

For $A \subseteq \mathbb{N}$ we write $P_A^{(i)} = \sum_{k \in A} P_k^{(i)}$. In particular, if $k_1 \leq k_2$, we abbreviate $P_{\{k_1, \ldots, k_2\}}^{(i)}$ as $P_{[k_1, k_2]}^{(i)}$. 

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The domain $H_\mu$ will usually be implicitly assumed, and suppressed from the notation.

The most relevant example is $H = L^2(K)$ with $\varepsilon$ being the right regular representation. Then the weight spaces are the spaces $L^2(\mathcal{X}; E_{-\mu})$, for which we get projections $P_k^{(i)} \in B(L^2(\mathcal{X}; E_{-\mu}))$.

**Remark 3.2.** In this case, the projections $P_k^{(i)}$ are precisely the spectral projections of the tangential Hodge-Laplace operators $\Delta_i = X'_i X_i$ on $E_{-\mu}$, tangential along the fibration $\mathcal{X} \to \mathcal{Y}_i$. The analysis that follows can be understood as a study of the simultaneous spectral theory of these non-commuting differential operators.

A section $s \in L^2(\mathcal{X}; E_{-\mu})$ will be said to be of right $K_i$-type $k$ if it is in the range of $P_k^{(i)}$. Equivalently, $s$ has right $K_i$-type $k$ if has a Peter-Weyl transform

$$\sum (\xi^* \otimes \xi) \in \bigoplus_{\sigma \in \hat{K}} V^{\sigma^\perp} \otimes (V^{\sigma})_{\mu}$$

where each $\xi_m$ belongs to an irreducible $K_i$-subrepresentation of highest weight $k$.

Any $K$-homogeneous vector bundle over $\mathcal{X}$ admits a $K$-equivariant decomposition $E = \bigoplus_{\mu} E_{\mu}$ into line bundles. We thus define projections $P_k^{(i)}$ on $L^2(\mathcal{X}; E) = \bigoplus_{\mu} L^2(\mathcal{X}; E_{\mu})$ to be the direct sum of the corresponding projections $P_k^{(i)}$ on each component.

A section of $E_0$ has $K_i$-type 0 if and only if, as a function on $K$, it is invariant under $M_i$. Thus, $P_0^{(i)} L^2(\mathcal{X}; E_0) = L^2(\mathcal{Y}_i)$. Since $\pi_k \otimes \pi_0 = \pi_k$, for any $k \in \mathbb{N}$, it follows that the spaces $P_k^{(i)} L^2(\mathcal{X}; E_{-\mu})$ are preserved by pointwise multiplication by continuous sections $f \in C(\mathcal{Y}_i)$.

Now let $\pi$ be an irreducible representation of $M_i$ on a vector space $W^\pi$, and let $F_{\pi^\perp}$ be the $K$-homogenous bundle over $\mathcal{Y}_i$ induced from the contragredient representation $\pi^\perp$. Thus

$$L^2(\mathcal{Y}_i; F_{\pi^\perp}) = \{ f \in L^2(K, W^{\pi^\perp}) \mid f(km_1) = \pi^\perp(m_1)^{-1} f(k) \ \forall k \in K, m_1 \in M_1 \}.$$  

(3.1)

These spaces are, of course, also modules over $C(\mathcal{Y}_i)$. Their relationship with the modules $P_k^{(i)} L^2(\mathcal{X}; E_{-\mu})$ is as follows.

**Lemma 3.3.** Let $\pi$ be a representation of $M_i$ whose restriction to $K_i$ is $\pi_k$, and let $\mu \in m^\perp$ be a weight of $M_i$ appearing with nonzero multiplicity in $\pi$. Then $P_k^{(i)} L^2(\mathcal{X}; E_{-\mu})$ is $C(\mathcal{Y}_i)$-linearly isomorphic to $L^2(\mathcal{Y}_i; F_{\pi^\perp})$.

**Proof.** By the Peter-Weyl theorem,

$$L^2(K, W^{\pi^\perp}) \cong \bigoplus_{\sigma \in \hat{K}} V^{\sigma^\perp} \otimes V^{\sigma} \otimes W^{\pi^\perp} \cong \bigoplus_{\sigma \in \hat{K}} V^{\sigma^\perp} \otimes \text{End}(W^\pi, V^{\sigma}).$$

The $M_1$-equivariance condition in (3.1) implies that

$$L^2(\mathcal{Y}_i; F_{\pi^\perp}) \cong \bigoplus_{\sigma \in \hat{K}} V^{\sigma^\perp} \otimes \text{Hom}_{M_i}(W^\pi, V^{\sigma}),$$

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where \( \text{Hom}_{M_i} \) denotes the space of intertwiners of \( M_i \)-representations.

Let \( v \in W^\pi \) be of weight \( \mu \). The map

\[
\text{Hom}_{M_i}(W^\pi, V^\sigma) \rightarrow (V^\sigma)_\mu
A \mapsto Av
\]

is an isomorphism, by the irreducibility of \( W^\pi \). Therefore the map

\[
1 \otimes 1 \otimes v : \bigoplus_{\sigma \in \mathcal{K}} V^\sigma \otimes V^\pi \to \bigoplus_{\sigma \in \mathcal{K}} V^\sigma \otimes V^\sigma
\xi \otimes \eta \otimes w \mapsto (w, v) \xi \otimes \eta,
\]

which is clearly \( C(\mathcal{Y}_i) \)-linear, restricts to an isomorphism

\[
L^2(\mathcal{Y}_i; F_{\pi^1}) \to P_i^{(i)} L^2(\mathcal{X}; E_{-\mu}).
\]

This implies the following useful finite generation property.

**Corollary 3.4.** Fix \( \mu \in \Lambda_W, k \in \mathbb{N} \). There exists a finite collection of continuous sections \( s_1, \ldots, s_n \in P_k^{(i)} C(\mathcal{X}; E_{\mu}) \) and bounded linear maps \( \varphi_1, \ldots, \varphi_n : P_k^2(\mathcal{X}; E_{\mu}) \to L^2(\mathcal{Y}_i) \) such that \( \sum_{j=1}^n s_j \varphi_j(s) = s \) for all \( s \in P_k^2(\mathcal{X}; E_{\mu}) \).

### 3.2 Spectrally proper and spectrally finite operators

**Definition 3.5.** Let \( \alpha = \alpha_i \) be a simple root of \( \mathcal{K} \). Let \( \mu, \nu \in \Lambda_W \), and let \( T : L^2(\mathcal{X}; E_{\mu}) \to L^2(\mathcal{X}; E_{\nu}) \) be a bounded operator. The \( \alpha \)-support of \( T \) is defined as

\[
\alpha\text{-Supp } T = \{ (k, k') \mid P_k^{(i)} T P_k^{(i)} \neq 0 \} \subseteq \mathbb{N} \times \mathbb{N}.
\]

(In this, and other similar expressions, we use the domain and range of the operator \( T \) to specify the domains of the projections \( P_k^{(i)} \) and \( P_k^{(i)} \).)

Recall that a subset \( S \) of \( \mathbb{N} \times \mathbb{N} \) is **proper** if for every \( k \in \mathbb{N} \), the sets

\[
\{ k' \in \mathbb{N} \mid (k, k') \in S \}
\]

and

\[
\{ k' \in \mathbb{N} \mid (k', k) \in S \}
\]

are finite.

**Definition 3.6.** A bounded operator \( T : L^2(\mathcal{X}; E_{\mu}) \to L^2(\mathcal{X}; E_{\nu}) \) is

- **spectrally proper for** \( \alpha \) if \( \alpha\text{-Supp } T \) is a proper subset of \( \mathbb{N} \times \mathbb{N} \).

- **spectrally finite for** \( \alpha \) if \( \alpha\text{-Supp } T \) is finite.

**Definition 3.7.** With the above notation, set

\[
\mathcal{A}_i(E_{\mu}, E_{\nu}) = \{ T : L^2(\mathcal{X}; E_{\mu}) \to L^2(\mathcal{X}; E_{\nu}) \text{ spectrally proper for } \alpha_i \}^\|,\]

...
\( \mathcal{K}_i(E_\mu, E_\nu) = \{ T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu) \text{ spectrally finite for } \alpha_i \}^\perp \). 

(Closures are taken in the operator norm.) Define also

\[ \mathcal{A}(E_\mu, E_\nu) = \mathcal{A}_1(E_\mu, E_\nu) \cap \mathcal{A}_2(E_\mu, E_\nu), \]

and

\[ \mathcal{K}(E_\mu, E_\nu) = \mathcal{K}_1(E_\mu, E_\nu) \cap \mathcal{K}_2(E_\mu, E_\nu). \]

In Proposition 3.17 we will see that \( \mathcal{K}(E_\mu, E_\nu) \) is in fact the space of compact operators from \( L^2(\mathcal{X}; E_\mu) \) to \( L^2(\mathcal{X}; E_\nu) \), justifying the notation.

In the case \( \mu = \nu \), the spectrally proper operators for \( \alpha_i \) form an algebra, and the spectrally finite operators for \( \alpha_i \) form an ideal in that algebra. Their norm-closures are therefore a \( C^* \)-algebra and a \( C^* \)-ideal, respectively. If \( \mu \) and \( \nu \) are allowed to vary, it is convenient to think of Definition 3.7 as defining \( C^* \)-categories \( \mathcal{A}_i, \mathcal{K}_i, \mathcal{A} \) and \( \mathcal{K} \). (For definitions and properties of \( C^* \)-categories, we refer the reader to [Mit02].) We similarly use the notation \( \mathcal{B}(E_\mu, E_\nu) \) to denote the set of bounded linear operators between section spaces \( L^2(\mathcal{X}; E_\mu) \) and \( L^2(\mathcal{X}; E_\nu) \).

Remark 3.8. The reader familiar with Roe algebras will recognize spectral properness as a propagation condition—the same as that which would appear in the definition of the Roe algebra of the spectrum \( \mathbb{N} \) if it were endowed with the indiscrete coarse structure (Example 2.8 of [Roe03]). From this point of view, \( \mathcal{K}_i \) should be compared with the ideal of that Roe algebra which is associated to the subspace \( \{0\} \) of \( \mathbb{N} \), as in [HRY93, Section 5].

In the previous section it was noted that the projections \( P^{(i)}_{k}[0,k] \) can be defined on section spaces of \( \mathcal{K} \)-homogeneous vector bundles of \( \mathcal{X} \) of arbitrary dimension. Using this allows one to extend the definition of the above \( C^* \)-categories in a similar fashion. Equivalently, a bounded operator between \( L^2 \) sections of \( E = \oplus_\mu E_\mu \) and \( E' = \oplus_\nu E_\nu \) belongs to \( \mathcal{A}_i(E, E') \) if and only if each of the entries in its matrix representation with respect to those direct sums belongs to \( \mathcal{A}_i(E_\mu, E_\nu) \), for the appropriate \( \mu \) and \( \nu \). The analogous statement also holds for the other \( C^* \)-categories defined above.

We now describe some alternative characterizations of these \( C^* \)-categories.

**Lemma 3.9.** Let \( i = 1 \) or \( 2 \) and let \( T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu) \) be a bounded linear map. The following are equivalent:

(i) \( T \in \mathcal{K}_i \),

(ii) \( (P^{(i)}_{\{0,k\}})_* T \to 0 \) and \( T(P^{(i)}_{\{0,k\}})^\perp \to 0 \) in norm as \( k \to \infty \),

(iii) \( P^{(i)}_{\{0,k\}} T P^{(i)}_{\{0,k\}} \to T \) in norm as \( k \to \infty \).

**Proof.** Property (ii) is immediate if \( T \) is spectrally finite for \( \alpha_i \), and hence holds for all \( T \in \mathcal{K}_i \) by density. The implications (ii)\( \Rightarrow \) (iii) and (iii)\( \Rightarrow \) (i) are straightforward.

\[ \square \]

**Lemma 3.10.** Let \( i = 1 \) or \( 2 \) and let \( T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu) \) be a bounded linear map. The following are equivalent:

(i) \( T \) is spectrally finite for \( \alpha_i \),

(ii) \( (P^{(i)}_{\{0,k\}})_* T \to 0 \) and \( T(P^{(i)}_{\{0,k\}})^\perp \to 0 \) in norm as \( k \to \infty \),

(iii) \( P^{(i)}_{\{0,k\}} T P^{(i)}_{\{0,k\}} \to T \) in norm as \( k \to \infty \).

**Proof.** The implications (i)\( \Rightarrow \) (ii) and (ii)\( \Rightarrow \) (iii) are straightforward. The implications (i)\( \Rightarrow \) (i) and (ii)\( \Rightarrow \) (ii) are immediate if \( T \) is spectrally finite for \( \alpha_i \), and hence are valid for all \( T \in \mathcal{K}_i \) by density.

\[ \square \]
(i) \( T \in A_i \),

(ii) For any \( k \in \mathbb{N} \), \((P^{(i)}_{[0,l]})^\perp TP^{(i)}_{[0,k]} \to 0 \) and \( P^{(i)}_{[0,k]} T(P^{(i)}_{[0,l]})^\perp \to 0 \) in norm as \( l \to \infty \),

(iii) \( T \) is a multiplier of \( K_i \), ie \( TK \) and \( KT \) are in \( K_i \) for all \( K \in K_i \) (when appropriately composable).

Proof. If \( T \) is spectrally proper for \( \alpha_i \) then (ii) is immediate, so by density, (ii) holds for all \( T \in A_i \). If \( T \) satisfies (ii), then \( TK \) and \( KT \) satisfy (ii) of Lemma \ref{lem:3.9} for any \( K \) which is spectrally finite for \( \alpha_i \), from which (iii) follows by density again.

Finally, let \( T \) be a multiplier of \( K_i \). We will show that for any \( \epsilon > 0 \), we can approximate \( T \) within \( \epsilon \) in norm by an operator \( S \) which is spectrally proper for \( \alpha_i \). We construct \( S \) by an inductive process: starting with \( S_0 = T \), we will construct multipliers \( S_n \) of \( K_i \) such that
\[
\|S_n - S_{n-1}\| < \epsilon 2^{-n}, \tag{3.2}
\]
as well as a strictly increasing sequence \((a_n) \subseteq \mathbb{N} \) such that
\[
(P^{(i)}_{[0,a_n]})^\perp S_n P^{(i)}_{[0,k]} = 0 \tag{3.3}
\]
and
\[
P^{(i)}_{[0,k]} S_n (P^{(i)}_{[0,a_n]})^\perp = 0, \tag{3.4}
\]
for each \( 0 \leq k \leq n - 1 \). The norm-limit of these \( S_n \) will then be the desired approximating operator \( S \).

Suppose we have defined \( S_{n-1} \). Both \( S_{n-1} P^{(i)}_{[0,n]} \) and \( P^{(i)}_{[0,a_n]} S_{n-1} \) are in \( K_i \), so by Lemma \ref{lem:3.9} there is an integer \( a_n \) (larger than \( a_{n-1} \)) such that the operators
\[
U_n = (P^{(i)}_{[0,a_n]})^\perp S_{n-1} P^{(i)}_{[0,n]}
\]
and
\[
V_n = P^{(i)}_{[0,n]} S_{n-1} (P^{(i)}_{[0,a_n]})^\perp,
\]
have norm less than \( \epsilon 2^{-n-1} \). If we now put
\[
S_n = S_{n-1} - U_n - V_n,
\]
one can directly check the properties \((3.2)\), \((3.3)\) and \((3.4)\).

3.3 Multiplication operators and group representations

**Proposition 3.11.** Let \( s \in C(\mathcal{X}; E_{-\lambda}) \), for some weight \( \lambda \). For any \( \mu \in \Lambda_W \), the multiplication operator \( s : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+\lambda)}) \) belongs to \( A \).

Proof. Fix \( i = 1 \) or 2. Suppose first that \( s(k) = (\xi^\dagger, \sigma(k) \xi) \), is a matrix unit, with \( \sigma \in K, \xi^\dagger \in V_j^\dagger \) and \( \xi \in (V_j)_{\lambda} \). Suppose further that \( \xi \) is of \( K_i \)-type \( j \). For each \( k \in \mathbb{N} \), the \( K_i \)-types appearing in \( \pi_j \otimes \pi_k \) lie between \( |k-j| \) and \( k+j \). Thus,
\[
\alpha\text{-Supp}(s) \subseteq \{(k,k') \mid |k-k'| \leq j\},
\]
which is proper. Such \( s \) span a dense subspace of \( C(\mathcal{X}; E_\lambda) \), so we are done.
To prove the analogous result for the group representations we need a realization of the operators \(U_{-\mu}(g) \ (g \in G)\) under the Peter-Weyl transform. If \(g = k \in K\) this is immediate: \(U_{-\mu}(k)\) is given by the left-regular representation of \(K\) on \(L^2(K)\), so it preserves right \(K\)-types for both \(i = 1\) and \(2\), and hence lies in \(\mathcal{A}\). Thanks to the decomposition, \(G = KAK\), it now suffices to understand the operators \(U_{-\mu}(a)\), for \(a \in A\). It is easier to work with the infinitesimal operators \(U_{-\mu}(A)\) with \(A \in a\).

Let \(s(k) = (\xi^\dagger, \sigma(k)\xi)\) in \(L^2(X; E_{-\mu})\) be a matrix unit. Using the Iwasawa decomposition, we define functions \(\kappa, a\) and \(n\) by

\[
g = \kappa(g) a(g) n(g) \in KAN.
\]

For \(a \in A\), the \(\mathcal{B}\)-equivariance property of (2.1) gives

\[
U_{-\mu}(a)s(k) = s(k k^{-1} a^{-1} k) = e^{-\rho(a(k^{-1} a^{-1} k))} s(k \kappa(k^{-1} a^{-1} k)) = e^{-\rho(a(k^{-1} a^{-1} k))} (\xi^\dagger, \sigma(k) \sigma(k^{-1} a^{-1} k)) \xi),
\]

for any \(k \in K\). Now put \(a = \exp(tA)\), and take the derivative with respect to \(t\) at \(t = 0\). We get

\[
U_{-\mu}(A)s(k) = \rho(da_e(Ad k^{-1}(A))) (\xi^\dagger, \sigma(k) \xi) - (\xi^\dagger, \sigma(k) \sigma(\kappa^{-1} a^{-1} k)) \xi). \quad (3.5)
\]

The derivatives \(d\kappa_e\) and \(da_e\) at the identity \(e \in G\) are the projections of \(\mathfrak{g}\) onto the \(\mathfrak{t}\) and \(\mathfrak{a}\) parts, respectively, of the decomposition \(\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}\), respectively. However, the formula (3.5) uses \(d\kappa_e\) and \(da_e\) only on the real subspace \(\mathfrak{p}\) of self-adjoint matrices in \(\mathfrak{g}\), since \(A \in \mathfrak{p}\) and the adjoint representation preserves \(\mathfrak{p}\). This observation allows us to replace the maps \(d\kappa_e\) and \(da_e\), which are only \(\mathbb{R}\)-linear on \(\mathfrak{g}\), by more convenient \(\mathbb{C}\)-linear maps, which we now describe.

For each root \(\alpha\), let us denote by \(X_\alpha\) the elementary matrix in the \(\alpha\)-root space of \(\mathfrak{c}\), \(i.e,\) in the notation of Section 2.1 \(X_\alpha = X_1, X_{-\alpha} = X_{1}^\dagger, \epsilon= \text{etc.}\) The roots of \(K\) divide into positive and negative, and we define \(\text{sign}(\alpha)\) to be \(+1\) or \(-1\) accordingly. Let \(H_1, H_2 \in \mathfrak{a}\) be a basis for the Cartan subalgebra \(\mathfrak{h}\). Denote by \(X_\alpha^\dagger, H_j^\dagger\) the elements of the basis of \(\mathfrak{g}^\dagger\) dual to the above.

**Lemma 3.12.** On the subspace \(\mathfrak{p} \subseteq \mathfrak{g}\), \(d\kappa_e\) and \(da_e\) agree with the maps

\[
- \sum_{\alpha \in \Delta} \text{sign}(\alpha) X_\alpha \otimes X_\alpha^\dagger \in \mathfrak{g} \otimes \mathfrak{g}^\dagger = \text{End}(\mathfrak{g})
\]

and

\[
\sum_{i=1,2} H_i \otimes H_i^\dagger \in \mathfrak{g} \otimes \mathfrak{g}^\dagger = \text{End}(\mathfrak{g}),
\]

respectively.

**Proof.** One can check this directly on the basis

\[
X_\alpha + X_{-\alpha}, \quad (\alpha \in \Delta^+) \\
i X_\alpha - iX_{-\alpha}, \quad (\alpha \in \Delta^+) \\
H_i \quad (i = 1, 2).
\]

for \(\mathfrak{p}\). 

\[\square\]
We obtain the following formula for the group representation:

\[ U_{-\mu}(A)s(k) = \sum_{i=1,2} \rho(H_i) (H_i^\dagger, \text{Ad} k^{-1}(A)) (\xi^\dagger, \sigma(k)\xi) \]
\[ + \sum_{\alpha \in \Delta} \text{sign}(\alpha) (X^\dagger_\alpha, \text{Ad} k^{-1}(A)) (\xi^\dagger, \sigma(k)\sigma(X_\alpha)\xi). \]
\[ = (\xi^\dagger \otimes A, (\sigma \otimes \text{Ad}^\dagger) (k) \Xi(\xi)), \]

where

\[ \Xi(\xi) = \sum_{i=1,2} \rho(H_i) \xi \otimes H_i^\dagger + \sum_{\alpha \in \Delta} \text{sign}(\alpha) (X_\alpha \xi) \otimes X_\alpha^\dagger \in V^\sigma \otimes g^\dagger. \]

(We are now suppressing explicit mention of \( \sigma \) for notational convenience.) Thus, in the Peter-Weyl picture, the representation of \( a \) is described in terms of tensor products with the co-adjoint representation \( \text{Ad}^\dagger \) of \( K \).

In what follows, we will use the trace form

\[ B_0(W_1, W_2) = \text{Tr}(W_1 W_2), \quad (W_1, W_2 \in g) \]

to identify \( g \) and \( g^\dagger \). In particular, \( X_\alpha^\dagger \) corresponds to \( X_{-\alpha} = X_{\alpha}' \).

**Proposition 3.13.** For any \( g \in G \), and any weight \( \mu \) of \( \mathfrak{t} \), \( U_{-\mu}(g) \in A \).

**Proof.** We prove that \( U_{-\mu}(g) \in A_1 \). The proof that \( U_{-\mu}(g) \in A_2 \) is analogous.

Let \( A \in a \). Given the formula \((3.13)\), the effect of \( U_{\mu}(A) \) on right \( K_1 \)-types is governed by the maps \( \Xi : V^\sigma \rightarrow V^\sigma \otimes g^\dagger \) above. Suppose \( \xi \in V^\sigma \) is a vector of \( K_1 \)-type \( k \). We will split \( \Xi \) into two pieces to be analyzed separately. Let us fix a choice of \( H_1, H_2 \in a \), namely,

\[ H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \]

which are orthonormal for the trace form. Note also that \( H_1 \) is of \( K_1 \)-type 2, and \( H_2 \) is of \( K_1 \)-type 0.

We first claim that the vector

\[ \Xi_1(\xi) = \rho(H_1)\xi \otimes H_1^\dagger + (X_1\xi) \otimes X_1^\dagger - (X'_1\xi) \otimes (X'_1)^\dagger \]

has a norm-bound depending only on the \( K_1 \)-type of \( \xi \). This is clear since \( H_1 \), \( X_1 \) and \( X'_1 \) all belong to \( \mathfrak{t}_1 \), and hence act with fixed norm on vectors of a given \( K_1 \)-type. In fact, if we identify \( \mu|_{K_1} = m \in \mathbb{Z} \), then the well-known formulae for irreducible unitary representations of \( \mathfrak{su}(2) \) give

\[ \| \rho(H_1)\xi \| = \frac{1}{\sqrt{2}} |m| \|\xi\|; \]
\[ \| \sigma(X_1)\xi \| = \frac{1}{2} \sqrt{(k-m)(k+m+2)} \|\xi\|; \]
\[ \| \sigma(X'_1)\xi \| = \frac{1}{2} \sqrt{(k-m+2)(k+m)} \|\xi\|. \]
and hence
\[ \| \Xi_1(\xi) \| \leq \left( \frac{1}{2} k + \frac{1}{2} (k + 1) + \frac{1}{2} (k + 1) \right) \| \xi \| \leq (2k + 1) \| \xi \|. \]

So the norm-bound actually depends linearly on \( k \). Note also that the vectors \( H_1, X_1, \) and \( X'_1 \) all have \( K_1\)-type 2. By the fusion rules for \( SU(2) \) representations, this implies that only possible \( K_1 \)-types appearing in the vector \( \Xi_1(\xi) \) are \( k - 2, k \) and \( k + 2 \).

We remark that this estimate on \( \Xi_1(\xi) \in V^\sigma \otimes g^\dagger \) does not immediately carry over to a norm estimate on the corresponding part of \( U_{-\mu}(A) \), because of the factor \(|\sigma|^2\) which appears in the Peter-Weyl transform (2.36). However, the irreducible representations \( \sigma' \) appearing in \( \sigma \otimes Ad^\dagger \) have dimension \(|\sigma'| \geq \| \sigma \| \), as we will argue shortly. The section \( k \mapsto (\xi^\dagger \otimes A, (\sigma \otimes Ad^\dagger)(k)\Xi_1(\xi)) \) therefore has \( L^2 \)-norm bounded by \( \sqrt{18(2k + 1)} \| s \| \).

To obtain the stated dimension bound, suppose that \( \sigma \) has highest weight \( \beta = b_1 \delta_1 - b_2 \delta_2 \). Then \( |\sigma| = \frac{1}{2} (b_1 + 1)(b_2 + 1)(b_1 + b_2 + 2) \). The highest weight of \( \sigma' \) is \( \beta + \alpha \), for \( \alpha \in \Delta \cup \{ 0 \} \). Note that these weights \( \alpha \) have \( \alpha = a_1 \delta_1 - a_2 \delta_2 \), with \( |a_1|, |a_2| \leq 2 \), from which the estimate can be readily deduced.

Next, we claim that the remaining part of \( \Xi(\xi) \),
\[ \Xi_2(\xi) = \rho(H_2)\xi \otimes H_2 + X_2 \xi \otimes X'_2 - X'_2 \xi \otimes X_2 + X_\rho \xi \otimes X'_\rho - X'_\rho \xi \otimes X_\rho \]
is of \( K_1 \)-type \( k \) only. For the first term on the right this is immediate, since \( H_2 \) has \( K_1 \)-type 0. For the latter four terms, a computation must be done. A vector has a unique \( K_1 \)-type if and only if it is an eigenvector of the Casimir operator \( \Omega_{K_1} = 2X'_1X_1 + \frac{1}{2}H_1^2 + H_1 \) for \( K_1 \), and the \( K_1 \)-type is then uniquely determined by its eigenvalue. Since we are working in a fixed weight space for \( K \), the element \( H_1 \) acts as a fixed scalar, so it suffices to consider the action of \( X'_1X_1 \).

One can compute
\[
(\sigma \otimes ad)(X'_1X_1)(X_2 \xi \otimes X'_2 - X'_2 \xi \otimes X_2 + X_\rho \xi \otimes X'_\rho - X'_\rho \xi \otimes X_\rho)
= X'_1X_1X_2 \xi \otimes X'_2 - X'_1X_1X'_2 \xi \otimes X_2
+ X'_1X_2 \xi \otimes X'_\rho - X'_1X_1X'_\rho \xi \otimes X_\rho
- X'_2 \xi \otimes X_\rho - X'_1X_2 \xi \otimes X'_\rho - X_1X_2 \xi \otimes X'_\rho - X'_1X'_2 \xi \otimes X_2
- X'_2 \xi \otimes X_2 + X_\rho \xi \otimes X'_\rho. \tag{3.7}
\]
The first four terms on the right hand side of (3.7) can be rewritten as
\[
X_2X'_1X_1 \xi \otimes X'_2 - X'_2 X'_1X_1 \xi \otimes X_2 + X_\rho X'_1X_1 \xi \otimes X'_\rho - X'_\rho X'_1X_1 \xi \otimes X_\rho
+ X'_1X_\rho \xi \otimes X'_2 + X'_\rho X'_1X_1 \xi \otimes X_2 + X_2X_1 \xi \otimes X'_\rho + X'_1X'_2 \xi \otimes X_\rho.
\]
Hence, (5.7) equals
\[
X_2X'_1X_1 \xi \otimes X'_2 - X'_2X'_1X_1 \xi \otimes X_2 + X_\rho X'_1X_1 \xi \otimes X'_\rho - X'_\rho X'_1X_1 \xi \otimes X_\rho.
\]
We see that \( \Xi_2(\xi) \) is an eigenvector of \( X'_1X_1 \) with exactly the same eigenvalue as \( \xi \), as claimed.

Therefore, \( U_{-\mu}(A) \) satisfies the hypotheses on \( Q \) in the following lemma, which will complete the proof.
Lemma 3.14. Fix $i = 1$ or $2$. Let $Q$ be an unbounded skew-adjoint operator on $L^2(\mathcal{X}; E_{\mu})$, such that

(i) $P_j^{(i)} Q P_k^{(i)} = 0$ whenever $|j - k| > 2$, and

(ii) $P_k^{(i)} Q P_{k+2}^{(i)}$ and $P_{k+2}^{(i)} Q P_k^{(i)}$ are bounded operators for each $k \in \mathbb{N}$, and their norms are bounded by $C(k+1)$ for some universal constant $C$.

Then $e^Q \in A_i$.

Remark 3.15. Such an operator $Q$ has the flavour of a ‘discrete wave operator’ on the space $\mathbb{N}$, or more accurately, on the space $\log \mathbb{N}$. The following proof is an obvious generalization of the proof of finite propagation speed for ordinary wave operators.

Proof. We will prove that for any $\epsilon > 0$ and any $k \in \mathbb{N}$, there exists $k' > k$ such that

$$\| (P_{[0,k']}^{(i)})^{-1} e^Q P_{[0,k]}^{(i)} \| < \epsilon.$$ 

We may also apply this with $-Q$ in place of $Q$, and thus we will obtain property (ii) of Lemma 3.10 for $e^Q$.

Fix $k \in \mathbb{N}$ and choose any unit vector $u \in P_{[0,k]}^{(i)} L^2(\mathcal{X}; E_{\mu})$. Put $u_s = e^{sQ} u$ ($0 \leq s \leq 1$).

Let $h_n = \sum_{j=1}^n j^{-1}$ be the $n$th harmonic sum, and define $\phi : \mathbb{N} \to [0,1]$ by

$$\phi(n) = \begin{cases} 1, & n \leq k \\ \max\{0, 1 - \frac{e^2}{8C} (h_n - h_k)\}, & n > k \end{cases}.$$ 

Define an operator on $L^2(\mathcal{X}; E_{\mu})$, diagonal with respect to the $K_i$-type decomposition, by

$$\Phi = \sum_{n \in \mathbb{N}} \phi(n) P_n^{(i)}.$$ 

Now we decompose $Q$ into its diagonal and off-diagonal components. For brevity, let us put

$$Q_{m,n} = P_m^{(i)} Q P_n^{(i)}.$$ 

Then $Q = Q_- + Q_d + Q_+$, where

$$Q_- = \sum_{n \in \mathbb{N}} Q_{n,n+2}, \quad Q_d = \sum_{n \in \mathbb{N}} Q_{n,n}, \quad Q_+ = \sum_{n \in \mathbb{N}} Q_{n+n+2}.$$ 

(Note that all $K_i$-types appearing nontrivially in $L^2(\mathcal{X}; E_{-\mu})$ have the same parity as $\mu|_{K_i}$.) The diagonal component $Q_d$ commutes with $\Phi$. Meanwhile,

$$\| [Q_{n,n+2}, \Phi] \| = \| (\phi(n + 2) - \phi(n)) Q_{n,n+2} \| \leq \frac{e^2}{4C (n+1)} \|Q_{n,n+2}\|,$$

so

$$\| [Q_-, \Phi] \| = \sup_{n \in \mathbb{N}} \| Q_{n,n+2}, \Phi \| \leq \frac{1}{4} e^2.$$
Similarly for $Q_+$. Hence, $\| [Q, \Phi] \| \leq \frac{1}{2} \epsilon^2$.

We now have

$$\left| \frac{d}{ds} (\Phi u_s, u_s) \right| = \left| \langle \Phi Q u_s, u_s \rangle + \langle \Phi u_s, Qu_s \rangle \right| = \left| \langle [\Phi, Q] u_s, u_s \rangle \right| \leq \frac{1}{2} \epsilon^2,$$

for all $s \in [0, 1]$. Therefore,

$$\left| \langle \Phi u_1, u_1 \rangle \right| = \left| \langle \Phi u_0, u_0 \rangle + \int_0^1 \frac{d}{ds} (\Phi u_s, u_s) \right| \geq 1 - \frac{1}{2} \epsilon^2.$$

Let $k'$ be the smallest integer for which $\phi(k') < \frac{1}{2}$. Put $v = P_{[0,k']}^{(i)} u_1$ and $w = (P_{[0,k']}^{(i)})^\perp u_1$. Then $\|v\|^2 + \|w\|^2 = 1$, but also,

$$\|v\|^2 + \frac{1}{2} \|w\|^2 > \langle \Phi v, v \rangle + \langle \Phi w, w \rangle = \langle \Phi u, u \rangle \geq 1 - \frac{1}{2} \epsilon^2.$$

Therefore, $\|w\| = \| (P_{[0,k']}^{(i)})^\perp e^Q u \| \leq \epsilon$. Since $u$ was arbitrary in $P_{[0,k')}^{(i)} L^2 (\mathcal{X}; E_\mu)$, this proves the claim.

\[\square\]

### 3.4 Relationship with compact operators

In this section we will need to make use of the Gel’fand-Tsetlin bases for irreducible representations of SU(3). We a brief overview here for the sake of fixing notation, and refer the reader to [Mol06] for a full introduction.

Integral weights for the Lie group U(3) are parameterized by triples of integers:

$$\mu = (\mu_1, \mu_2, \mu_3) = \sum_{j=1}^3 \mu_j \delta_j.$$

Dominant weights are those for which $\mu_1 \geq \mu_2 \geq \mu_3$. We use the same notation for the weights of SU(3), acknowledging now that two triples represent the same weight if their difference is in $\mathbb{Z}(1,1,1)$.

The Gel’fand-Tsetlin basis vectors of an irreducible representation of U(3) with highest weight $\mu$ are indexed by patterns of integers

$$M = \begin{pmatrix} m_{31} & m_{32} & m_{33} \\ m_{21} & m_{22} & m_{11} \end{pmatrix}$$

where $m_{3k} = \mu_k$, and the entries satisfy the ‘betweenness conditions’ $m_{j+1,k} \geq m_{jk}$ and $m_{j+1,k+1}$. We denote the unit vector corresponding to a pattern $M$ by $(M)$. (If $M$ is a pattern which does not satisfy the betweenness conditions, we take $(M)$ to denote the zero vector.) Identifying U(1) and U(2) with the ‘upper-left’ subgroups

$$\begin{pmatrix} U(1) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} U(2) & 0 \\ 0 & 0 \end{pmatrix}$$

of U(3), the Gel’fand-Tsetlin vector $(M)$ is defined, up to phase, by the property that for each $j = 1,2,3$, $(M)$ belongs to an irreducible U($j$)-subrepresentation.
with highest weight equal to the \(j\)th row of \(M\). If \(S_j\) is the sum of the entries of the \(j\)th row of \(M\) (and \(S_0 = 0\) by convention), then the weight of \((M)\) is \((S_1 - S_0, S_2 - S_1, S_3 - S_2)\). When working instead with SU(3)-representations, two Gel’fand-Tsetlin patterns describe the same vector if they differ in each entry by an overall constant.

One could also define a Gel’fand-Tsetlin-type basis using the ‘lower-right’ subgroups

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & U(1) \end{pmatrix} \subseteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & U(2) \end{pmatrix} \subseteq U(3).
\]

This is most easily achieved as follows. Let

\[
\tilde{w} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in G,
\]

which is a representative of the Weyl group element

\[
w \cdot \mu = (\mu_1, \mu_2, \mu_3) \mapsto (\mu_3, \mu_2, \mu_1)
\]

Note that conjugation by \(\tilde{w}\) interchanges the upper-left and lower-right subgroups. Given a representation \(\sigma\) of \(G\), let \(\sigma'\) be the representation

\[
\sigma'(g) = \sigma(\tilde{w}g\tilde{w}^{-1}),
\]

which is isomorphic to \(\sigma\). The ordinary Gel’fand-Tsetlin basis for \(\sigma'\) is an alternative basis for \(V^\sigma\), whose basis vectors we denote by \((M)\)’.

Explicit formulae for the irreducible representations of \(g\) in the Gel’fand-Tsetlin basis can be found, for instance, in [Mol06].

**Lemma 3.16.** Fix \(\mu \in \Lambda_W\). For any finite sets \(A, B \subseteq \mathbb{N}\), \(P^{(2)}_B P^{(1)}_A\) is a compact operator on \(L^2(\mathcal{X}; E_\mu)\).

**Proof.** It suffices to prove the result with \(A = \{k\}\) and \(B = \{l\}\) singleton sets. We begin with the case \(\mu = 0, l = 0\), but \(k\) arbitrary.

Suppose \(s \in P^{(1)}_k L^2(\mathcal{X}; E_0)\) and \(t \in P^{(2)}_0 L^2(\mathcal{X}; E_0)\). Under the Peter-Weyl transform, \(t \mapsto \sum_{\eta} \eta^\dagger \otimes \eta\), with each \(\eta\) of weight 0 and \(K_2\)-type 0. The only such vectors are, up to scalar multiple,

\[
\eta_{n,0} \overset{\text{def}}{=} \begin{pmatrix} n & 0 & -n \\ 0 & 0 & 0 \end{pmatrix}' \in V^{(n,0,-n)},
\]

with \(n = 0, 1, \ldots\). Likewise, \(s \mapsto \sum_{\xi} \xi^\dagger \otimes \xi\), with each \(\xi\) of weight 0 and \(K_1\)-type \(k\). The Gel’fand-Tsetlin vectors of weight 0 in \(V^{(n,0,-n)}\) are

\[
\xi_{n,2j} \overset{\text{def}}{=} \begin{pmatrix} n & 0 & -n \\ j & -j & 0 \end{pmatrix}.
\]
for \( j = 0, \ldots, n \), and here \( \xi_{n,2j} \) is of \( K_1 \)-type \( 2j \). In particular, if \( k \) is odd then \( P_0^{(2)}P_k^{(1)} = 0 \) on \( L^2(\mathcal{X}; E_0) \). We therefore restrict attention to the even case, \( k = 2j \).

Let us write \( \eta_{n,0} = \sum_{j=0}^{n} c_{n,j} \xi_{n,2j} \), for some coefficients \( c_{n,j} \in \mathbb{C} \). Being of \( K_2 \)-type 0, \( \eta_{n,0} \) is annihilated by \( X_2 \). By the Gel’fand-Tsetlin formulae,

\[
X_2 \left( \begin{pmatrix} n & 0 & -n \\ j & -j \\ 0 & \end{pmatrix} \right) = (j+1) \left( \begin{pmatrix} (n-j)(n+j+2) \\ (2j+1)(2j+2) \end{pmatrix} \right)^{\frac{1}{2}} \left( \begin{pmatrix} n & 0 & -n \\ j+1 & -j \\ 0 & \end{pmatrix} \right) + j \left( \begin{pmatrix} (n-j+1)(n+j+1) \\ 2j(2j+1) \end{pmatrix} \right)^{\frac{1}{2}} \left( \begin{pmatrix} n & 0 & -n \\ j & -j+1 \\ 0 & \end{pmatrix} \right).
\]

Solving for the coefficients \( c_{n,j} \) we obtain, up to phase,

\[
\eta_{n,0} = \sum_{j=0}^{n} \frac{\sqrt{2j+1}}{n+1} \xi_{n,2j}. \tag{3.8}
\]

Therefore,

\[
|\langle \xi_{n,2j}, \eta_{n,0} \rangle| = \frac{\sqrt{2j+1}}{n+1}.
\]

Let \( R_N \in \mathcal{B}(L^2(\mathcal{X}; E_0)) \) be the projection onto the subspace spanned by sections of \( K \)-type \( (n,0,-n) \), for \( n = 0, \ldots, N \). Note that \( R_N \) commutes with \( P_k^{(1)} \) and \( P_0^{(2)} \). If \( s \in P_k^{(1)}L^2(\mathcal{X}; E_0) \), \( t \in P_0^{(2)}L^2(\mathcal{X}; E_0) \), then by (3.4),

\[
|\langle R_N t, s \rangle| \leq \frac{\sqrt{k+1}}{N+1} \|s\| \|t\|.
\]

Therefore \( \|R_N P_0^{(2)}P_k^{(1)}\| \leq \sqrt{k+1}/(N+1) \). Since \( N \) is arbitrary and \( R_N \) is finite rank, this proves that \( P_0^{(2)}P_k^{(1)} \) is compact.

Now let \( \mu, k \) and \( l \) be arbitrary. Use Lemma 3.3 to find a finite collection of continuous sections \( s_j \in P_k^{(1)}C(\mathcal{X}; E_\mu) \), \( t_{j'} \in P_l^{(2)}C(\mathcal{X}; E_\mu) \) and bounded linear maps

\[
\varphi_j : P_k^{(1)}L^2(\mathcal{X}; E_\mu) \to P_0^{(1)}L^2(\mathcal{X}; E_0),
\]

\[
\psi_{j'} : P_l^{(2)}L^2(\mathcal{X}; E_\mu) \to P_0^{(2)}L^2(\mathcal{X}; E_0),
\]

such that \( \sum_j s_j \varphi_j = \text{Id} \), \( \sum_{j'} t_{j'} \psi_{j'} = \text{Id} \). Then

\[
P_l^{(2)}P_k^{(1)} = \sum_{j,j'} P_l^{(2)}(\psi_{j'}^{*}P_0^{(2)}t_{j'})(s_j P_0^{(1)} \varphi_j P_k^{(1)}).
\]

For each \( j, j' \), we have \( t_{j'}s_j \in C(\mathcal{X}; E_0) \subseteq \mathcal{A} \). Thus, \( t_{j'}s_j P_k^{(1)} \) can be approximated arbitrarily well in norm by \( P_k^{(1)}t_{j'}s_j P_0^{(1)} \), for some \( k' \in \mathbb{N} \). Since \( P_0^{(2)}P_k^{(1)} \) is compact, the result follows.

\[ \square \]

**Proposition 3.17.** The \( C^* \)-category \( K = K_1 \cap K_2 \) is the category of compact operators between the section spaces \( L^2(\mathcal{X}; E_\mu) \).
Proof. For \( i = 1 \) or \( 2 \), the projections \( (P_{0,k}^{(i)})^\perp \) converge strongly to 0 as \( k \to \infty \). Thus \( (P_{0,k}^{(i)})^\perp K \) and \( K(P_{0,k}^{(i)})^\perp \) converge to zero in norm for any compact operator \( K \). By Lemma 3.9, the compact operators belong to \( K_1 \cap K_2 \).

If \( T_1 \in K_1(E_\lambda, E_\mu) \) and \( T_2 \in K_2(E_\mu, E_\nu) \), are spectrally finite for \( \alpha_1 \) and \( \alpha_2 \), respectively, then for some \( k_1, k_2 \in \mathbb{N} \),

\[
T_2 T_1 = T_2 P_{0,k_2}^{(2)} P_{0,k_1}^{(1)} T_1,
\]

which is compact by the previous lemma. A density argument completes the proof. \( \square \)

A consequence of Proposition 3.17 is that elements of \( K_1 \) are multipliers of \( K_2 \). Therefore, by Lemma 3.10, \( K_1 \subseteq A_2 \) and hence \( K_1 \vartriangleleft A \). Similarly, \( K_2 \vartriangleleft A \).

Summarizing, we have the following important lattice of ideals:

\[
\begin{array}{c}
A \\
\downarrow \\
K_1 \\
\downarrow \\
K_2 \\
\downarrow \\
K
\end{array}
\]

### 4 Normalized differential operators

#### 4.1 Tangential pseudodifferential operators

In this section we will show how the ideals \( K_i \) relate to Connes’ foliation \( C^* \)-algebra for the fibrations \( X_i \to Y_i = G/P_i \). For background to the material used here, we refer to [MS06].

For \( i = 1 \) or \( 2 \), let \( F_i \) denote the foliation of \( X \) associated to the fibration \( X_i \to Y_i \). Let \( E \) and \( E' \) be bundles over \( X \). We denote the algebra of order zero pseudodifferential operators from \( E \) to \( E' \), tangential along \( F_i \), by \( \Psi_0^{(i)}(E, E') \). If \( E = E' \) we abbreviate this to \( \Psi_0^{(i)}(E) \). Let \( S^* F_i \) be the cosphere bundle of the foliation \( F_i \). Recall ([Con79]) that the tangential principal symbol map

\[
\text{Symb}_0 : \Psi_0^{(i)}(E) \to C(S^* F_i, \text{End}(E))
\]

extends to the norm-closure \( \overline{\Psi_0^{(i)}(E)} \) in \( B(L^2(X; E)) \), and yields a short exact sequence of \( C^* \)-algebras

\[
0 \longrightarrow C^*_r(G_i) \longrightarrow \overline{\Psi_0^{(i)}(E)} \overset{\text{Symb}_0}{\longrightarrow} C(S^* F_i, \text{End}(E)) \longrightarrow 0.
\]

Here \( C^*_r(G_i) \) is the \( C^* \)-algebra of the foliation groupoid \( G_i \) of \( F_i \), represented on sections of the bundle \( E \).

Since \( F_i \) is in fact a fibration, \( C^*_r(G_i) \) has a realization in terms of Hilbert module operators. The section space \( C(X; E) \) becomes a pre-Hilbert module over \( C(Y_i) \) if we define a \( C(Y_i) \)-valued inner product by \( L^2 \)-integration along the fibres. Specifically, in the case \( E = E_\mu \ (\mu \in \Lambda_W) \), the inner product is

\[
\langle s_1, s_2 \rangle_{C(Y_i)}(k) = \int_{K_i} \overline{s_1(kk_1)} s_2(kk_1) dk_1 \quad (k \in K),
\]

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for $s_1, s_2 \in C(\mathcal{X}; E_\mu)$. Denote the Hilbert module completion by $\mathcal{E}_i(\mathcal{X}; E_\mu)$. If $E = \bigoplus_\mu E_\mu$, then $\mathcal{E}_i(\mathcal{X}; E) = \bigoplus_\mu \mathcal{E}_i(\mathcal{X}; E_\mu)$. The algebra $C^*_r(\mathcal{G}_i)$ is precisely the algebra of compact Hilbert-module operators on $\mathcal{E}_i(\mathcal{X}; E)$.

The next proposition describes the relationship between $C^*_r(\mathcal{G}_i)$ and $\mathcal{K}_i(E, E)$.

**Proposition 4.1.** Let $\mu, \nu \in \Lambda_W$. If $T : \mathcal{E}_i(\mathcal{X}; E_\mu) \to \mathcal{E}_i(\mathcal{X}; E_\nu)$ is a compact operator in the sense of Hilbert $C(\mathcal{Y}_i)$-modules, then its extension to an operator $L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu)$ belongs to $\mathcal{K}_i(E_\mu, E_\nu)$.

**Proof.** If $t_1 \in C(\mathcal{X}; E_\mu)$, $t_2 \in C(\mathcal{X}; E_\nu)$ are each of a single $\mathcal{K}_i$-type, then the ‘rank-one’ operator

$$s \mapsto \langle t_1, s \rangle_{C(\mathcal{Y}_i)} t_2$$

clearly satisfies condition (iii) of Lemma 3.9. Such operators span a dense subspace of the $C(\mathcal{Y}_i)$-compact operators, and extension of compact Hilbert module operators to bounded $L^2$-operators is continuous with respect to the norm topologies.

\[\square\]

### 4.2 Normalized BGG operators

In this section we prove various properties of the phases $X^n_i$ of the $K$-invariant differential operators $X^n_i : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu + n \alpha_i)})$. To begin with, we note that most of the analysis will reduce to the case $n = 1$. This is because $X^n_i = (X_i)^n$, where the right-hand side here denotes the composition of the operators $X_i : L^2(\mathcal{X}; E_{-(\mu + j \alpha_i)}) \to L^2(\mathcal{X}; E_{-(\mu + j \alpha_i)})$ for $j = 1, \ldots, n$.

Fix $i = 1$ or 2, and fix a weight $\mu \in \Lambda_W$. Let $D_i$ be the essentially self-adjoint tangentially elliptic first-order differential operator

$$D_i = \begin{pmatrix} 0 & X_i' \\ X_i & 0 \end{pmatrix}$$

on sections of $E = E_{-\mu} \oplus E_{-(\mu + \alpha_i)}$. In what follows, we will make use of symbolic calculus, and since the operators $\Phi(D_i)$ are defined using functional calculus, some remarks are in order. Firstly, since the spectrum of $D_i$ is discrete, there is some smooth function $\phi : \mathbb{R} \to [-1, 1]$ such that $\Phi(D_i) = \phi(D_i)$. Applying Theorem XII.1.3 of [Tay81] fibrewise, we see that $\Phi(D_i)$ is a tangential pseudodifferential operator of order zero, and moreover that its principal symbol is the order zero homogeneous part of $\phi(\text{Symb}(D_i))$, i.e., $\text{Symb}(D_i) \text{Symb}(D_i)^{-1}$.

Let $N_i$ (respectively, $Z_i$) be the element of $n$ (respectively, $\mathfrak{g}$) which is represented by the same matrix as defines $X_i$ (respectively, $X_i'$) in $\mathfrak{t}_\mathbb{C}$. Let $J$ denote the operation of multiplication by $\sqrt{-1}$ in $\mathfrak{g}$. For any $V \in \mathfrak{g}$, let

$$V^h = \frac{1}{2}(V - iJV), \quad \overline{V}^h = \frac{1}{2}(V + iJV) \quad \in \mathfrak{g}_\mathbb{C}.$$ 

Recall that in Section 3.3 we defined functions $\kappa$, $\alpha$ and $\eta$ by the Iwasawa decomposition: $g = \kappa(g)\alpha(g)\eta(g) \in K\mathbb{A}N$. We abbreviate $\alpha \eta(g) = \alpha(g)\eta(g)$.

**Lemma 4.2.** For each $g \in G$, the differential operator $X_i : C^\infty(\mathcal{X}; E_{-\mu}) \to C^\infty(\mathcal{X}; E_{-(\mu + \alpha_i)})$ satisfies

$$U_{-(\mu + \alpha_i)}(g)X_i U_{-\mu}(g^{-1}) = c_g X_i + d_g,$$

where $c_g(k) = e^{\alpha_i}(\alpha(g^{-1}k))$ for $k \in K$, and $d_g$ is some smooth section of $E_{-\alpha_i}$. 

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Note that $c_g$ is a smooth positive function on $X$ and is independent of $\mu$.

Remark 4.3. If one puts a holomorphic structure on the bundles $E_\mu$ by transferring the natural holomorphic structure from the complex Lie group $Z$, using the nilpotent picture, then both $X_i$ and $U_{-(\mu+a_1)}(g)X_iU_{-\mu}(g^{-1})$ are anti-holomorphic differential operators along the complex one-dimensional fibres of $F_i$. Granted this, the lemma is then trivial, at least for some positive function $c_g$. However, the independence of $c_g$ with respect to $\mu$ is important to us, so we include a proof which calculates it.

Proof. Let $s \in C^\infty(X; E_{-\mu})$, considered as a $B$-equivariant function on $G$. Since $X_i = N_i - \overline{Z_i}^\mu$, and $s$ is $N$-invariant, we have $X_iz = -\overline{Z_i}^\mu s$. Recall from Remark 4.3 that to realize the section $X_i s \in C^\infty(X; E_{-(\mu+a_1)})$ on $G$, one needs to extend $X_i s$ by $B$-equivariance from $K$: $X_i s(g) = e^{-\rho(a)} X_i s(\kappa(a))$.

For any $k \in K$ and $g \in G$,

$$ (Z_iU_{-\mu}(g^{-1})s)(\kappa(g^{-1}k)) = \frac{d}{dt}s(g \kappa(g^{-1}k)e^{tZ_i})|_{t=0} = \frac{d}{dt}s(k \kappa(g^{-1}k)e^{tZ_i})|_{t=0} = e^\rho(a(g^{-1}k)) (Ad(\kappa(g^{-1}k))Z_i)s(k). $$

For any $a \in A$ and $n \in N$, $Ad(an)Z_i = e^{-\alpha(a)}(a)Z_i + B$, for some $B \in b$. By the $B$-equivariance of $s$, we obtain

$$ (Z_iU_{-\mu}(g^{-1})s)(\kappa(g^{-1}k)) = e^\rho(a(g^{-1}k)) \kappa(a(g^{-1}k)) Z_is(k) + d'_g(k)s(k) $$

for some smooth function $d'_g$ on $K$. Therefore,

$$ (U_{-(\mu+a_1)}(g)Z_iU_{-\mu}(g^{-1})s)(k) = e^{\alpha(a(g^{-1}k))} Z_is(k) + d''_g(k)s(k), $$

for some smooth $d''_g$ on $K$. The same is true with $JZ_i$ in place of $Z_i$ (if we replace $d''_g$ by $id''_g$), and the result follows.

Let $\varphi_\mu : Z \otimes \mathbb{C} \to E_\mu$ be the trivializing coordinate patch of $E_\mu$ given by the nilpotent picture, which is to say $\varphi_\mu^*s = s|_Z$ for $s \in C(X; E_\mu)$. We refer to $\varphi$ as the ‘standard chart’ on $E_\mu$. The next lemma describes $X_i$ in these coordinates.

Lemma 4.4. For $s \in C^\infty(X; E_\mu)$,

$$ \varphi_{-(\mu+a_1)}^*X_is = (-f_0\overline{Z_i}^\mu + f_1)\varphi_{-\mu}^*s, $$

for some smooth functions $f_0$, $f_1$ on $Z$. Moreover, $f_0$ is everywhere positive, and is independent of $\mu$.

Proof. At the identity in $G$, $X_is = -\overline{Z_i}^\mu s$, as observed in the proof of the previous lemma. In the standard chart, the representation $U_{-\mu}(z)$, with $z \in Z$, is just left-translation by $z$. Now apply the preceding lemma to see that the left-invariant vector field $-\overline{Z_i}^\mu$ is everywhere linearly related to the differential operator $X_i$ in these coordinates.
Proposition 4.5. For $\mu, \nu \in \Lambda_W$. Let $s \in C(\mathcal{X}; E_{-\nu})$ define a multiplication operator

$$s : L^2(\mathcal{X}; E_{-\mu} \oplus E_{-(\mu+\alpha_i)}) \to L^2(\mathcal{X}; E_{-(\mu+\nu)} \oplus E_{-(\mu+\nu+\alpha_i)}).$$

Then $sD_i - D_is \in \mathcal{K}_i$.

**Proof.** In the standard chart for these bundles, $s$ is multiplication by $\varphi^*_\nu s$ (independent of $\mu$). By Lemma 4.4, the principal symbol of $D_i$, and hence of $\text{Ph}(D_i)$, is the same on both the domain and range of $s$. Thus, $\text{Symb}(s \text{Ph}(D_i) - \text{Ph}(D_i)s) = 0$ on the standard chart, which has dense image. \hfill \Box

Proposition 4.6. Let $\mu \in \Lambda_W$. For any $g \in G$,

$$(U_{-\mu} \oplus U_{-(\mu+\alpha_i)})(g), D_i) \in \mathcal{K}_i.$$ 

**Proof.** For brevity, let us write $U = U_{-\mu} \oplus U_{-(\mu+\alpha_i)}$. From Lemma 4.2 $U(g)D_iU(g^{-1}) = c_g D_i + R_g$, for some order zero operator $R_g$ on $E_{-\mu} \oplus E_{-(\mu+\alpha_i)}$. Therefore, the principal symbol of $U(g) \text{Ph}(D_i)U(g^{-1}) = \text{Ph}(U(g)D_iU(g^{-1}))$ is

$$c_g \text{Symb}(D_i) \left| c_g \text{Symb}(D_i) \right|^{-1} = \text{Symb}(D_i) \| \text{Symb}(D_i) \|^{-1} = \text{Symb}(D_i).$$

Thus, $U(g) \text{Ph}(D_i)U(g^{-1}) - \text{Ph}(D_i) \in \mathcal{K}_i$, and since $U(g) \in \mathcal{A}_i$, we are done. \hfill \Box

Looking at the matrix entries of $D_i$, the preceding two lemmas imply that the operators $sX_i - X_is$ and $U_{-(\mu+\alpha_i)}(g)X_i - X_iU_{-\mu}(g)$ are also in $\mathcal{K}_i$.

Proposition 4.7. The operator

$$X_i : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+\alpha_i)})$$

belongs to $\mathcal{A}_i$.

**Proof.** We prove the case $i = 1$, the other case being analogous. Since $X_1$ preserves $K_1$-types, it is clearly in $\mathcal{A}_i$, and we need only show that it belongs to $\mathcal{A}_2$. For this, we begin with a specific computation showing that, for the operator $X_1 : L^2(\mathcal{X}; E_{0}) \to L^2(\mathcal{X}; E_{\alpha_1})$, given any $\epsilon > 0$ there is $k' \in \mathbb{N}$ such that

$$\| (P^{(2)}_{[0,k']})^{-1} X_1 P^{(2)}_{0} \| < \epsilon. \quad (4.1)$$

Let $V^{(\mu_1, \mu_2, \mu_3)}$ denote $V^\sigma$, where $\sigma$ is the irreducible representation of $K$ with highest weight $(\mu_1, \mu_2, \mu_3)$. Put

$$\xi_{n,j} = \begin{pmatrix} n & 0 & -n \\ j & -j & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \eta_{n,j} = \begin{pmatrix} n & 0 & -n \\ j & -j & 0 \\ 0 & 0 & 0 \end{pmatrix}^t,$$

which are the Gel’fand-Tsetlin vectors in $(V^{(n,0,-n)})_0$ of $K_1$-type $2j$ and $K_2$-type $2j$, respectively (see Section 4.4). Similarly, we have two Gel’fand-Tsetlin bases for the weight spaces $(V^{(n,0,-n)})_{\alpha_1}$, comprised respectively of the vectors

$$\xi'_{n,j} = \begin{pmatrix} n & 0 & -n \\ j & -j & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \eta'_{n,j} = \begin{pmatrix} n & 0 & -n \\ j & -j & 0 \\ 1 & 0 & 0 \end{pmatrix}^t,$$
for $j = 1, \ldots, n$. We begin by estimating the quantity $\langle \eta'_{n,j}, X_1 \eta_{n,0} \rangle$.

From Equation (3.8),

$$\eta_{n,0} = \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{2a+1} \xi_{n,j}$$

(up to phase). From the Gel’fand-Tsetlin formulae,

$$X_1 \xi_{n,a} = \sqrt{a(a+1)} \xi'_{n,a} = \sqrt{a(a+1)} X_1 \xi_{n,a}.$$

The automorphism $g \mapsto \tilde{w} g \tilde{w}^{-1}$, which was used to define the ‘lower-right’ Gel’fand-Tsetlin basis in Section 3.4, interchanges $X_1'$ and $X_2$. Therefore, using the Gel’fand-Tsetlin formula for the action of $X_2$,

$$\langle \eta'_{n,b}, X_1 \eta_{n,0} \rangle = \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{2a+1} (\langle \eta'_{n,b}, X_1 \xi_{n,a} \rangle)$$

$$= \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{\frac{2a+1}{a(a+1)}} (\langle X_1' \eta'_{n,b}, \xi_{n,a} \rangle)$$

$$= \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{\frac{2a+1}{a(a+1)}} \frac{b}{2} (n-b+1)(n+b+1)^{\frac{1}{2}} \left( \frac{1}{\sqrt{2b+1}} \langle \xi_{n,b}, \xi_{n,a} \rangle + \frac{1}{\sqrt{2b-1}} \langle \eta'_{n,b-1}, \xi_{n,a} \rangle \right)$$

$$= \frac{1}{n+1} \sqrt{\frac{b}{2}} (n-b+1)(n+b+1) \sum_{a=0}^{n} \frac{2a+1}{a(a+1)}(x_{n,a,b} - x_{n,a,b-1}), \quad (4.2)$$

where we have put $x_{n,a,b} = (2a+1)^{\frac{1}{2}} (2b+1)^{-\frac{1}{2}} \langle \eta_{n,b}, \xi_{n,a} \rangle$. Let us now estimate $x_{n,a,b}$ for large $n$.

By the Gel’fand-Tsetlin formulae,

$$a(a+1)x_{n,a,b}$$

$$= (2a+1)^{-\frac{1}{2}} (2b+1)^{-\frac{1}{2}} (X_1' X_1 \xi_{n,a}, \eta_{n,b})$$

$$= (2a+1)^{-\frac{1}{2}} (2b+1)^{-\frac{1}{2}} (\xi_{n,a}, X_1' X_1 \eta_{n,b})$$

$$= \frac{b(n-b+1)(n+b+1)}{2(2b+1)} x_{n,a,b-1} + \frac{1}{2} (n(n+2) - b(b+1)) x_{n,a,b}$$

$$+ (b+1)(n-b)(n+b+2) x_{n,a,b+1}.$$

This yields the recurrence relation

$$b(n-b+1)(n+b+1) x_{n,a,b-1}$$

$$+(2b+1)(n(n+2) - b(b+1) - 2a(a+1)) x_{n,a,b}$$

$$+(b+1)(n-b)(n+b+2) x_{n,a,b+1} = 0, \quad (4.3)$$

which can solved, in principle, from the initial condition $x_{n,a,0} = (n+1)^{-1}$ of Equation (4.8). (One initial condition suffices since when $b = 0$, the first term in

\footnote{To avoid repeating this phrase throughout we may adjust the phase of the highest-weight vector in the ‘lower right’ Gel’fand-Tsetlin basis to correct the phase error.}
Lemma 3.3 vanishes.) However, if $n$ is large in comparison with $b$ (and $a$ is arbitrary), then after dividing by $n^2$, (4.3) is well approximated by

$$b \left( x_{n,a,b-1} + (2b+1)(1-2n^{-2}a(a+1)) x_{n,a,b} + (b+1) x_{n,a,b+1} \right) = 0. \quad (4.4)$$

The solution to (4.4) is $x_{n,a,b} = (-1)^b (n+1)^{-1} P_b(1-2n^{-2}a(a+1))$, where $P_b$ is the $b$th Legendre polynomial.

Therefore,

$$\lim_{n \to \infty} \sum_{a=0}^{n} \left( \frac{2a+1}{a(a+1)} \right) x_{n,a,b} = \begin{cases} 1 & \text{if } b = 0, \\ 0 & \text{otherwise}. \end{cases}$$

By (4.2), then,

$$\lim_{n \to \infty} \int_0^1 2P_b(1-2t^2) \, dt = \begin{cases} \frac{2}{b+1} & \text{if } b = 0, \\ \frac{1}{2b-1} - \frac{1}{2b+1} & \text{otherwise}. \end{cases}$$

So, for any $k' \in \mathbb{N}$,

$$\lim_{n \to \infty} \left| \langle \eta_{n,b}, X_1 \eta_{n,0} \rangle \right|^2 = \lim_{n \to \infty} \left( \|X_1 \eta_{n,0}\|^2 - \sum_{b=0}^{k'} \left| \langle \eta_{n,b}, X_1 \eta_{n,0} \rangle \right|^2 \right) = \frac{1}{(2k'-1)^2}. \quad (4.5)$$

Looking now to sections, we note that a section $s$ in $P_0^{(2)} L^2(\mathcal{X}; E_0)$ has Peter-Weyl transform $\sum_n \beta_n^s \otimes \eta_{n,0}$, for some vectors $\beta_n^s \in \mathcal{V}^{(n,0,-n)}$. Therefore, $X_1 s = \sum_n \beta_n^s \otimes X_1 \eta_{n,0}$. Let $R_N$ denote the finite-rank projection onto the subspace of $L^2(\mathcal{X}; E_{-\alpha_1})$ spanned by sections of $K$-type $(n,0,-n)$, for $n = 0, \ldots, N$. By the above computation, for any $\varepsilon > 0$, one can choose $N$ and $k'$ large enough such that $\| (R_N)^\perp (P_{[0,k']}^{(2)})^\perp X_1 P_0^{(2)} \| < \varepsilon$ on $L^2(\mathcal{X}; E_0)$. But the $K$-representation of type $(n,0,-n)$ contains $K_2$-types only up to $2n$, so if we enlarge $k'$ to be greater than $2N$, we have $\| (P_{[0,k']}^{(2)})^\perp X_1 P_0^{(2)} \| < \varepsilon$ on $L^2(\mathcal{X}; E_0)$, as claimed.

With this base case completed, we now let $\mu \in \Lambda_W$, and $k \in \mathbb{N}$ be arbitrary. Choose sections $s_j \in P_0^{(2)} C(\mathcal{X}; E_{-\mu})$ and maps $\phi_j : P_k^{(2)} L^2(\mathcal{X}; E_{-\mu}) \to P_0^{(2)} L^2(\mathcal{X}; E_0)$ as in Lemma 3.3. Then

$$X_1 P_k^{(2)} = \sum_j X_1 s_j P_0^{(2)} \phi_j P_k^{(2)} = \sum_j (s_j X_1 + [X_1, s_j]) P_0^{(2)} \phi_j P_k^{(2)}$$

Let $\varepsilon > 0$. Choose $k'$ satisfying (4.1). By Propositions 3.11 and 4.5, we can also find $k''$ sufficiently large such that $\| (P_{k''}^{(2)})^\perp s_j P_k^{(2)} \| < \varepsilon$ and $\| (P_{k''}^{(2)})^\perp [X_1, s_j] \| < \varepsilon$, for each $j$. Thus, as an operator from $L^2(\mathcal{X}; E_0)$ to $L^2(\mathcal{X}; E_{(\mu+\alpha_1)^{\perp}})$,

$$\| (P_{k''}^{(2)})^\perp (s_j X_1 + [X_1, s_j]) P_0^{(2)} \| \leq \| (P_{k''}^{(2)})^\perp s_j P_k^{(2)} X_1 P_0^{(2)} \| + \| (P_{k''}^{(2)})^\perp s_j P_k^{(2)} [X_1, P_0^{(2)}] \| + \| (P_{k''}^{(2)})^\perp [X_1, s_j] P_0^{(2)} \| \leq (\| s_j \| + 2) \varepsilon.$$
and hence, as an operator from $L^2(\mathcal{X}; E_{-\mu})$ to $L^2(\mathcal{X}; E_{-(\mu+\alpha_i)})$,

$$\| (P_k^{(2)})' X_1 P_k^{(2)} \| < C\epsilon,$$

for some constant $C$.

The same kind of estimates can be proven for $X'_1 = X_1^*$ in place of $X_1$. (The analogue of (4.5) for $X'_1$ can be most easily obtained by switching the representations $V^{(n,0,-m)}$ with their (unitarily equivalent) contragredient representations, which transforms $X'_1$ to $-X_1$. The rest of the argument goes through by a mere substitution of $X'_1$ for $X_1$.) Therefore, by Lemma 3.9 $X_1 \in A(E_{-\mu}, E_{-(\mu+\alpha_i)})$ for each $\mu \in \Lambda_W$.

\[ \text{Proposition 4.8.} \text{ Let } i = 1 \text{ or } 2, \mu, \nu \in \Lambda_W, \text{ and } n \in \mathbb{N}. \text{ For any } s \in L^2(\mathcal{X}; E_{\nu}), \text{ and any } g \in \mathfrak{g}, \text{ the ‘commutators’}
\]

$$X_i^n s - s X_i^n : L^2(\mathcal{X}; E_{-\mu}) \rightarrow L^2(\mathcal{X}; E_{-(\mu+\nu+n\alpha_i)}),$$

and

$$X_i^n U_{-\mu} - U_{-(\mu+n\alpha_i)} X_i^n : L^2(\mathcal{X}; E_{-\mu}) \rightarrow L^2(\mathcal{X}; E_{-(\mu+\alpha_i)})$$

belong to $K_i$.

\[ \text{Proof.} \text{ Expand } X_i^n s - s X_i^n = \sum_{j=1}^n X_i^{j-1}(X_is - sX_i)X_i^{n-j}, \text{ and apply Lemma 4.5.} \text{ The group representation case follows in a similar fashion from Lemma 4.6.} \]

We conclude this section by remarking that Lemma 5.5 of [AS68] applies to tangential pseudodifferential operators:

\[ \text{Lemma 4.9.} \text{ Let } A \in \mathfrak{g}. \text{ The one-parameter family of operators}
\]

$$t \mapsto U_{-(\mu+n\alpha_i)}(\exp(tA))X_i U_{-\mu}(\exp(-tA))$$

is continuous in the norm topology.

In the language of Kasparov, $X_i$ is $G$-continuous.

4.3 Weyl commutation relations

The representations $U_\mu$ are unitary principal series representations of $G$. The representation $U_\mu$ is irreducible for any $\mu \in \Lambda_W$. Moreover, two such representations, $U_\mu$ and $U_{\mu'}$ are unitarily equivalent if and only if $\mu$ and $\mu'$ are in the same orbit of the Weyl group. If $\mu' = w_i \cdot \mu$, where $w_i \in W$ is the reflection associated to the simple root $\alpha_i$, then the intertwining operator between the two representations can be described explicitly.

\[ \text{Proposition 4.10.} \text{ Suppose } \mu' = w_i \cdot \mu, \text{ and let } n \text{ be the integer such that }
\]

$$\mu - \mu' = n\alpha_i.$$ The unitary intertwiner

$$I = I_{\mu,\alpha_i} : L^2(\mathcal{X}; E_{-\mu}) \rightarrow L^2(\mathcal{X}; E_{-\mu'})$$

is $I = X_i^n$ if $n \geq 0$, and $I = (X'_i)^n$ if $n \leq 0$.  

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These intertwining operators satisfy the ‘commutation relation’
\[ I_{w_{21},\mu,\alpha_1} I_{w_{11},\mu,\alpha_2} I_{w_{21},\mu,\alpha_1} = I_{w_{11},w_{21},\mu,\alpha_2} I_{w_{21},\mu,\alpha_1}, \]
for any \( \mu \in \Lambda_W \) (KS61). In particular,
\[ X_1 X_2^2 X_1 = X_2 X_1^2 X_2 : L^2(\mathcal{X}; E_\rho) \to L^2(\mathcal{X}; E_{-\rho}). \quad (4.7) \]
Note that the weights \( \pm \rho \) defining the domain and range are crucial in this identity. Nevertheless, if the domain is changed, a weaker commutation property still holds.

**Proposition 4.11.** As an operator from \( L^2(\mathcal{X}; E_0) \) to \( L^2(\mathcal{X}; E_{-2\rho}) \),
\[ X_1 X_2^2 X_1 - X_2 X_1^2 X_2 \in K_1 + K_2. \]

**Proof.** Using a trivializing partition of unity for \( E_\rho \), one can find a finite collection of sections \( s_1, \ldots, s_2 \in C(\mathcal{X}; E_\rho) \) such that \( \sum_{j=1}^n s_j s_j = 1 \in C(\mathcal{X}) \). Then, as an operator from \( L^2(\mathcal{X}; E_0) \) to \( L^2(\mathcal{X}; E_{-2\rho}) \),
\[ X_1 X_2^2 X_1 = \sum_{j=1}^n X_1 X_2^2 X_1 s_j s_j \]
\[ = \sum_{j=1}^n \overline{s}_j X_1 X_2^2 X_1 s_j \quad \text{(modulo } K_1 + K_2), \]
by repeatedly applying Proposition 3.11. In this last equation, \( X_1 X_2^2 X_1 \) is an operator from \( L^2(\mathcal{X}; E_\rho) \) to \( L^2(\mathcal{X}; E_{-\rho}) \), so is equal to \( X_2 X_1^2 X_2 \). Reversing the same process, we see that
\[ X_1 X_2^2 X_1 = X_2 X_1^2 X_2 : L^2(\mathcal{X}; E_0) \to L^2(\mathcal{X}; E_{-2\rho}) \]
modulo \( K_1 + K_2 \).

## 5 Construction of the gamma element

Let \( n \in \mathbb{Z}^+ \) and \( \mu \in \Lambda_W \). Using the standard formulae for irreducible representations of \( \mathfrak{sl}(2, \mathbb{C}) \), it is readily observed that the operators \( X^n : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+n\alpha_i)}) \) and \( X^{-n} : L^2(\mathcal{X}; E_{-(\mu+n\alpha_i)}) \to L^2(\mathcal{X}; E_{-\mu}) \) are inverses modulo \( K_i \), since their product in either order equals \( P^{(i)}_{[0,k]} \) for some \( k \). Let us define operators
\[ T_1 = -X_1 X_2 X_1' : L^2(\mathcal{X}; E_{-\alpha_1}) \to L^2(\mathcal{X}; E_{-(2\alpha_1+\alpha_2)}), \]
\[ T_2 = -X_2 X_1 X_1' : L^2(\mathcal{X}; E_{-\alpha_2}) \to L^2(\mathcal{X}; E_{-(\alpha_1+2\alpha_2)}). \]
Thanks to Proposition 4.11, these operators are defined precisely so that each of the four diamonds
\[ \begin{array}{c}
\text{•} \\
\text{•} \\
\text{•} \\
\text{•}
\end{array} \]
in the diagram

\[
\begin{align*}
L^2(\mathcal{X}; E_{-\alpha_1}) & \xrightarrow{T_1} L^2(\mathcal{X}; E_{-(2\alpha_1+\alpha_2)}) \\
L^2(\mathcal{X}; E_0) & \xrightarrow{T_2} L^2(\mathcal{X}; E_{-(\alpha_1+2\alpha_2)}) \\
L^2(\mathcal{X}; E_{-\alpha_2}) & \xrightarrow{X_1} L^2(\mathcal{X}; E_{-(2\alpha_1+\alpha_2)}) \\
L^2(\mathcal{X}; E_{-\alpha_2}) & \xrightarrow{X_2} L^2(\mathcal{X}; E_{-(\alpha_1+2\alpha_2)})
\end{align*}
\]

anticonnutes modulo \(K_1 + K_2\).

Let \(\Upsilon = \{ w \cdot (-\rho) + \rho \mid w \in \mathcal{W} \} \subseteq \Lambda_W\). These are the negatives of the weights appearing in the diagram above. Recall that there is a length function \(l : W \to \mathbb{N}\) on the Weyl group, defined by letting \(l(w)\) be the length of the shortest word in simple reflections which represents \(w\). We partition \(\Upsilon\) according to the length of the Weyl group elements: \(\Upsilon_p = \{ w \cdot (-\rho) + \rho \mid w \in \mathcal{W}, \ l(w) = p \}\). Let \(H_p\) denote the Hilbert space

\[H_p = \bigoplus_{\mu \in \Upsilon_p} L^2(\mathcal{X}; E_{-\mu}),\]

and let \(H = \bigoplus_{p=0}^3 H_p\), endowed with the \(\mathbb{Z}/2\mathbb{Z}\)-grading coming from the parity of \(p\).

If \(\mu, \nu \in \Upsilon\), we identify an operator in \(\mathcal{B}(E_{-\mu}, E_{-\nu})\) with the operator on \(H\) obtained by extending it trivially on each \(L^2(\mathcal{X}; E_{-\mu'})\) for \(\mu' \neq \mu\). Let \(\mathcal{A}(H), K_1(H), K_2(H), K(H)\) denote the above defined \(C^*\)-algebras of operators on \(H = \bigoplus_{\mu \in \Upsilon} L^2(\mathcal{X}; E_{-\mu})\).

Denote by \(Q_{\mu}\) the projection onto a component \(L^2(\mathcal{X}; E_{-\mu})\) of \(H\). Let the continuous functions \(f \in C(\mathcal{X})\) act ‘diagonally’ as multiplication operators on each component of \(H\), and let group elements be represented on \(H\) by \(U(g) = \bigoplus_{\mu \in \Upsilon} U_{-\mu}(g)\).

The following is a simple generalization of a key step in the construction of the Kasparov product (see [HR00, Proposition 9.2.5]).

**Lemma 5.1.** There exist positive operators, \(M_1, M_2 \in \mathcal{B}(H)\) with \(M_1^2 + M_2^2 = 1\), such that

(i) \(M_i K_i(H) \subseteq K(H)\) for \(i = 1, 2\),

(ii) \(M_1\) and \(M_2\) commute, modulo \(K(H)\), with all multiplication operators \(f \in C(\mathcal{X})\), all representations of group elements \(U(g)\) for \(g \in G\), and all operators appearing in the normalized BGG complex \(\mathcal{L}_N\).

(iii) \(M_1\) and \(M_2\) commute on the nose with the representation of the compact group \(U(K)\), and with the projections \(Q_{\mu}\).

**Proof.** Let \(\mathcal{S} \subseteq \mathcal{B}(H)\) be the set consisting of all multiplication operators \(f \in C(\mathcal{X})\), all \(U(g)\) for \(g \in G\), all operators appearing in the normalized BGG complex, and all projections \(Q_{\mu}\) for \(\mu \in \Upsilon\). With \(i = 1, 2\), let \(K_3^i(H)\) be the smallest \(C^*\)-subalgebra of \(\mathcal{B}(H)\) which contains the projections \(P_k^{(i)}\) and
Theorem 5.2. The operator $F$ on the graded Hilbert space $H$, together with the multiplication representation of $C(X)$ and the unitary representation $U(G)$, defines a cycle $\theta$ of $KK(C(X), \mathbb{C})$.

**Proof.** Since $[X_i, f], [X_i, U(g)] \in \mathcal{K}_i(H)$ for all $f \in C(X), g \in G$, it is straightforward to see that each component of $F$ commutes with each $f$ and $U(g)$ modulo compact operators. The diamonds in (5.2) anti-commute modulo $\mathcal{K}(H)$, and hence the components of $F^2$ which alter the degree $p$ are all compact. We consider now those components which preserve the degree. Let $\sim$ denote equality modulo compact operators.

- The component of $F^2$ preserving $H_0 = L^2(X; E_0)$ is
  \[ X'_1 M_1^2 X_1 + X'_2 M_2^2 X_2 \sim M_1^2 (X'_1 X_1 - 1) + M_2^2 (X'_2 X_2 - 1) + 1 \sim 1. \]

- The component preserving $H_1 = L^2(X; E_{\alpha_1}) \oplus L^2(X; E_{\alpha_2})$ is given by a $2 \times 2$-matrix. Modulo compacts, the component mapping $L^2(X; E_{\alpha_1})$ to itself is
  \[ M_2^2 X_1X'_1 + M_1^2 X'_2 X_2 + M_1^2 M_2^2 T_1T_1 \sim M_1^2 + M_2^2 + M_1^2 M_2^2 = 1. \]

and the component mapping $L^2(X; E_{\alpha_1})$ to $L^2(X; E_{\alpha_2})$ is
\[
M_1 M_2 X_2X'_1 + M_1^3 M_2 X_1X'_1 X_1' + M_1 M_2^3 T_1 T_1 \sim M_1 (M_1 M_2 (X_2X'_1 + X'_1 + T_1 T_1)) + M_2 (M_1 M_2 (X_2X'_1 + T_1 T_1)) \sim 0.
\]

Amend the diagram (5.1) as follows:

![Diagram](5.2)

Let $F$ denote the operator on $H$ obtained by adding together all the operators of this diagram, plus their adjoints.
The other two components can be computed similarly, with the result that the diagonal components equal 1 and off-diagonal components equal 0 modulo compacts.

- On $H_2$ and $H_3$, analogues of the above calculations similarly show that $F^2$ equals the identity modulo compact operators.

Finally, by Lemma 4.9, $F$ is $G$-continuous.

**Proposition 5.3.** The $K$-equivariant index of $\theta$ is 1, and hence, by Theorem 1.1, it is a model for $\gamma \in KK^G(C, C)$.

**Proof.** The operator $F$ is $K$-equivariant on the nose, so its $K$-index is the sum of the $K$-indices of each $K$-isotypical component. Since these components are all finite-dimensional, this amounts to simply determining the graded dimension of each component of $H$. These graded dimensions can be deduced immediately by observing that, as $U(K)$-representations, the spaces $L^2(X; E_{-\mu})$ appearing in (5.2) are exactly the same as (the $L^2$-completions of) those appearing in the BGG resolution of the trivial representation for $G$.

**Remark 5.4.** If one does not wish to appeal to the BGG resolution, one can instead make a direct computation of the graded dimensions by using the Weyl character formula or the more elementary remarks of [FH91, p. 184].

**References**


