INTERTWINING OPERATORS OF IRREDUCIBLE REPRESENTATIONS FOR EXPONENTIAL SOLVABLE LIE GROUPS

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Abstract. Let G be a real solvable exponential Lie group with Lie algebra $\mathfrak{g}$ and let $f \in \mathfrak{g}^*$. We take two polarizations $\mathfrak{p}_j, j = 1, 2$, at f which meet Pukanszky's condition. Let $P_j := \exp(\mathfrak{p}_j), j = 1, 2$, be the associated subgroups in G. The linear functional f defines unitary characters $\chi_j(\exp(X)) := e^{-i\langle f,X\rangle}$, $X \in \mathfrak{p}_j$, of P_j. Let $\tau_j := \mathrm{ind}_{P_j}^G \chi_j$, $j = 1, 2$, be the corresponding induced representations, which are unitary and irreducible. It is well known that τ_1 and τ_2 are unitary equivalent. The description of the intertwining operator of such an equivalence is given via an abstract integral $I_{\mathfrak{p}_2,\mathfrak{p}_1}$ over the homogeneous space $P_2/P_1 \cap P_2$ and one of the main problems is the convergence of such an integral. In this paper, we show that the product P_1P_2 is closed in G. This then implies that our integral converges at least on a dense subspace of elements of the space of τ_1 and we can prove that this formal integral gives us a concrete intertwining operator. We can in this way avoid the use of a third Pukanszky polarization, which was necessary in the approach made in [1]. Finally, given three Pukanszky polarizations $\mathfrak{p}_i, i = 1, 2, 3$ at f, we accurately determine the composition formula of $I_{\mathfrak{p}_3,\mathfrak{p}_2} \circ I_{\mathfrak{p}_2,\mathfrak{p}_1} \circ I_{\mathfrak{p}_1,\mathfrak{p}_3}$ using Maslov's index.

1. Introduction

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. *Exponential* means that the exponential mapping $\exp : \mathfrak{g} \to G$ is a C^∞ diffeomorphism. Therefore the group G is connected and simply connected. The orbit method makes it possible to parametrize the unitary dual $\hat{G}$ of G by the space of coadjoint orbits of G in $\mathfrak{g}^*$, the dual vector space of $\mathfrak{g}$. It is well known that for such groups, any unitary and irreducible representation π is monomial and is precisely of the form $\pi_{f,\mathfrak{h}} := \mathrm{ind}_H^G \chi_f$ where $f \in \mathfrak{g}^*$ and $H = \exp(\mathfrak{h})$ stands for a polarization at f which fulfills the Pukanszky condition. The next section is devoted to providing broader explanations of these facts. However, it is also required that such a representation does depend only on the orbit of the linear form f through the coadjoint action, and that the choice of the polarizing subgroups at f is somehow irrelevant. This means that for any other polarization $H' = \exp(\mathfrak{h}')$ at f meeting the Pukanszky condition, the monomial representations $\pi_{f,\mathfrak{h}}$ and $\pi_{f,\mathfrak{h}'}$ are unitarily equivalent. It is then so natural to seek whether it is possible to concretely build a unitary intertwining operator which realizes such an equivalence. This problem which was advocated first by M. Vergne, is solved in the setup where G is nilpotent in [9] by G. Lion. He shows that for any function $\xi \in \mathcal{H}^{+\infty}_{\pi_{f,\mathfrak{h}'}}$, the integral function $T_{H',H}\xi$

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defined by

$$T_{H',H}\xi(g)=\int_{H/H\cap H'}\xi(gh)\chi_f(h)dh, g\in G \tag{1.1}$$

defines a C^∞ vector of the representation $\pi_{f,\mathfrak{h}}$ and that the operator $T_{H',H}$ continuously extends to be an isometric intertwining operator up to a normalization of the involved measure classes in question. Nilpotent Lie groups are unimodular and it is pretty easy to see that the direct product $H\cdot H'$ is closed in G. So no hurdle stands to the convergence of the integral 1.1 defined above when restricted to the subspace of $\mathcal{H}^{+\infty}_{\pi_{f,\mathfrak{h}}}$ of $C^{+\infty}$ functions with compact support modulo H. Beyond this setup, evidence has accumulated that the difficulties involved in this context are considerable for several structural and technical reasons. In [5], the second author of the present paper suggests a formal candidate to the intertwining operator, still denoted by $T_{H',H}$, for the context of exponential Lie groups:

$$T_{H',H}\xi(g)=\int_{H/H\cap H'}\xi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)dh, g\in G, \tag{1.2}$$

ξ is merely taken in addition with compact support modulo H. When restricted to a class of exponential Lie groups for which the pair $(\mathfrak{h},\mathfrak{h}')$ of the polarizing subalgebras meets a structural condition denoted by $\mathcal{N}$, he also drew a prototypical proof of the unitarity of such an operator as well. This condition on $(\mathfrak{h},\mathfrak{h}')$ is settled in a way such that the simple product of their Lie groups is closed in G. Later, the two last authors together with D. Arnal showed in [1] that the operator $T_{H',H}$ extends to be an isometric intertwining operator for arbitrary exponential Lie groups whenever one the polarizations in question is of M. Vergne. This somehow permits them to consider a general instance making use of a third Vergne type polarization. This approach was quite efficient to a certain extent, and this is mainly due to the irrelevance of that third polarization involved through the composition formula.

In this paper, we again pay attention to this problem. We show first that the direct product of two Pukanszky polarizations is closed in G. Such a fundamental upshot gives rise to the convergence of the integral 1.2 at least on a dense subspace of the $C^{+\infty}$ vector space of the first representation in question. This allows us to overcome all the difficulties involved in this setup and to prove that the operator 1.2 can be extended to a unitary intertwining operator. We will finally be devoted to show that given three Pukanszky polarizations $\mathfrak{h}_i, i=1,2,3$ at $f\in\mathfrak{g}^*$ in $\mathfrak{g}$, the composition formula using the Maslov index of the Kashiwara quadratic form holds. Such a formula was obtained first for nilpotent Lie groups in [9] and later in [5] for exponential solvable Lie groups such that the pairs of polarizations $(\mathfrak{h}_i,\mathfrak{h}_j)(1\le i,j\le 3)$ in question fulfill the condition $\mathcal{N}$. The last section of the paper aims to prove that the formula can be settled to encompass arbitrary exponential solvable Lie groups.

2. Preliminaries and notations

We begin this section by reviewing some useful facts and notations regarding some aspects of monomial representations of an exponential solvable Lie group. The material dealt with here is quite standard, we refer the reader to the references [1,2,4,5,6,7,8,12,16] for more complete details. Throughout, $\mathfrak{g}$ will be a n-dimensional real exponential Lie algebra, G will be the associated connected and simply connected exponential Lie group. The Lie group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G and on $\mathfrak{g}^*$, the dual vector space by the coadjoint representation Ad_G^*. The space of coadjoint orbits is denoted therefore by $\mathfrak{g}^*/Ad_G^*$. Let $\hat{G}$ be the unitary dual of G, the set of all equivalence classes of unitary and irreducible representations of G. Let $f \in \mathfrak{g}^*$. We then have the skew symmetric bilinear form B_f on $\mathfrak{g}$ defined by $B_f(X,Y) := \langle f, [X,Y]\rangle$, for $X, Y \in \mathfrak{g}$. Let $\mathfrak{g}(f)$ be the radical of this bilinear form, which is just the Lie algebra of the stabilizer $G(f)$ of f in G, i.e. $G(f) = \exp(\mathfrak{g}(f))$. Let us define for a subspace $\mathfrak{a}$ of $\mathfrak{g}$

$$\mathfrak{a}^{\perp,\ \mathfrak{g}^*} := \{\ell \in \mathfrak{g}^*; \ell|_{\mathfrak{a}} = 0\},$$

$\ell|_{\mathfrak{a}}$ being the restriction of ℓ to $\mathfrak{a}$. Let also

$$\mathfrak{a}^f = \{X \in \mathfrak{g}; B_f(X,Y) = 0,\ \forall Y \in \mathfrak{a}\}.$$

To simplify notations, we shall write $\mathfrak{a}^{\perp}$ instead of $\mathfrak{a}^{\perp,\ \mathfrak{g}^*}$ wherever there is no possible confusion.
If $\mathfrak{a} \subset \mathfrak{a}^f$ (resp. $\mathfrak{a} = \mathfrak{a}^f$), then we say that $\mathfrak{a}$ is isotropic (resp. Lagrangian) for B_f. It is well known that an isotropic subspace $\mathfrak{a}$ is Lagrangian, if and only if

$$\dim(\mathfrak{a}) = \frac{1}{2}(\dim(\mathfrak{g}) + \dim(\mathfrak{g}(f))).$$

We denote by $S(f,\mathfrak{g})$ (resp. $M(f,\mathfrak{g})$) the family of all isotropic, (resp. Lagrangian) subspaces for B_f.
Let dg be a left Haar measure on G and let Δ_G be the modular function of G. Thus

$$\int_G \varphi(gx^{-1})dg = \Delta_G(x)\int_G \varphi(g)dg \quad (x \in G)$$

for every φ belonging to the space $C_c(G)$ of continuous functions with compact support on G. We have

$$\Delta_G(x) = |\det(\mathrm{Ad}x)|^{-1} \quad (x \in G).$$

Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$ and we denote by $\Delta_{H,G}$ the real character of H defined by

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)} \quad (h \in H).$$

Hence for $X \in \mathfrak{h}$, we have

$$\Delta_{H,G}(\exp(X)) = \exp(\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}X).$$

Let $\mathcal{E}(G/H)$ be the space of all complex-valued continuous functions with compact support modulo H, which satisfy the covariance condition

$$\varphi(gh) = \Delta_{H,G}(h)\varphi(g)$$

for all $g \in G$ et $h \in H$. The group G acts by left translation on $\mathcal{E}(G/H)$ and there exists a unique (up to a positive multiple) translation invariant positive linear functional on this space. We shall denote it by the symbol $\nu = \nu_{G,H}$ and we write it in the form of an integral

$$\nu_{G,H}(\varphi) = \oint_{G/H} \varphi(g)d\nu_{G,H}(g).$$

If $\Delta_H = \Delta_G$ on H, then $\nu_{G,H}$ is simply an invariant measure on the homogeneous space G/H. It will sometimes be denoted by $d\dot{g}$.

We take now another closed subgroup K of H. The next lemma will play an important role in the sequel.

Lemma 2.1. *(Transitivity Lemma [2], chap. V)*
Let $K \subset H$ be two closed subgroups of G and let φ be a $\nu_{G/H}$-integrable function. Then the set of all the g's in G, for which the function

$$h \mapsto \varphi(gh)\Delta_{H,G}^{-1}(h)$$

of H is not $\nu_{H,K}$-integrable is $\nu_{G,H}$ negligible. The function defined on G by

$$g \mapsto \oint_{H/K} \varphi(gh)\Delta_{H,G}^{-1}(h)d\nu_{H,K}(h)$$

is $\nu_{G,H}$-integrable and, up to a normalization, we have the formula

$$\oint_{G/K} \varphi(x)d\nu_{G,K}(x) = \oint_{G/H} d\nu_{G,H}(g) \oint_{H/K} \varphi(gh)\Delta_{H,G}^{-1}(h)d\nu_{H,K}(h). \tag{2.3}$$

Let σ be a unitary representation of H on a Hilbert space $\mathcal{H}$. We denote by $C_c(G/H,\sigma)$ the space of all continuous mappings $\varphi : G \to \mathcal{H}$, compactly supported modulo H, which satisfy the covariance condition

$$\varphi(gh) = \Delta_{H,G}^{\frac{1}{2}}(h)\sigma(h)^{-1}\varphi(g) \quad (g \in G,\ h \in H).$$

When φ belongs to $C_c(G/H,\sigma)$, the function $g \mapsto \|\varphi(g)\|_{\mathcal{H}}^2$ belongs to the space $\mathcal{E}(G/H)$ and we let

$$\|\varphi\|_2^2 := \oint_{G/H} \|\varphi(g)\|_{\mathcal{H}}^2 d\nu(g).$$

The induced representation $\mathrm{ind}_H^G\sigma$ of G is realized by left translation on the Hilbert space completion $L^2(G/H,\sigma)$ of $C_c(G/H,\sigma)$ equipped with the norm $\|\|_2$.
For an element $\mathfrak{h}$ in $S(f,\mathfrak{g})$, we denote by χ_f the unitary character of $H = \exp(\mathfrak{h})$ defined by

$$\chi_f(\exp(X)) = e^{-if(X)} \quad (X \in \mathfrak{h}).$$

This character induces a unitary representation $\pi := \pi_{f,\mathfrak{h}} := \mathrm{ind}_H^G\chi_f$ of G on the Hilbert space $L^2(G/H,\chi_f)$, the completion of the vector space $\mathcal{C}_c(G/H,\chi_f)$ of the complex-valued continuous functions of G, which are compactly supported modulo H and which satisfying the covariance condition

$$\varphi(gh) = \chi_f^{-1}(h)\Delta_{H,G}(h)^{1/2}\varphi(g) \quad (g \in G,\ h \in H).$$

We denote by $I(f,\mathfrak{g})$ the subset of $S(f,\mathfrak{g})$ consisting of all the subalgebras $\mathfrak{h}$ for which the representation $\pi_{f,\mathfrak{h}}$ is irreducible. We then have Pukanszky's criterion [2]:

Theorem 2.2. *Let $\mathfrak{h}$ be an element of $S(f,\mathfrak{g})$. The following conditions are equivalent:*

1. $H{\cdot}f = f + \mathfrak{h}^\perp$
2. $\mathfrak{h} \in M(f,\mathfrak{g})$ *and* $f + \mathfrak{h}^\perp \subset G{\cdot}f$*;*
3. $\mathfrak{h} \in M(\ell,\mathfrak{g})$ *for all* $\ell \in f + \mathfrak{h}^\perp$*;*
4. $\mathfrak{h} \in I(f,\mathfrak{g})$.

When the conditions of Theorem 2.2 are satisfied, then we say that $\mathfrak{h}$ satisfies Pukanszky's condition or simply that $\mathfrak{h}$ is a *Puk-polarization at f*.

In order to construct a Puk-polarization at f, we can use the standard method of M. Vergne. Let $(\mathfrak{g}_j)_{j=0}^n$ be a good sequence of subalgebras of $\mathfrak{g}$, i. e. a sequence of subalgebras of $\mathfrak{g}$,

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},\ \dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1\ (1 \leq j \leq n),$$

and such that if $\mathfrak{g}_j$ is not an ideal of $\mathfrak{g}$, then $\mathfrak{g}_{j-1}$ and $\mathfrak{g}_{j+1}$ are both ideals of $\mathfrak{g}$ and the quotient space $\mathfrak{g}_{j+1}/\mathfrak{g}_{j-1}$ is an irreducible $\mathfrak{g}$ module. For $1 \leq j \leq n$, let f_j be the restriction of f to $\mathfrak{g}_j$. Then the subspace $\sum_{j=1}^n \mathfrak{g}_j(f_j)$ is a polarization at f which satisfies Pukanszky's condition. The polarizations constructed in this fashion are called Vergne Polarizations.

Let us introduce other ingredients. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. An ordered set $\{X_1, X_2, \ldots, X_m\}$ of elements of $\mathfrak{g}$ is called an co exponential basis (relatively to $\mathfrak{k}$) if the mapping

$$((x_1, x_2, \ldots, x_m), Y) \mapsto \left(\prod_{j=1}^{m} \exp(x_j X_j)\right) \exp(Y)$$

is a diffeomorphism of $\mathbb{R}^m \times \mathfrak{k}$ onto the group G. Such a basis always exists (cf. [2]).

If π is a unitary representation of the group G in a Hilbert space $\mathcal{H}_\pi$, we denote by $\mathcal{H}_\pi^\infty$ the space of the C^∞ vectors of π equipped with its usual topology (cf. [3], [15]) and by $\mathcal{H}_\pi^{-\infty}$ its anti-dual. Given a complex character σ of a subgroup K of G, let

$$\left(\mathcal{H}_\pi^{-\infty}\right)^{K,\sigma} := \{a \in \mathcal{H}_\pi^{-\infty}; \pi(k)a = \sigma(k)a,\ \forall k \in K\}.$$

The determination of these spaces is an interesting problem. For instance, if $K = \exp(\mathfrak{k})$ where $\mathfrak{k} \in S(f, \mathfrak{g})$, we consider the monomial representation $\tau = \mathrm{ind}_K^G \chi_f$ of G and the Dirac measure $\delta_\tau : \mathcal{H}_\tau^\infty \to \mathbb{C}$ defined by $\delta_\tau : \varphi \mapsto \overline{\varphi(e)}$, where e denotes the identity element of G. By construction, δ_τ belongs to $(\mathcal{H}_\tau^{-\infty})^{K,\chi_f \Delta_{K,G}^{1/2}}$ and would like to obtain an explicit disintegration of δ_τ using elements in $(\mathcal{H}_\pi^{-\infty})^{K,\chi_f \Delta_{K,G}^{1/2}}$ for representations $\pi \in \hat{G}$ appearing in the disintegration of τ (cf. [7], [8], [14]). When G is nilpotent and $\mathfrak{k} \in M(f, \mathfrak{g})$, if we realize τ on the space $L^2(\mathbb{R})^m$ with the help of a co-exponential basis relative to $\mathfrak{k}$, we see that [4] the space of the C^∞-vectors $\mathcal{H}_\tau^\infty$ coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ of the rapidly decreasing C^∞-functions. In this case the anti-dual $\mathcal{H}_\tau^{-\infty}$ can be identified with the space $\mathcal{S}'(\mathbb{R}^m)$ of the tempered distributions.

We now come back to our main problem. Let as before $G = \exp(\mathfrak{g})$ be a solvable exponential Lie group with Lie algebra $\mathfrak{g}$. We take $f \in \mathfrak{g}^*$ and two Puk-polarizations $\mathfrak{p}_1,\ \mathfrak{p}_2$ in $I(f, \mathfrak{g})$. It follows that the representation $\rho(f, \mathfrak{p}_1, G)$ is equivalent to $\rho(f, \mathfrak{p}_2, G)$. We propose to construct an explicit intertwining operator between them. The first essential step in [5] had been the discovery of the relation

$$\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_1/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X + \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X = 0, X \in \mathfrak{p}_1 \cap \mathfrak{p}_2. \tag{2.4}$$

(see Théorème 2 in [5])

For the convenience of the reader, we give a (new) proof of this relation 2.4. In fact we can reformulate this relation in the following way.

Lemma 2.3. *Let $f \in \mathfrak{g}^*$, $\mathfrak{p}_1 \in I(f, \mathfrak{g})$ and $\mathfrak{p}_2 \in M(f, \mathfrak{g})$. We write $\mathfrak{p}_1 = \mathfrak{m}_1 \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2)$ and $\mathfrak{p}_2 = \mathfrak{m}_2 \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2)$ for some subspaces $\mathfrak{m}_i \subset \mathfrak{p}_i, i = 1, 2$. Let $\{\hat{e}_j; 1 \leq j \leq m\}$ be a* *basis of* $\mathfrak{m}_2$ *and* $\{e_i; 1 \leq i \leq m\}$ *the dual basis of* $\mathfrak{m}_1$ *with respect to the form* B_f*. Then the vector* $W := \sum_{i=1}^m [e_i, \hat{e}_i]$ *is contained in* $\mathfrak{p}_1 + \mathfrak{p}_2$*. In particular for every* $X \in \mathfrak{p}_1 \cap \mathfrak{p}_2$*, the relation 2.4 holds.*

Proof. The claim is clear if when $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{g}$. So we can assume that $f_{|\mathfrak{p}_1+\mathfrak{p}_2} \neq \{0\}$. We compute for $X \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ the number $\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X + \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X$ in the following way:

$$\begin{aligned}
\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X + \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X &= \sum_{i=1}^m -B_f(\hat{e}_i, [X, e_i]) + \sum_{i=1}^m B_f(e_i, [X, \hat{e}_i]) \\
&= -\sum_{i=1}^m \langle f, [\hat{e}_i, [X, e_i]]\rangle + \sum_{i=1}^m \langle f, [e_i, [X, \hat{e}_i]]\rangle \\
&= \sum_{i=1}^m \langle f, [X, [e_i, \hat{e}_i]]\rangle \\
&\quad -\langle f, [\hat{e}_i, [X, e_i]] + [X, [e_i, \hat{e}_i]] + [e_i, [\hat{e}_i, X]]\rangle \\
&= \sum_{i=1}^m \langle f, [X, [e_i, \hat{e}_i]]\rangle.
\end{aligned}$$

Hence

$$\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X + \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X = B_f(X, W), \tag{2.5}$$

where $W := \sum_{i=1}^m [e_i, \hat{e}_i]$. Whence, if $W \in \mathfrak{p}_1 + \mathfrak{p}_2$, then we see that $\langle f, [X, W]\rangle = 0$ for every $X \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ and so relation 2.4 is true.
Suppose now that W is not contained in $\mathfrak{p}_1 + \mathfrak{p}_2$. Hence there exists $X \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, such that $\langle f, [X, W]\rangle = 1$. Let $\mathfrak{k} := [\mathfrak{m}_1, \mathfrak{m}_2] + \mathfrak{p}_1 + \mathfrak{p}_2 = [\mathfrak{p}_1, \mathfrak{p}_2] + \mathfrak{p}_1 + \mathfrak{p}_2$. It is clear that $W \in \mathfrak{k} \setminus (\mathfrak{p}_1 + \mathfrak{p}_2)$ and that $\mathrm{a}dX(\mathfrak{k}) \subset \mathfrak{k}$. Choose a vector W_0 in $\mathfrak{p}_1 + \mathfrak{p}_2$, such that for $U := W + W_0$ we have that $\langle f, U\rangle = 0$. Let $q := -\mathrm{a}d^*(X)f$. Then the relations $\langle f, U\rangle = 0, \langle q, U\rangle = \langle f, [X, U]\rangle = \langle f, [X, W]\rangle = 1$ hold. Let $V := [X, U] + \gamma U \in \mathfrak{k}$, where $\gamma \in \mathbb{R}$ is chosen such that $\langle q, V\rangle = 0$. It is clear that $\langle f, V\rangle = 1$ and that $\langle f, [X, V]\rangle = \langle q, V\rangle = 0$. Let

$$\mathfrak{w} = \{S \in \mathfrak{g}, \langle f, [X, S]\rangle = 0\} \cap \ker(f).$$

Then $\mathfrak{w}$ contains $(\mathfrak{p}_1 + \mathfrak{p}_2) \cap \ker(f)$ and $\mathfrak{g}$ is the direct sum of $\mathbb{R}U, \mathbb{R}V$ and $\mathfrak{w}$. Therefore we have two linear functionals λ, μ on $\mathfrak{g}$ and a linear mapping $w : \mathfrak{g} \to \mathfrak{w}$, such that

$$S = \lambda(S)U + \mu(S)V + w(S), S \in \mathfrak{g}. \tag{2.6}$$

For $c \in \mathbb{R}$, let $\ell_c \in \mathfrak{g}^*$ be the linear functional which coincides with f on $\mathfrak{w} + \mathbb{R}V$ and for which $\ell_c(U) = c$. For the particular value $c = 0$, we get obviously $\ell_0 = f$. In particular

$$\langle \ell_c, S\rangle = c\lambda(S) + \mu(S), S \in \mathfrak{g}\ (c \in \mathbb{R}). \tag{2.7}$$

Using Pukanszky's condition, we have that $\mathfrak{p}_1 \in M(\ell_c, \mathfrak{g}), c \in \mathbb{R}$. Since $\langle \ell_c, [\mathfrak{p}_2, \mathfrak{p}_2]\rangle = \{0\}$ and since $\dim(\mathfrak{p}_2) = \dim(\mathfrak{p}_1)$, $\mathfrak{p}_2$ is in $M(\ell_c, \mathfrak{g})$ for every $c \in \mathbb{R}$. Hence the spaces $\mathfrak{p}_1/(\mathfrak{p}_1 \cap \mathfrak{p}_2)$ and $\mathfrak{p}_2/(\mathfrak{p}_1 \cap \mathfrak{p}_2)$ are dual one to each other with respect to the bilinear form B_{ℓ_c} for all $c \in \mathbb{R}$.

Let $\{e_k(c); 1 \leq k \leq m\}$ be the basis of $\mathfrak{m}_1$ which is dual to the basis $\{\hat{e}_j; 1 \leq j \leq m\}$ with respect to the form B_{ℓ_c}. We write

$$e_k(c) := \sum_{j=1}^{m} \lambda_{kj}(c) e_j$$

with $\lambda_{kj}(c) \in \mathbb{R}$ $(1 \leq j,\ k\ \leq m)$.
Letting $W(c) := \sum_{j=1}^{m} [e_j(c), \hat{e}_j]$ $(c \in \mathbb{R}), W := W(0)$, we get as in 2.5 that for any $X \in \mathfrak{p}_1 \cap \mathfrak{p}_2$

$$\beta := \text{tr ad}_{\mathfrak{p}_1/(\mathfrak{p}_1 \cap \mathfrak{h}_2)} X + \text{tr ad}_{\mathfrak{p}_2/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X = B_{\ell_c}(X, W(c)). \tag{2.8}$$

Let us write according to equation 2.6

$$[e_i, \hat{e}_j] = \beta_{j,i} U + \delta_{ij} V + w_{j,i}. \tag{2.9}$$

where $w_{j,i} \in \mathfrak{w}$, where $\beta_{j,i} \in \mathbb{R}$ $(1 \leq i,\ j \leq m)$ and where δ_{ij} denotes the Kronecker symbol. Therefore

$$\begin{aligned} \beta = \text{tr ad}_{\mathfrak{p}_1/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X + \text{tr ad}_{\mathfrak{p}_2/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X &= \sum_{i=1}^{m} \langle f, [X, [e_i, \hat{e}_i]] \rangle \\ &= \sum_{i=1}^{m} \langle f, [X, \beta_{i,i} U] \rangle \\ &= \sum_{i=1}^{m} \beta_{ii}. \end{aligned} \tag{2.10}$$

Furthermore, we have for any $c \in \mathbb{R}$ that

$$\begin{aligned} W(c) &= \sum_{k=1}^{m} \sum_{j=1}^{m} \lambda_{kj}(c) [e_j(c), \hat{e}_k] \\ &= \sum_{k=1}^{m} \sum_{j=1}^{m} \lambda_{kj}(c) (\beta_{k,j} U + \delta_{jk} V + w_{k,j}) \\ &= \left(\sum_{k=1}^{m} \sum_{j=1}^{m} \lambda_{kj}(c) \beta_{k,j} \right) U + \left(\sum_{j=1}^{m} \lambda_{jj}(c) \right) V + \sum_{k=1}^{m} \sum_{j=1}^{m} \lambda_{k,j}(c) w_{k,j}. \end{aligned} \tag{2.11}$$

We have $m^2 + 1$ equations involving the coefficients $\beta_{i,j}$. Namely for $1 \leq k, i \leq m$

$$\begin{aligned} \delta_{ki} &= B_{\ell_c}(e_k(c), \hat{e}_i) \\ &= \langle \ell_c, [e_k(c), \hat{e}_i] \rangle \\ &= \sum_{j=1}^{m} \lambda_{k,j}(c) \langle \ell_c, [e_j, \hat{e}_i] \rangle \\ &= \sum_{j=1}^{m} \lambda_{k,j}(c) \langle \ell_c, \beta_{i,j} U + \delta_{ij} V + w_{i,j} \rangle \\ &= \sum_{j=1}^{m} \lambda_{k,j}(c) (c\beta_{i,j} + \delta_{ij}). \end{aligned}$$

Therefore

$$\sum_{j=1}^{m} c\beta_{i,j}\lambda_{k,j}(c) \ + \lambda_{k,i}(c) = \delta_{k,i} \quad (1 \le i, k \le m) \tag{2.12}$$

If we consider the functionals $\ell_c,\ c \in \mathbb{R}$, then equations 2.8 and 2.11 tell us that

$$\begin{aligned} \beta &= B_{\ell_c}(X, \sum_{k=1}^{m}[e_k(c), \hat{e}_k]) \\ &= \sum_{j,k=1}^{m} \lambda_{k,j}(c)\beta_{k,j}. \end{aligned} \tag{2.13}$$

In order to show that $\beta = 0$, consider the m by m matrices $B := (\beta_{i,j})_{1\le i,j\le m}$ and $\Lambda(c) := (\lambda_{i,j}(c))_{1\le i,j\le m}$. Equations 2.12 and 2.13 then read

$$(cB + \mathbb{I}_m)\Lambda^t(c) = \mathbb{I}_m,$$

i.e.

$$\Lambda^t(c) = (\mathbb{I}_m + cB)^{-1}$$

and

$$\beta = \mathrm{tr}(B\Lambda^t(c)) = tr(B(\mathbb{I}_m + cB)^{-1}), c \in \mathbb{R}.$$

In particular, equation 2.12 tells us that the matrix $\mathbb{I}_m + cB$ is invertible for every real number c and so no element of the spectrum $\sigma(B)$ of B can be a real number different from 0. We now write

$$\beta = \mathrm{tr}(B(\mathbb{I}_m + cB)^{-1}) = \sum_{\lambda\in\sigma(B)} \frac{\lambda}{1 + c\lambda}, c \in \mathbb{R}.$$

Hence $\beta = \lim_{c\to\infty} \sum_{\lambda\in\sigma(B)} \frac{\lambda}{1+c\lambda} = 0$, which achieves the proof of the proposition.

□

Lemma 2.3 implies the identity

$$\Delta_{P_1}(h)\Delta_{P_2}(h) = \Delta^2_{P_1\cap P_2}(h),\ \ h \in P_1 \cap P_2.$$

and

$$\Delta_{P_1,G}(h) = \Delta_{P_2,G}(h)\Delta^2_{P_1\cap P_2,P_2}(h) \quad (h \in P_1 \cap P_2) \tag{2.14}$$

where $P_j = \exp(\mathfrak{p}_j)$ $(j = 1,\ 2)$. For a function $\varphi \in L^2(G/P_1, \chi_f)$ and $g \in G$, the function Φ_g on P_2 defined by

$$\Phi_g(h) := \varphi(gh)\chi_f(h)\Delta^{-1/2}_{P_2,G}(h)$$

satisfies the covariance relation

$$\Phi_g(hx) = \Delta_{P_1\cap P_2,P_2}(x)\Phi_g(h) \quad (h \in P_2, x \in P_1 \cap P_2).$$

This allows us to write down the formal integral

$$\left(I^G_{\mathfrak{p}_2\mathfrak{p}_1}\varphi\right)(g) := \left(I_{\mathfrak{p}_2\mathfrak{p}_1}\varphi\right)(g) := \oint_{P_2/(P_2\cap P_1)} \varphi(gh)\chi_f(h)\Delta^{-1/2}_{P_2,G}(h)d\nu(h) \quad (g \in G). \tag{2.15}$$

At least formally speaking, it is clear that the function $I_{\mathfrak{p}_2\mathfrak{p}_1}\varphi$ satisfies the covariance condition which is necessary to belong to the space $L^2(G/P_2, \chi_f)$ and that $I_{\mathfrak{p}_2\mathfrak{p}_1}$ commutes with left translations.

We therefore can assert that the convergence of the integral 2.15 is one of the main hurdles we face up. Remark that the Pukanszky condition implies that the product H_2H_1 is locally closed in G and therefore the space P_2P_1/H_1 is homeomorphic to the homogeneous space $P_2/(P_1\cap P_2)$. If now P_2P_1 is closed in G, which is true when G is nilpotent or one of our two polarizations is a Vergne polarization, then the integral (2.15) converges for every function $\varphi\in C_c(G/P_1,\chi_f)$. When we studied this problem in [1], we didn't know whether P_2P_1 is closed or not. In [1] we asked the question:

Question. Let G be an exponential solvable Lie group. Let $f\in\mathfrak{g}^*$ and let $\mathfrak{p}_j$ be Puk-polarizations at f and $P_j=\exp(\mathfrak{p}_j)$ $(j=1,\ 2)$. Is the product P_2P_1 closed in G?

We now give a positive answer to this question.

3. A closed subset of G

We need a few lemmas in order to prove that the product of two Puk-polarizations is closed.

Lemma 3.1. *Let $G=exp(\mathfrak{g})$ be an exponential solvable Lie group, $f\in\mathfrak{g}^*$, $\mathfrak{p}_j, j=1,2$, two Puk-polarizations at f and $P_j=exp(\mathfrak{p}_j), j=1,2$. Then*

$$P_1P_2=\{g\in G;\ \mathrm{Ad}^*(g)f-f\in(\mathrm{Ad}(g)\mathfrak{p}_2\cap\mathfrak{p}_1)^\perp\}\tag{3.16}$$

Proof. Let

$$B:=\{g\in G;\ \mathrm{Ad}^*(g)f-f\in(\mathrm{Ad}(g)\mathfrak{p}_2\cap\mathfrak{p}_1)^\perp\}.\tag{3.17}$$

We must show that $B=P_1P_2$. If $g\in B$, then $\mathrm{Ad}^*(g)f=f+\mathrm{Ad}^*(g)q_2+q_1$, where $q_2\in\mathfrak{p}_2^\perp$ and $q_1\in\mathfrak{p}_1^\perp$, since $(\mathrm{Ad}(g)\mathfrak{p}_2\cap\mathfrak{p}_1)^\perp=(\mathrm{Ad}^*(g)\mathfrak{p}_2)^\perp+\mathfrak{p}_1^\perp$. Hence $\mathrm{Ad}^*(g)(f-q_2)=f+q_1$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are Puk-polarizations, there exists $u_2\in P_2$ and $u_1\in P_1$, such that

$$\mathrm{A}d^*(u_2)f=f-q_2,\ \mathrm{A}d^*(u_1)f=f+q_1$$

and therefore

$$\mathrm{A}d^*(gu_2)f=\mathrm{Ad}^*(g)\circ\mathrm{Ad}^*(u_2)f=\mathrm{Ad}^*(u_1)f.$$

Hence $u_1\in gu_2G(f)\subset gP_2$ and so $g\in P_1P_2$.
On the other hand, if $g=u_1u_2\in P_1P_2$, then

$$\mathrm{Ad}^*(g)f=\mathrm{Ad}^*(g)\circ\mathrm{Ad}^*(u_2^{-1})(\mathrm{Ad}^*(u_2)f)=\mathrm{Ad}^*(u_1)(f+q_2)=\mathrm{Ad}^*(u_1)f+\mathrm{Ad}^*(g)(\mathrm{Ad}^*(u_2^{-1})q_2)$$

for some $q_2\in\mathfrak{p}_2^\perp$. Whence

$$\mathrm{Ad}^*(g)f-f\in\mathrm{Ad}^*(g)\mathfrak{p}_2^\perp+\mathrm{Ad}^*(u_1)f-f\in\mathrm{Ad}^*(g)\mathfrak{p}_2^\perp+\mathfrak{p}_1^\perp,$$

which gives $g\in B$. □

Let now $\mathfrak{n}$ be the nil-radical of $\mathfrak{g}$ and let $N:=\exp(\mathfrak{n})$. Choose a subspace $\mathfrak{v}_1$ of $\mathfrak{p}_1$ such that $\mathfrak{p}_1=\mathfrak{p}_1\cap\mathfrak{n}\oplus\mathfrak{v}_1$ and take a subspace $\mathfrak{v}_2$ of $\mathfrak{p}_2$, such that $\mathfrak{p}_2+\mathfrak{n}+\mathfrak{p}_1=(\mathfrak{p}_1+\mathfrak{n})\oplus\mathfrak{v}_2$. Then $G_0:=P_1NP_2$ is a closed connected normal subgroup of G containing P_1 and P_2. Furthermore the mapping

$$\Theta:\mathfrak{v}_1\times N\times\mathfrak{v}_2\to G_0;(X_1,n,X_2)\mapsto\exp(X_1)n\exp(X_2)\tag{3.18}$$

is a diffeomorphism.

Lemma 3.2. *The product P_1P_2 is closed in G if and only if $P_1P_2\cap N$ is closed in N.*

Proof. Suppose that the subset $A := P_1P_2 \cap N$ is closed in G. Let $g \in G$ be in the closure of P_1P_2. Then $g \in G_0$ and $g = \lim_{n\to\infty} g_n$, fore some sequence $(g_n)_n \subset P_1P_2$. We write $g_n = \exp(X_n)u_n\exp(Y_n)$ with $u_n \in N, X_n \in \mathfrak{v}_1, Y_n \in \mathfrak{v}_2$, $g = \exp(X)u\exp(Y), X \in \mathfrak{v}_1, u \in N, Y \in \mathfrak{v}_2$, according to (3.18) $(n \in \mathbb{N})$. Then $u_n = \exp(-X_n)g_n\exp(-Y_n) \in P_1P_2 \cap N = A$, $n \in \mathbb{N}$, and $Y_n \to Y \in \mathfrak{v}_2, X_n \to X \in \mathfrak{v}_1, u_n \to u$ as n tends to ∞, since Θ is a diffeomorphism. Hence $u \in P_1P_2 \cap N$, since A is closed and so $g = \exp(X)u\exp(Y) \in P_1(P_1P_2 \cap N)P_2 \subset P_1P_2$. Hence P_1P_2 is closed. ☐

We prove now that A is closed. Denote by $\overline{A}$ the closure of A in G. It is easy to see that

Lemma 3.3.

$$\overline{A} \setminus A = \{g \in \overline{A}; \mathit{dim}(\mathit{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1) > \mathit{dim}(\mathfrak{p}_2 \cap \mathfrak{p}_1) := d\}.$$

Proof. Indeed, if $g = u_1u_2 \in P_1P_2$, then

$$\dim(\mathrm{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1) = \dim(\mathrm{Ad}(u_1)\mathfrak{p}_2 \cap \mathfrak{p}_1) = \dim(\mathrm{Ad}(u_1)(\mathfrak{p}_2 \cap \mathfrak{p}_1)) = \dim(\mathfrak{p}_2 \cap \mathfrak{p}_1) = d.$$

Let now $g \in \overline{A}$, such that $\dim(\mathrm{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1) = d$. Let (g_n) be a sequence taken in A, with $\lim_{n\to\infty} g_n = g$. Then, for a subsequence, the subspaces $\mathrm{Ad}(g_n)\mathfrak{p}_2 \cap \mathfrak{p}_1$ converge in the subspace topology to a subspace S of $\mathrm{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1$ and $\dim(S) = d = \dim(\mathrm{A}d(g)\mathfrak{p}_2 \cap \mathfrak{p}_1)$. Hence $S = \mathrm{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1$. We know now that $\mathrm{Ad}^*(g)f - f \in S^{\perp}$, since

$$\mathrm{Ad}^*(g)f - f = \lim_{n\to\infty} \mathrm{Ad}^*(g_n)f - f \in (\lim_{n\to\infty} \mathrm{Ad}(g_n)\mathfrak{p}_2 \cap \mathfrak{p}_1)^{\perp}.$$

Hence $g \in P_1P_2$ by Lemma 3.1. Since N is also closed, we have that $g \in P_1P_2 \cap N = A$. ☐

Proposition 3.4. *Let f be an element of the dual space of the exponential solvable Lie algebra $\mathfrak{g} = Lie(G)$. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two Puk-polarizations at f. Then the product P_2P_1 of the two closed connected subgroups $P_j := exp(\mathfrak{p}_j), j = 1, 2,$ is a closed subset of G.*

Proof. Let $\mathcal{V} := \{\psi_1, \cdots, \psi_m\}$ be a basis of the subspace $\mathfrak{p}_1^{\perp}$ and let $\mathfrak{Y} := \{Y_1, \cdots, Y_r\}$ be a basis of $\mathfrak{p}_2$. Let $g \in G$. Then

$$\begin{aligned} \dim(\mathrm{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1) &= \dim(\mathrm{A}d(g)\mathfrak{p}_2) + \dim(\mathfrak{p}_1) - \dim(\mathrm{A}d(g)\mathfrak{p}_2 + \mathfrak{p}_1) \\ &= \dim(\mathrm{A}d(g)\mathfrak{p}_2) - \dim(\mathrm{A}d(g)\mathfrak{p}_2 + \mathfrak{p}_1)/\mathfrak{p}_1 \\ &= \dim(\mathfrak{p}_2) - \dim(\mathfrak{p}_1^{\perp,\ (\mathrm{A}d(g)\mathfrak{p}_2+\mathfrak{p}_1)^*}) \\ &= \dim(\mathfrak{p}_2) - \dim(\mathfrak{p}_1^{\perp,\mathfrak{g}}{}_{|\mathrm{A}d(g)\mathfrak{p}_2}) \end{aligned}$$

Hence the dimension of $\mathrm{Ad}(g)\mathfrak{p}_2 \cap \mathfrak{p}_1$ is equal to the number $\dim(\mathfrak{p}_2) - \mathrm{rank}(M(g))$, where

$$M(g) = \begin{pmatrix} \langle\psi_1, Ad(g)Y_1\rangle & \dots & \langle\psi_1, Ad(g)Y_r\rangle \\ \vdots & \dots & \vdots \\ \langle\psi_m, Ad(g)Y_1\rangle & \dots & \langle\psi_m, Ad(g)Y_r\rangle \end{pmatrix} \in M_{m,r}(\mathbb{R}).$$

We shall show, that we can find a basis $\mathcal{V}$ and a basis $\mathfrak{Y}$, such that the corresponding matrix $M(g)$ admits a minor $m(g)$ of order $\dim(\mathfrak{p}_2) - d = \dim(\mathfrak{p}_2/\mathfrak{p}_1 \cap \mathfrak{p}_2)$, which is equal to one for $g \in A$, hence also for every $g \in \overline{A}$. Hence for $g \in \overline{A}$, $\dim(Ad(g)\mathfrak{p}_2 \cap \mathfrak{p}_1) = \dim(\mathfrak{p}_2) - \mathrm{rank}(M(g)) \le \dim(\mathfrak{p}_2) - (\dim(\mathfrak{p}_2) - d) = d$. By Lemma 3.3, g is then contained in A.

Let $\mathfrak{g} = \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n \supset \{0\}$ be a Jordan-Hölder sequence of $\mathfrak{g}$ for the Ad-action of N. That means that the $\mathfrak{g}_i$'s are subalgebras of $\mathfrak{g}$, that $[\mathfrak{n}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ and that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$ for every i. Define for a subspace $\mathfrak{v} \subset \mathfrak{g}$ the index set

$$(3.19)\quad I^{\mathfrak{v}} := \{1 \leq i \leq n; \mathfrak{g}_i + \mathfrak{v} = \mathfrak{g}_{i+1} + \mathfrak{v}\} = \{1 \leq i \leq n; \mathfrak{g}_{i+1} + \mathfrak{v} \supset \mathfrak{g}_i\} \subset \{1, \cdots, n\}.$$

So, obviously $I^{\mathfrak{p}_1\cap\mathfrak{p}_2} \subset I^{\mathfrak{p}_2} \subset I^{\mathfrak{p}_2+\mathfrak{p}_1}$. We choose for every $i \in I^{\mathfrak{p}_1} \setminus I^{\mathfrak{p}_2}$ an element $Z_i \in \mathfrak{p}_1 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$. Likewise, for $i \in I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1}$, we take $Z_i \in \mathfrak{p}_2 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$.
Let now $i \in I^{\mathfrak{p}_2} \cap I^{\mathfrak{p}_1} \setminus I^{\mathfrak{p}_2\cap\mathfrak{p}_1}$. Then given $X_i^1 \in \mathfrak{g}_i \cap \mathfrak{p}_1 \setminus \mathfrak{g}_{i+1}$, there exists $X_i^2 \in \mathfrak{g}_i \cap \mathfrak{p}_2 \setminus \mathfrak{g}_{i+1}$, such that $Z_j := X_i^1 + X_i^2 \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$ for some $j > i$. We choose now X_i^1 and X_i^2, such that j is maximal. This index j is then unique and we put $j(i) := j$. Then it follows that $j \in I^{\mathfrak{p}_2+\mathfrak{p}_1}$ and the maximality of j implies that $j \notin I^{\mathfrak{p}_2} \cup I^{\mathfrak{p}_1}$.
Let us verify that the mapping

$$I^{\mathfrak{p}_2} \cap I^{\mathfrak{p}_1} \setminus I^{\mathfrak{p}_1\cap\mathfrak{p}_2} \longrightarrow I^{\mathfrak{p}_2+\mathfrak{p}_1} \setminus (I^{\mathfrak{p}_1} \cup I^{\mathfrak{p}_2}); i \mapsto j = j(i),$$

which had been defined above, is a bijection.
If for $i < i'$ we have that $j = j'$, then there exists $X_i^1 \in \mathfrak{p}_1 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, X_{i'}^1 \in \mathfrak{p}_1 \cap \mathfrak{g}_{i'} \setminus \mathfrak{g}_{i'+1}$, $X_i^2 \in \mathfrak{p}_2 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, X_{i'}^2 \in \mathfrak{p}_2 \cap \mathfrak{g}_{i'} \setminus \mathfrak{g}_{i'+1}$, such that $Z_j := X_i^1 + X_i^2 \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$ and $Z'_j := X_{i'}^1 + X_{i'}^2 \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$. For a scalar $\lambda \in \mathbb{R}^*$, we then have that $Z_j - \lambda Z'_j \in \mathfrak{g}_{j+1}$ and so $X_i^1 - \lambda X_{i'}^1 \in \mathfrak{g}_i \cap \mathfrak{p}_1 \setminus \mathfrak{g}_{i+1}, X_i^2 - \lambda X_{i'}^2 \in \mathfrak{g}_i \cap \mathfrak{p}_2 \setminus \mathfrak{g}_{i+1}$ and $(X_i^1 - \lambda X_{i'}^1) + (X_i^2 - \lambda X_{i'}^2 0 \in \mathfrak{g}_{j+1}$, contradicting the maximality of j.
For the indices $j \in I^{\mathfrak{p}_1+p_2} \setminus (I^{\mathfrak{p}_1} \cup I^{\mathfrak{p}_2})$, we have an index $i \leq j$, and elements $X_i^1 \in \mathfrak{p}_1 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}, X_i^2 \in \mathfrak{p}_2 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$, such that $X_i^1 + X_i^2 := Z_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$. Since $j \notin (I^{\mathfrak{p}_1} \cup I^{\mathfrak{p}_2})$, the index i must be strictly smaller than the index j. We choose i maximal with these properties. Then $i \in I^{\mathfrak{p}_1} \cap I^{\mathfrak{p}_2}$. Furthermore $j = j(i)$, since letting $j' := j(i)$, we have that $j' \geq j$ and in the case where $j' > j$, there exists $U_i^1 \in \mathfrak{g}_i \cap \mathfrak{p}_1 \setminus \mathfrak{g}_{i+1}, U_i^2 \in \mathfrak{g}_i \cap \mathfrak{p}_2 \setminus \mathfrak{g}_{i+1}$, such that $Z_{j'} := U_i^1 + U_i^2 \in \mathfrak{g}_{j'} \setminus \mathfrak{g}_{j'+1}$. But we can take then a scalar $\lambda \in \mathbb{R}^*$, such that $X_i^1 - \lambda U_i^1 \in \mathfrak{g}_{i+1}$. This implies that $X_i^2 - \lambda U_i^2 \in \mathfrak{g}_{i+1}$ too and so $(X_i^1 - \lambda U_i^1) + (X_i^2 - \lambda U_i^2) = Z_j - \lambda Z_{j'} \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$, contradicting the maximality of i. This shows that our mapping is also surjective.
We can now define a basis of $\mathfrak{g}$ containing our basis $\mathfrak{Y}$ of $\mathfrak{p}_2$. For $i \in I^{\mathfrak{p}_1\cap\mathfrak{p}_2}$ we take $Z_j \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$. For $i \in I^{\mathfrak{p}_1} \setminus I^{\mathfrak{p}_2}$ or $i \in I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1}$, we take our Z_i defined above. For $i \in I^{\mathfrak{p}_1} \cap I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1\cap\mathfrak{p}_2}$, we have the vector $Z_i := X_i^2 \in \mathfrak{p}_2 \cap \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$. We add the vectors $X_i^1 + X_i^2 = Z_j \in (\mathfrak{p}_1 + \mathfrak{p}_2) \cap \mathfrak{g}_{j(i)}$ defined above for $j \in I^{\mathfrak{p}_2+\mathfrak{p}_1} \setminus (I^{\mathfrak{p}_1} \cup I^{\mathfrak{p}_2})$ and a vector $Z_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$ for $i \notin I^{\mathfrak{p}_2+\mathfrak{p}_1}$. This gives us a Jordan-Hölder basis $\mathcal{Z}$ of $\mathfrak{g}$ and $\mathfrak{Y} = \mathcal{Z} \cap \mathfrak{p}_2$.
The basis $\mathcal{V}$ of $\mathfrak{p}_1^\perp$ we need is constructed with the vectors $\varphi_i, i \in \{1, \cdots, n\}$ in the following way.
For $i \notin I^{\mathfrak{p}_2+\mathfrak{p}_1}$, we let $\varphi_i(Z_k) = \delta_{i,k}, k = 1, \cdots, n$.
For $i \in I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1}$ we put $\varphi_i(Z_k) = \delta_{i,k}, k \notin I^{\mathfrak{p}_1}, \varphi(\mathfrak{p}_1) = \{0\}$.
For $i \in I^{\mathfrak{p}_1} \cap I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1\cap\mathfrak{p}_2}$ we have $j = j(i) \in I^{\mathfrak{p}_2+\mathfrak{p}_1} \setminus (I^{\mathfrak{p}_1} \cup I^{\mathfrak{p}_2})$ and we let $\varphi_i(Z_k) = 0$ if $k \notin \{i, j\}$ and $\varphi_i(Z_j) = \varphi_i(X_i^2) = 1$ and $\varphi_i(Z_k) = 0$ for all $k \in I^{\mathfrak{p}_1}$.
We obtain in this way $\dim(\mathfrak{g}/\mathfrak{p}_1)$ linearly independent functionals in $\mathfrak{p}_1^\perp$, i.e. a basis of $\mathfrak{p}_1^\perp$. These functionals have the property that $\varphi_i(Z_k) = \delta_{i,k}, k \geq i$, if $i \notin (I^{\mathfrak{p}_1} \cap I^{\mathfrak{p}_2})$ and $\varphi_i(Z_k) = \delta_{j(i),k}, k \geq j(i)$. This gives us the relative minor of the matrix $M(g)$ we are looking for. It suffices to take the sub-matrix $S(g)$ of $M(g)$ formed by the rows R_i, $i \in I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1\cap\mathfrak{p}_2}$, and the columns C_i, $i \in I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1}$ and the columns $C_{j(i)}$, $i \in I^{\mathfrak{p}_1} \cap I^{\mathfrak{p}_2} \setminus I^{\mathfrak{p}_1\cap\mathfrak{p}_2}$. This square sub-matrix $S(g)$ is then of order $\dim(\mathfrak{p}_2/\mathfrak{p}_1 \cap \mathfrak{p}_2)$ and it is

lower triangular with 1's on the diagonal, since $Ad(g)Z_k = \sum_{l\geq k} a_{l,k}(g)Z_l$, $k = 1, \cdots, n$, where $a_{k,k}(g) = 1, g \in N$. □

4. An intertwining operator

Proposition 3.4 now allows us to write down explicitly our intertwining operator.

Proposition 4.1. *Let $G = exp(\mathfrak{g})$ be an exponential Lie group. Let $f \in \mathfrak{g}^*$ and let $\mathfrak{p}_1, \mathfrak{p}_2$ be two Puk-polarizations at f. For every η in $C_c(G/P_1, \chi_f)$, the integral*

$$I_{\mathfrak{p}_2,\mathfrak{p}_1}\eta(g) := \oint_{P_2/P_1\cap P_2} \eta(gp_2)\chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2),\ g \in G, \tag{4.20}$$

converges and defines a continuous function on G/P_2 satisfying the covariance condition of the space $L^2(G/P_2, \chi_f)$.

Proof. We have for $\eta \in C_c(G/P_1, \chi_f)$, $p_2 \in P_2, q \in P_1 \cap P_2$ and $g \in G$ that

$$\eta(gp_2q)\Delta_{P_2,G}^{-1/2}(p_2q) = \chi_f(q^{-1})\Delta_{P_2,G}^{-1/2}(q)\Delta_{P_1,G}^{1/2}(q)\eta(gp_2) = \chi_f(q^{-1})\frac{\Delta_{P_1\cap P_2}(q)}{\Delta_{P_2}(q)}\eta(gp_2),$$

which merely stems from the relation

$$\Delta_{P_1,G}(q)\Delta_{P_2,G}^{-1}(q) = \Delta_{P_1\cap P_2,P_2}^2(q)$$

by 2.14. Since the subset P_2P_1 is closed in G, for every function $\eta \in C_c(G/P_1, \chi_f)$, its restriction to gP_2P_1, $g \in G$, has compact support modulo P_1. Hence the function $p_2 \mapsto \chi_f(p_2)\Delta_{P_2,G}(p_2)^{-1/2}\eta(gp_2)$ is contained in $\mathcal{E}(P_2/P_1 \cap P_2)$ and the integral (4.20) converges for every $g \in G$. Furthermore, the function $I_{\mathfrak{p}_2,\mathfrak{p}_1}\eta$ is continuous by the theorem of dominated convergence. It is immediately checked that $I_{\mathfrak{p}_2,\mathfrak{p}_1}\eta$ satisfies also the covariance relation which is necessary to belong to the space $C_c(G/P_2, \chi_f)$. □

We shall directly show in this section that the mapping $I_{\mathfrak{p}_2,\mathfrak{p}_1}$ is an isometry. Similar computations had already be done in [5] and [1]. Therefore we present here only a full proof for completely solvable groups. This proof includes some details which had been forgotten in [1], Thï¿½orï¿½me 6.1 (see in particular Lemma 4.4).

Lemma 4.2. *Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two Puk-polarizations at $f \in \mathfrak{g}^*$. Then for every $\varphi \in C_c^\infty(G/P_1, \chi_f)$, the function $I_{\mathfrak{p}_2,\mathfrak{p}_1}(\varphi)$ is contained in $L^2(G/P_2, \chi_f)$ and for the right choice of the linear form $\oint_{P_2/P_1\cap P_2} d\nu$, we have that $\|I_{\mathfrak{p}_2,\mathfrak{p}_1}(\varphi)\|_2 = \|\varphi\|_2$ for every $\varphi \in C_c^\infty(G/P_1, \chi_f)$.*

Proof. We proceed by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g}$ is abelian, there is nothing to prove. If there exists an ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that f vanishes on $\mathfrak{a}$, we pass to the quotient $\mathfrak{g}/\mathfrak{a}$ and we are done. If there exists a proper subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{p}_1 + \mathfrak{p}_2$, then the induction hypothesis applied to $\mathfrak{b}$ tells us that for every $\varphi \in C_c^\infty(G/P_1, \chi_f)$

$$\begin{aligned}
\oint_{G/P_2} |I_{\mathfrak{p}_2,\mathfrak{p}_1}\varphi(u)|^2 d\nu(u) &= \oint_{G/B}\oint_{B/P_2} |I_{\mathfrak{p}_2,\mathfrak{p}_1}\varphi(ub)|^2 \Delta_{B,G}^{-1}(b) d\nu_{B,P_2}(b) d\nu_{G,B}(u) \\
&= \oint_{G/B}\oint_{B/P_2} |\Delta_{B,G}^{-\frac{1}{2}}(b) I_{\mathfrak{p}_2,\mathfrak{p}_1}\varphi(ub)|^2 d\nu_{B,P_2}(b) d\nu_{G,B}(u) \\
&= \oint_{G/B}\oint_{B/P_2} \\
&\quad |\oint_{P_2/P_1\cap P_1} \Delta_{B,G}^{-\frac{1}{2}}(bp_2)\varphi(ubp_2)\chi_f(p_2)\Delta_{P_2,B}^{-1/2}(p_2) d\nu(p_2)|^2 \\
&\quad d\nu_{B,P_2}(b) d\nu_{G,B}(u) \\
&= \oint_{G/B}\oint_{B/P_1} \Delta_{B,G}^{-1}(b)|\varphi(ub)|^2 d\nu_{B,P_1}(b) d\nu_{G,B}(u) \\
&= \oint_{G/P_1} |\varphi(g)|^2 d\nu(g).
\end{aligned}$$

From now on, we can suppose that f does not vanish on any proper ideal of $\mathfrak{g}$ and that $\mathfrak{p}_1 + \mathfrak{p}_2$ is not contained in any proper subalgebra. We can assume also that $[\mathfrak{g}, \mathfrak{g}]$ is not included in the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$, since otherwise $\mathfrak{g}$ is nilpotent and the lemma is well known in that case. Let $\mathfrak{a}$ be a minimal non central ideal of $\mathfrak{g}$ contained in $[\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{z} := \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is one or 0 dimensional. If $\mathfrak{z}$ is reduced to $\{0\}$, then $\mathfrak{p}_1 + \mathfrak{p}_2$ is contained in the centralizer $\mathfrak{g}_0$ of $\mathfrak{a}$, since $\mathfrak{p}_i, i = 1, 2$, are Puk-polarizations. Hence we can suppose that $\mathfrak{z} = \mathbb{R}Z$ is one dimensional and $\mathfrak{a} \supset \mathfrak{z}$ has dimension 2 or 3. We can also assume that $f(Z) = 1$. Let

$$\mathfrak{g}^0 := \{U \in \mathfrak{g}; \langle f, [U, \mathfrak{a}]\rangle = \{0\}\} = \mathfrak{a}^f.$$

Let $\mathfrak{p}_i^0 := \mathfrak{p}_i \cap \mathfrak{g}^0,\ i = 1, 2$. We have by assumption that $\mathfrak{p}_1 + \mathfrak{p}_2 \not\subset \mathfrak{g}^0$. Suppose first that $\mathfrak{p}_1 \not\subset \mathfrak{g}^0$ and $\mathfrak{p}_2 \subset \mathfrak{g}^0$. Let $\mathfrak{p}_1' := \mathfrak{g}^0 \cap \mathfrak{p}_1 + \mathfrak{a}$. Choose a subspace $\mathfrak{y}$ in $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{y} \oplus \mathfrak{p}_1 \cap \mathfrak{a}$ and $\mathfrak{y} \subset \ker(f)$. Then $\mathfrak{p}_2/\mathfrak{p}_1 \cap \mathfrak{p}_2 \simeq \mathfrak{p}_2/\mathfrak{p}_1' \cap \mathfrak{p}_2 \oplus \mathfrak{y}$. Let us write now and the other cases to come (for simplicity of notations) $I_{2,1} := I_{\mathfrak{p}_2,\mathfrak{p}_1}, I_{2,1'} := I_{\mathfrak{p}_2,\mathfrak{p}_1'}, I_{1,1'} = I_{\mathfrak{p}_1,\mathfrak{p}_1'}$ and so on. We know that $I_{1',1}$ and $I_{2',2}$ are in fact Fourier transforms and hence

$$\begin{aligned}
I_{2,1'} \circ I_{1',1}(\varphi)(g) &= \oint_{P_2/P_2\cap P_1'} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2) \int_{\mathfrak{y}} \chi_f(\exp(Y))\varphi(gp_2\exp(Y)) dY d\nu(p_2) \\
&= \oint_{P_2/P_2\cap P_1'} \int_{\mathfrak{y}} \chi_f(p_2\exp(Y))\Delta_{P_2,G}^{-1/2}(p_2\exp(Y))\varphi(gp_2\exp(Y)) dY d\nu(p_2) \\
&= \oint_{P_2/P_2\cap P_1} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)\varphi(gp_2) d\nu(p_2) \\
&= I_{2,1}\varphi(g).
\end{aligned}$$

Hence $I_{2,1}$ is an isometry too.

If now $\mathfrak{p}_2 \not\subset \mathfrak{g}^0$ and $\mathfrak{p}_1 \subset \mathfrak{g}^0$, then let $\mathfrak{p}_2' := \mathfrak{g}^0 \cap \mathfrak{p}_2 + \mathfrak{a}$. Choose a subspace $\mathfrak{x}$ in $\mathfrak{p}_2$ such that $\mathfrak{p}_2 = \mathfrak{x} \oplus \mathfrak{p}_2^0$ and $\mathfrak{x} \subset \ker(f)$. Then $\mathfrak{p}_2/\mathfrak{p}_1 \cap \mathfrak{p}_2 \simeq \mathfrak{x} \oplus \mathfrak{p}_2'/\mathfrak{p}_2' \cap \mathfrak{p}_1$ and

$$\begin{aligned}
I_{2,2'} \circ I_{2',1}(\varphi)(g) &= \int_{\mathfrak{x}} \chi_f(\exp(X))\Delta_{P_2,G}^{-1/2}(\exp(X)) \\
&\quad \oint_{P_2'/P_2'\cap P_1} \chi_f(p_2)\Delta_{P_2',G}^{-1/2}(p_2)\varphi(g\exp(X)p_2)d\nu(p_2)dX \\
&= \int_{\mathfrak{x}}\oint_{P_2'/P_2'\cap P_1} \chi_f(\exp(X)p_2)\Delta_{P_2,G}^{-1/2}(\exp(X)p_2)\varphi(g\exp(X)p_2) \\
&\quad \Delta_{P_2^0,P_2}^{-1}(p_2)dX d\nu(p_2) \\
&= \oint_{P_2/P_2\cap P_1} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)\varphi(gp_2)d\nu(p_2) \\
&= I_{2,1}\varphi(g),
\end{aligned}$$

since by 2.14, $\Delta_{P_2',G}^{-1/2}(p_2) = \Delta_{P_2,G}^{-1/2}(p_2)\Delta_{P_2^0,P_2}^{-1}(p_2), p_2 \in P_2^0$. Hence $I_{2,1}$ is also in this case an isometry.

We now come to the case where $\mathfrak{p}_1 \not\subset \mathfrak{g}^0, \mathfrak{p}_2 \not\subset \mathfrak{g}^0$. We assume first that $\mathfrak{a}$ is two-dimensional, i.e $\mathfrak{a} = \mathbb{R}Y + \mathbb{R}Z$ for some $Y \in \mathfrak{a} \cap \ker(f)$. Then there exists a character $\alpha : \mathfrak{g} \to \mathbb{R}$ and a linear functional $0 \neq \beta : \mathfrak{g} \to \mathbb{R}$, such that $[U, Y] = \alpha(U)Y + \beta(U)Z$ for all $U \in \mathfrak{g}$. In particular $\mathfrak{g}^0 = \ker(\beta)$.

Suppose that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \not\subset \mathfrak{g}^0$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are Puk-polarizations, we have that $\mathrm{A}d^*(P_\cap P_2)f = f + (\mathfrak{p}_1 + \mathfrak{p}_2)^\perp$. This shows that the restrictions of α and β to $\mathfrak{p}_1 \cap \mathfrak{p}_2$ cannot be proportional, since $Y \notin \mathfrak{p}_1 + \mathfrak{p}_2$. We can therefore choose $X \in \ker(f) \cap \mathfrak{p}_1 \cap \mathfrak{p}_2 \setminus \mathfrak{g}^0$ such that $[X, Y] = Z$. We have $\mathfrak{a} \cap \mathfrak{p}_1 = \mathfrak{p}_2 \cap \mathfrak{a} = \mathbb{R}Z$. Let $\mathfrak{p}_i' := \mathfrak{p}_i \cap \mathfrak{g}^0 + \mathfrak{a},\ i = 1, 2$. Then $\mathfrak{p}_2/\mathfrak{p}_2 \cap \mathfrak{p}_1 \simeq \mathfrak{p}_2^0/\mathfrak{p}_1^0 \cap \mathfrak{p}_1^0 \simeq \mathfrak{p}_2^0/\mathfrak{p}_1 \cap \mathfrak{p}_2' \simeq \mathfrak{p}_2'/\mathfrak{p}_2' \cap \mathfrak{p}_1'$. For $\varphi \in C_0^\infty(G/P_1, \chi_f)$ and $g \in G$, let $I(g) = I_{2,1'} \circ I_{1',1}\varphi(g)$. We then have

$$\begin{aligned}
I(g) &= \oint_{P_2/P_2\cap P_1'} (I_{1',1}\varphi)(gp_2)\Delta_{P_2,G}^{-1/2}(p_2)\chi_f(p_2)d\nu(p_2) \\
&= \oint_{P_2/P_1\cap P_2} \int_{\mathbb{R}} (I_{1',1}\varphi)(gp_2\mathrm{exp}xX))\Delta_{P_1\cap P_2,P_2}^{-1}(\mathrm{exp}xX))\times \\
&\quad \times \Delta_{P_2,G}^{-1/2}(p_2\mathrm{exp}xX))\chi_f(p_2)dxd\nu(p_2) \\
&= \oint_{P_2/P_1\cap P_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(gp_2\mathrm{exp}xX)\mathrm{exp}yY))\Delta_{P_1\cap P_2,P_2}^{-1}(\mathrm{exp}xX))\times \\
&\quad \times \Delta_{P_2,G}^{-1/2}(p_2\mathrm{exp}xX))\chi_f(p_2)dydxd\nu(p_2) \\
&= \oint_{P_2/P_1\cap P_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(gp_2\mathrm{exp}yY))\Delta_{P_1,G}^{1/2}(\mathrm{exp}xX))\Delta_{P_1\cap P_2,P_2}^{-1}(\mathrm{exp}xX))\times \\
&\quad \times \Delta_{P_2,G}^{-1/2}(\mathrm{exp}xX))\Delta_{P_2,G}^{-1/2}(p_2)\chi_f(p_2)e^{-ixy}dydxd\nu(p_2) \\
&= \oint_{P_2/P_1\cap P_2} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(gp_2\mathrm{exp}yY))e^{-ixy}dydx\right) \Delta_{P_2,G}^{-1/2}(p_2)\chi_f(p_2)d\nu(p_2) \\
&= \oint_{P_2/P_1\cap P_2} \varphi(gp_2)\chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2) = (I_{2,1}\varphi)(g).
\end{aligned}$$

The next case arrives when $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset \mathfrak{g}^0$ which is equivalent to the fact $\mathfrak{a} \subset \mathfrak{p}_1 + \mathfrak{p}_2$. If $\mathfrak{a} \subset \mathfrak{p}_1^0 + \mathfrak{p}_2^0$, then we have $Y = U_2 - U_1$ for some vectors $U_i \in \mathfrak{p}_i^0 \cap \ker(f), i = 1, 2.$.

Lemma 4.3. *We can choose* $U_1, U_2 \in ker\alpha$.

Proof. Indeed, if $\mathfrak{p}_1 \cap \mathfrak{p}_2 \not\subset \mathfrak{g}_0$, then we take $T \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, such that $\alpha(T) = 1$ and we replace U_i by $[T, U_i]$, $i = 1, 2$. If $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset \mathfrak{g}_0$, then we choose $X_1 \in \ker(f) \cap \mathfrak{p}_1$ with $[X_1, Y] = Z$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset \mathfrak{g}^0$, the vector X_1 cannot be in $\mathfrak{p}'_1 + \mathfrak{p}_2$. Suppose now that $\alpha(U_2) = a \neq 0$. Then $\alpha(U_1) = a$ and $[U_1, X_1] = -aX_1 - W$ for some $W \in \mathfrak{p}_1^0 \cap \ker(f)$. We take an element $f' \in f + (\mathfrak{p}'_1 + \mathfrak{p}_2)^\perp$, such that $f'(X_1) = -\frac{1}{a}$. The Pukanszky condition tells us that $\mathfrak{p}'_1$ and $\mathfrak{p}_2$ are contained in $M(f', \mathfrak{g})$. But $[X_1, \mathfrak{p}_1 \cap \mathfrak{p}_2] \subset [X_1, \mathfrak{p}_1 \cap \mathfrak{g}_0] \subset \ker(f) \cap \mathfrak{p}_1^0$ and

$$B_{f'}(X_1, U_2) = f'([X_1, Y + U_1]) = f'(Z + [X_1, U_1]) = 1 + f'(aX_1 + W) = 0.$$

Hence $X_1 \in (\mathfrak{p}_1 \cap \mathfrak{p}_2 + \mathbb{R}U_2)^{f'} \cap \ker(f) \subset ((\mathfrak{p}_1)^{f'} + \mathfrak{p}_2^{f'}) \cap U_2^{f'} = \mathfrak{p}'_1 + \mathfrak{p}_2$, which leads to a contradiction. □

The preceding lemma implies that $[U_1, U_2] = [U_1, U_1 + Y] = 0$.
We have that $\mathfrak{p}'_1 \cap \mathfrak{p}_2 \simeq (\mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus \mathbb{R}U_2$, $\mathfrak{p}'_2 \cap \mathfrak{p}_1 \simeq (\mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus \mathbb{R}U_1$ and $\mathfrak{p}'_1 \cap \mathfrak{p}'_2 \equiv (\mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus \mathbb{R}U_1 \oplus \mathbb{R}U_2$. Furthermore $P'_2 = P_2^0 \exp(\mathbb{R}Y)$, $P'_1 \cap P'_2 = (P_1 \cap P_2)\exp(\mathbb{R}U_2)\exp(\mathbb{R}Y)$ and the space $P'_2/P'_1 \cap P'_2$ is isomorphic to $P_2^0/(P_1 \cap P_2)\exp(\mathbb{R}U_2)$. Hence using again 2.3 and 2.14 and the fact that

$$\mathrm{tr}(\mathrm{ad}_{\mathfrak{p}_1/\mathfrak{p}_1 \cap \mathfrak{p}_2}(U_1)) = -\mathrm{tr}(\mathrm{ad}_{\mathfrak{p}_2/\mathfrak{p}_1 \cap \mathfrak{p}_2}(U_2)),$$

we obtain for $\varphi \in C_0^\infty(G/P_1, \chi_f), g \in G$ and $I(g) := I_{2,2'} \circ I_{2',1'} \circ I_{1',1}\varphi(g)$ that

$$\begin{aligned} I(g) &= \oint_{P_2/P_2^0} \chi_f(k)\Delta_{P_2,G}^{-1/2}(k)d\nu(k) \oint_{P'_2/P'_2 \cap P'_1} \chi_f(p_2)\Delta_{P'_2,G}^{-1/2}(p_2)d\nu(p_2) \\ &\quad \int_{\mathbb{R}} \varphi(gkp_2\exp(yY))dy \\ &= \oint_{P_2/P_2^0} \chi_f(k)\Delta_{P_2,G}^{-1/2}(k)d\nu(k) \oint_{P'_2/P'_2 \cap P'_1} \chi_f(p_2)\Delta_{P'_2,G}^{-1/2}(p_2)d\nu(p_2) \\ &\quad \int_{\mathbb{R}} \varphi(gkp_2\exp(yU_2))\Delta_{P_1,G}^{-1/2}(\exp(yU_1))dy. \end{aligned}$$

Therefore

$$\begin{aligned}
I(g) &= \oint_{P_2/P_2^0} \chi_f(k)\Delta_{P_2,G}^{-1/2}(k)d\nu(k) \oint_{P_2^0/(P_2\cap P_1)\exp(\mathbb{R}U_2)} \int_{\mathbb{R}} \chi_f(p_2)\Delta_{P_2',G}^{-1/2}(\exp(yU_1))\times \\
&\times \Delta_{P_1\cap P_2',P_2'}^{-1}(\exp(yU_1))\Delta_{P_2^0,P_2}^{-1}(p_2)\Delta_{P_2,G}^{-1/2}(p_2)\varphi(gkp_2\exp(yU_2))dyd\nu(p_2) \\
&= \oint_{P_2/P_2^0} \chi_f(k)\Delta_{P_2,G}^{-1/2}(k)d\nu(k) \oint_{P_2^0/(P_2\cap P_1)\exp(\mathbb{R}U_2)} \int_{\mathbb{R}} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(\exp(yU_2))\times \\
&\times \Delta_{(P_2\cap P_1)\exp(\mathbb{R}U_2),P_2^0}^{-1}(\exp(yU_2))\Delta_{P_2^0,P_2}^{-1}(p_2)\Delta_{P_2,G}^{-1/2}(p_2)\varphi(gkp_2\exp(yU_2))dyd\nu(p_2) \\
&= \oint_{P_2/P_2^0} \chi_f(k)\Delta_{P_2,G}^{-1/2}(k)d\nu(k)\times \\
&\times \oint_{P_2^0/(P_2\cap P_1)} \chi_f(p_2)\Delta_{P_2^0,P_2}^{-1}(p_2)\Delta_{P_2,G}^{-1/2}(p_2)\varphi(gkp_2)dyd\nu(p_2) \\
&= I_{21}\varphi(g).
\end{aligned}$$

Hence $I_{2,1}$ is a multiple of an isometry.
If $\mathfrak{a} \subset \mathfrak{p}_1 + \mathfrak{p}_2 \setminus (\mathfrak{p}_1^0 + \mathfrak{p}_2^0)$, then we can find $X_i \in \mathfrak{p}_i \cap \ker(f), i = 1,2$ and $a \in \mathbb{R}^*$, such that $Y = a(X_2 - X_1)$, $[X_1, Y] = [X_2, Y] = \delta Y + Z$, $\mathfrak{p}_1 = \mathfrak{p}_1^0 \oplus \mathbb{R}X_1, \mathfrak{p}_2 = \mathfrak{p}_2^0 \oplus \mathbb{R}X_2$.

Lemma 4.4.

(1) $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset ker(\alpha)$.

(2) *We can choose* X_1, X_2 *in* $ker(\alpha)$ *such that the identity*

$$tr\ ad_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X_1 + tr\ ad_{\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X_2 = 0$$

holds.

Proof.
1) If $\mathfrak{p}_1 \cap \mathfrak{p}_2 \not\subset \mathfrak{g}_0$, then there exists $T \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, such that $[T, Y] = Y$. Hence $Y = [T, a(X_2-X_1)] = -a(X_2-X_1)+W_1+W_2 = -Y+W_1+W_2$ with $W_i \in \mathfrak{p}_i^0 \cap \ker(f), i = 1,2$ and $Y \in \mathfrak{p}_1^0 + \mathfrak{p}_2^0$, which leads to a contradiction.
2) Suppose that $\delta \neq 0$. Then $a[X_1, X_2] = [X_1, a(X_2-X_1)] = \delta Y + Z$. Since $\alpha(\mathfrak{p}_1) \neq \{0\}$, there exists $T_1 \in \mathfrak{p}_1^0$ such that $[T_1, Y] = Y$. Let $\mathfrak{b} := \mathbb{R}X_2 \oplus \mathfrak{p}_1^0$. We then have that $\mathfrak{b} \cap \mathfrak{p}_2 = \mathbb{R}X_2 \oplus \mathfrak{p}_1 \cap \mathfrak{p}_2, f([\mathfrak{b}, \mathfrak{b}]) = \{0\}$ and $[\mathfrak{b}, \mathfrak{b}] \subset \mathbb{R}[T_1, X_2] + \mathfrak{p}_1^0 \cap \mathfrak{g}_0$.
Suppose first that $[X_1, X_2] \in \mathfrak{p}_2 + [\mathfrak{b}, \mathfrak{b}]$. Then

$$\delta Y + Z = a[X_1, X_2] \in \mathfrak{p}_2 + [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{p}_2 + \mathbb{R}([T_1, X_1] + a^{-1}Y) + \mathfrak{p}_1^0 \cap \mathfrak{g}_0.$$

Hence there exists $\gamma \in \mathbb{R}$, such that $\delta Y - \gamma([T_1, X_1] + a^{-1}Y) \in \mathfrak{p}_2 + \mathfrak{p}_1^0 \cap \mathfrak{g}_0$. If $\gamma = a\delta$, then $[T_1, X_1] \in (\mathfrak{p}_2 + \mathfrak{p}_1^0) \cap \ker(\alpha)$, which has been excluded since we have at present $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset \mathfrak{g}^0$. Therefore $Y \in \mathfrak{p}_1 \cap \ker(\alpha) + \mathfrak{p}_2$, i.e. we can choose $X_i \in \mathfrak{p}_i \cap \ker(\alpha), i = 1,2$.
If $[X_1, X_2] \not\subset \mathfrak{p}_2 + [\mathfrak{b}, \mathfrak{b}]$, then we can find an element $f' \in f + (\mathfrak{p}_2 + [\mathfrak{b}, \mathfrak{b}])^\perp$, such that $f'([X_1, X_2]) = 0$. According to Pukanszky's condition, $\mathfrak{p}_2 \in M(\mathfrak{g}, f')$. Since $\mathfrak{b}$ is an isotropic subspace for the bilinear form $B_{f'}$ and since $\dim(\mathfrak{b}) = \dim(\mathfrak{p}_2)$, $\mathfrak{b}$ is maximal isotropic for $B_{f'}$. Because $f'([X_1, \mathfrak{p}_2 \cap \mathfrak{b}]) = \{0\}$ it follows then that $X_1 \in \mathfrak{p}_2 + \mathfrak{b}$ and therefore $\mathfrak{p}_1 \cap \mathfrak{p}_2 \not\subset \mathfrak{g}^0$, a contradiction.
We can take therefore $X_1, X_2 \in \ker(\alpha)$.
Let us now prove that we can choose also $X_1, X_2 \in \ker(\alpha)$ such that our trace relation

$$\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X_1 + \mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)}X_2 = \mathrm{tr}\ \mathrm{ad}_{{\mathfrak{p}^0}_1/({\mathfrak{p}^0}_1\cap{\mathfrak{p}^0}_2)}X_1 + \mathrm{tr}\ \mathrm{ad}_{{\mathfrak{p}^0}_2/({\mathfrak{p}^0}_1\cap{\mathfrak{p}^0}_2)}X_2 = 0.$$

holds.
If $Y \in \mathfrak{p}_1 + [\mathfrak{p}_2, \mathfrak{p}_2]$, our assertion is trivial, since we can choose $X_2 \in [\mathfrak{p}_2, \mathfrak{p}_2]$ in this case and then the both vectors $X_i,\ i = 1, 2$, are contained in the nilradical of $\mathfrak{g}$. We suppose therefore that $Y \notin \mathfrak{p}_1 + [\mathfrak{p}_2, \mathfrak{p}_2]$, and we will adapt the proof of Théorème 2 in [5].
If the character α vanishes on $\mathfrak{p}_2$, then we have that $[Y, \mathfrak{p}_2] \subset \mathfrak{p}_2$ and the subspace $\mathfrak{q}_2 := \mathbb{R}X_1 + \mathfrak{p}_2^0$ is a Puk-polarization at f and $X_1 \in \mathfrak{p}_1 \cap \mathfrak{q}_2$. Hence, according to relation 2.4 we have that

$$\text{tr ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{q}_2)} X_1 + \text{tr ad}_{\mathfrak{q}_2/(\mathfrak{p}_1\cap\mathfrak{q}_2)} X_1 = 0.$$

Our relation then follows from the fact that $\text{tr ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{q}_2)} X_1 = \text{tr ad}_{\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)} X_1$ and $\text{tr ad}_{\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)} X_2 = \text{tr ad}_{\mathfrak{q}_2/(\mathfrak{p}_1\cap\mathfrak{q}_2)} X_1$.
If $\alpha(\mathfrak{p}_2) \neq \{0\}$, then we can find $T_2' \in \mathfrak{p}_2 \cap \ker(f)$ which satisfies the relation

$$[T_2', Y] = Y + cZ$$

for some $c \in \mathbb{R}$ and then $T_2 := T_2' - cX$ has the property that $T_2 \in \mathfrak{p}_2^0 \cap \ker(f)$ and $[T_2, Y] = Y$. We can suppose that $T_2 \notin \mathfrak{p}_1 + \mathfrak{p}_2 \cap \ker(\alpha) + \mathbb{R}Y$, since otherwise

$$T_2 - U_2 = U_1 + sY$$

for some $s \in \mathbb{R}, U_1 \in \mathfrak{p}_1$ and some $U_2 \in \mathfrak{p}_2 \cap \ker(\alpha)$. We can apply this relation to the identity $Y = a(X_2 - X_1)$ and we get $X_i' \in \mathfrak{p}_i \cap [\mathfrak{g}, \mathfrak{g}], i = 1, 2$, such that $Y = a(X_2' - X_1'), [X_2', Y] = Z$.
We can therefore assume that $Y \notin \mathbb{R}T_2 + \mathfrak{p}_1 + \mathfrak{p}_2^0 \cap \ker(\alpha)$.
Let us fix a subspace $\mathfrak{w}$ of $\mathfrak{g}$ such that $\mathfrak{w}$ contains $\mathfrak{p}_1 + \mathfrak{p}_2^0 \cap \ker(\alpha) + \mathbb{R}T_2$ and such that we have $\mathfrak{g} = \mathfrak{w} \oplus \mathbb{R}Y$. Let

$$\mathfrak{w}_0 := \{U \in \mathfrak{w}; [X_1, U] \in \mathfrak{w}, \langle f, [X_1, U]\rangle = 0\} \cap \ker(f).$$

Then $\mathfrak{w}_0 + \mathbb{R}Z$ contains $\mathfrak{p}_1 + \mathfrak{p}_2^0 \cap \alpha$ and furthermore

$$\mathfrak{g} = \mathbb{R}Y \oplus \mathbb{R}T_2 \oplus \mathbb{R}Z \oplus \mathfrak{w}_0.$$

Indeed, there exist a linear functional $\gamma' \in \mathfrak{g}^*$ and a linear map $w' : \mathfrak{g} \to \mathfrak{w}$, such that for any $X \in \mathfrak{g}$, $[X_1, X] = \gamma'(X)Y + w'(X)$. Since $[X_1, T_2] \in \frac{1}{a}Y + [\mathfrak{p}_2, \mathfrak{p}_2]$, we have that

$$\begin{aligned}[X_1, X - a\gamma'(X)T_2 + \langle f, w'(X)\rangle Y] &\in \gamma'(X)Y + w'(X) - a\gamma'(X)[X_1, T_2] + \langle f, w'(X)\rangle Z \\ &\in \ker(f) \cap \mathfrak{w} + [\mathfrak{p}_2, \mathfrak{p}_2] = \ker(f) \cap \mathfrak{w}.\end{aligned}$$

Hence $X \in \mathbb{R}T_2 + \mathbb{R}Y + \mathbb{R}Z + \mathfrak{w}_0$. It is clear that the sum $\mathbb{R}T_2 + \mathbb{R}Y + \mathbb{R}Z + \mathfrak{w}_0$ is direct. We can therefore write:

$$X = \beta(X)Y + \lambda(X)Z + a\gamma(X)T_2 + w_0(X),\ X \in \mathfrak{g} \tag{4.21}$$

for certain linear forms $\lambda,\ \beta, \gamma \in \mathfrak{g}^*$ and for some element $w_0(X) \in \mathfrak{w}_0$. For $c, \tau \in \mathbb{R}$, let $\ell_{c,\tau} \in \mathfrak{g}^*$ be chosen such that

$$\ell_{c,\tau} = f \text{ on } \mathfrak{w}_0 + \mathbb{R}Z, \text{ such that } \ell_{c,\tau}(Y) = c \text{ and } \ell_{c,\tau}(T_2) = \frac{\tau}{a}.$$

In particular we have that $\ell_{0,0} = f$. Let $\mathfrak{q}_2 := \mathfrak{p}_2^0 + \mathbb{R}(aX_2 + Y)$. Then one checks that $\mathfrak{q}_2 \in S(\ell_{c,\tau}, \mathfrak{g})$ for every $c, \tau \in \mathbb{R}$. By Pukanszky's condition $\mathfrak{p}_1 \in I(\ell_{c,\tau}, \mathfrak{g})$. But then necessarily $\mathfrak{q}_2 \in M(\ell_{c,\tau}, \mathfrak{g})$, because $\dim(\mathfrak{p}_1) = \dim(\mathfrak{q}_2)$.
The two spaces $\mathfrak{p}_1^0/(\mathfrak{p}_1 \cap \mathfrak{p}_2))$ and $\mathfrak{p}_2^0/(\mathfrak{p}_1 \cap \mathfrak{p}_2))$ are dual to each other for the bilinear forms $B_{\ell_{c,\tau}}$ for all $c, \tau \in \mathbb{R}$. Let us write $\mathfrak{p}_1^0 = \mathfrak{m}_1 \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2)$ et $\mathfrak{p}_2^0 = \mathfrak{m}_2 \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2)$ for

certain subspaces $\mathfrak{m}_1,\ \mathfrak{m}_2$. Let $\{\hat{e}_j; 1 \leq j \leq m\}$ be a basis of $\mathfrak{m}_2$ and let $\{e_i; 1 \leq i \leq m\}$ (resp. $\{e_i(c,\tau); 1 \leq i \leq m\}$) be the dual basis of $\mathfrak{m}_1$ for the bilinear forms $B_{\ell_{c,\tau}}, c, \tau \in \mathbb{R}$. Writing

$$W := \sum_{j=1}^{m} [e_j, \hat{e}_j]$$

and

$$W_{c,\tau} := \sum_{j=1}^{m} [e_j(c), \hat{e}_j] := a^{-1}Z + W_{c,\tau},$$

we arrive by 2.5 at

$$\text{tr } \mathrm{ad}_{\mathfrak{p}_1/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X_1 + \text{tr } \mathrm{ad}_{\mathfrak{p}_2/(\mathfrak{p}_1 \cap \mathfrak{p}_2)} X_2 = B_f(X_1, W_{c,\tau}), c, \tau \in \mathbb{R}. \tag{4.22}$$

According to (4.21)

$$[e_i, \hat{e}_j] = a\gamma_{ji} T_2 + \beta_{ji} Y + \delta_{ji} Z + w_{ji}$$

where $\gamma_{ij},\ \beta_{ij} \in \mathbb{R}, w_{ji} \in \mathfrak{w}_0 (1 \leq i,\ j \leq m)$, and where δ_{ij} denotes the Kronecker symbol.

If we write $e_k(c,\tau) = \sum_{j=1}^{m} \lambda_{kj}(c,\tau) e_j$ with $\lambda_{kj}(c) \in \mathbb{R}$ $(1 \leq j,\ k\ \leq m)$, then we get

$$[e_k(c,\tau), \hat{e}_i] = \sum_{j=1}^{m} \lambda_{kj}(c,\tau) [e_j, \hat{e}_i] = \sum_{j=1}^{m} \lambda_{kj}(c,\tau) \left(a\gamma_{ij} T_2 + \beta_{ij} Y + \delta_{ij} Z + w_{ij} \right). \tag{4.23}$$

Hence

$$\delta_{ki} = \langle \ell_c, [e_k(c,\tau), \hat{e}_i] \rangle = \sum_{j=1}^{m} \lambda_{kj}(c,\tau) \left(c\beta_{ij} + \tau\gamma_{ij} \right) + \lambda_{ki}(c,\tau) (1 \leq j,\ k\ \leq m). \tag{4.24}$$

For the trace condition, we use 4.22 and 4.23 and we get

$$\begin{aligned} \beta &= B_{\ell_{c,\tau}}(X_1, \sum_{k=1}^{m} [e_k(c,\tau), \hat{e}_k]) \\ &= \langle \ell_{c,\tau}, \sum_{k=1}^{m} [X_1, [e_k(c,\tau), \hat{e}_k] \rangle \\ &= \langle \ell_{c,\tau}, \sum_{k=1}^{m} \sum_{j=1}^{m} [X_1, \lambda_{kj}(c,\tau) \left(a\gamma_{kj} X_2 + \beta_{kj} Y + \delta_{kj} Z + w_{kj} \right)] \rangle \\ &= \sum_{k,j=1}^{m} \left(\beta_{kj} + c\gamma_{kj} \right) \lambda_{kj}(c,\tau). \end{aligned} \tag{4.25}$$

We now interpret these equations 4.24 and 4.25 in the following way. Consider the $m \times m$ matrices $B := (\beta_{i,j})_{1 \leq i,j \leq m}$, $\Gamma := (\gamma_{i,j})_{1 \leq i,j \leq m}$ and $\Lambda(c,\tau) := (\lambda_{i,j}(c,\tau))_{1 \leq i,j \leq m}$. Equation 4.24 then reads

$$(cB + \tau\Gamma + \mathbb{I}_m) \Lambda^t(c,\tau) = \mathbb{I}_m,$$

i.e.

$$\Lambda^t(c,\tau) = (cB + \tau\Gamma + \mathbb{I}_m)^{-1}$$

and

$$\beta = tr((cB + \tau\Gamma + \mathbb{I}_m) \Gamma^t(c)) = tr((cB + \tau\Gamma + \mathbb{I}_m)^{-1}(B + c\Gamma)), c \in \mathbb{R}. \tag{4.26}$$

We must prove that $\beta = 0$.

Let $\mathcal{S} = \mathcal{S}^k$ be the family of all subsets S of the set $\{0, \cdots, k-1\} := E_k$ and by $\mathcal{S}_i = \mathcal{S}_i^k$ the family of elements $S \in \mathcal{S}$ containing i elements. We denote by $\delta_S, S \in \mathcal{S}$, its characteristic function, i.e $\delta_S(x) = 1$, if $x \in S$ and $\delta_S(x) = 0$, if $x \notin S$. For $S \in \mathcal{S}_i$, let M_S be the $m \times m$ matrix $M_S := B^{\delta_S(0)}\Gamma^{1-\delta_S(0)} \cdots B^{\delta_S(k-1)}\Gamma^{1-\delta_S(k-1)}$ and $M_i^k := \sum_{S\in\mathcal{S}_i} M_S,\ 0 \le i \le k$.
For c and τ small enough, we can express $(cB + \tau\Gamma + \mathbb{I}_m)^{-1}$ as a converging series in c, τ:

$$\begin{aligned}(cB+\tau\Gamma+\mathbb{I}_m)^{-1} &= \sum_{k=0}^{\infty}(-1)^k(cB+\tau\Gamma)^k \\ &= \sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{k}c^i\tau^{k-i}M_i^k \\ &= \sum_{i,j=0}^{\infty}(-1)^{i+j}c^i\tau^j M_i^{i+j}.\end{aligned}$$

If we plug this identity into relation 4.26, we get the identity

$$\mathrm{tr}B = \sum_{i,j=0}^{\infty}(-1)^{i+j}c^i\tau^j\mathrm{tr}(BM_i^{i+j}) + \sum_{i,j=0}^{\infty}(-1)^{i+j}c^{i+1}\tau^j\mathrm{tr}(\Gamma M_i^{i+j}), c, \tau \in \mathbb{R}. \tag{4.27}$$

Hence,

$$\mathrm{tr}(B\Gamma^j) = 0, j \in \mathbb{N}^*, \tag{4.28}$$

and

$$\mathrm{tr}(BM_{i+1}^{k+1}) = \mathrm{tr}(\Gamma M_i^k), k \ge i \ge 0. \tag{4.29}$$

Let Σ be the group of bijections of the set E_k generated by the cyclic permutation $\sigma(i) = i+1$ modulo $k(i \in E_k)$. This group has k elements and it acts on the space $\mathcal{S}_i$ in a natural way. Let for $S \in \mathcal{S}_i^k$, Σ_S be the stabilizer of S in Σ. Denote by d_S the number of elements of $\Sigma_S, S \in \mathcal{S}_i^k$. Let also Ω be the orbit space $\mathcal{S}_i/\Sigma$. We choose for every Σ-orbit O in $\mathcal{S}_i$ a representative S_O and we denote by $|O|$ the number of its elements. For any $S \in \mathcal{S}_i$, we have that

$$\begin{aligned}M_{\sigma S} &= B^{\delta_S(\sigma^{-1}(0))}\Gamma^{1-\delta_S(\sigma^{-1}(0))}\cdots B^{\delta_S(\sigma^{-1}(k-1))}\Gamma^{1-\delta_S(\sigma^{-1}(k-1))} \\ &= B^{\delta_S(k-1)}\Gamma^{1-\delta_S(k-1)}\cdots B^{\delta_S(0)}\Gamma^{1-\delta_S(0)}\cdots B^{\delta_S(k-2)}\Gamma^{1-\delta_S(k-2)}\end{aligned}$$

Hence

$$\mathrm{tr}(M_S) = \mathrm{tr}(M_{\sigma S}) = \mathrm{tr}(M_{\sigma^l S}),\ l \in \mathbb{N}.$$

This allows us to write

$$\mathrm{tr}(M_i^k) = \sum_{O\in\Omega}|O|\mathrm{tr}(M_{S_O}) = k\sum_{O\in\Omega}\frac{1}{d_{S_O}}\mathrm{tr}(M_{S_O}). \tag{4.30}$$

Consider the mapping $I : \mathcal{S}_i^{k-1} \to \mathcal{S}_i^k, T = (t_1 < t_2 \cdots < t_{i-1}) \mapsto (t_1+1 < \cdots < t_{i-1}+1)$. This mapping is a bijection of the set S_i^{k-1} onto the subset $\mathcal{S}_{i,1}^k := \{S \in \mathcal{S}_i^k, 0 \notin S\}$.

Lemma 4.5. *We have the formula*

$$tr(M_i^k) = \frac{k}{k-i}tr(\Gamma M_i^{k-1}) = \frac{k}{i}tr(BM_{i-1}^{k-1}), 1 \le i \le k. \tag{4.31}$$

Proof. Indeed, we have that

$$\mathrm{t}r(\Gamma M_i^{k-1}) = \sum_{T\in\mathcal{S}_i^{k-1}} \mathrm{t}r(\Gamma M_T) = \sum_{T\in\mathcal{S}_i^{k-1}} \mathrm{t}r(M_{I(T)}) = \sum_{S\in\mathcal{S}_{i,1}^{k}} \mathrm{t}r(M_S).$$

If $i<k$, then we can take every $S_O \in \mathcal{S}_{i,1}^k$ and we write

$$\mathrm{t}r(\Gamma M_i^{k-1}) = \sum_{O\in\Omega} |O\cap \mathcal{S}_{i,1}^k| \mathrm{t}r(M_{S_O}). \tag{4.32}$$

Now it is clear that for $\tau\in\Sigma$, necessarily $\tau S\in\mathcal{S}_{i,1}^k$ if and only if $\tau=\sigma^{-t}$, for some $t\in E_{k+1}\setminus S$. On the other hand the subset $\Sigma_{S,1} := \{\tau\in\Sigma, \tau S\in\mathcal{S}_{i,1}^k\}$ is invariant under multiplication by elements of Σ_S, since S is Σ_S-invariant. Hence $|O\cap\mathcal{S}_{i,1}^k| = \frac{k-i}{d_{S_O}}, O\in\Omega$. Comparing 4.30 with 4.32, we obtain the desired formula.
The case $i=k-1$ is easy and the formula for $\mathrm{t}r(BM_i^k)$ can be proved in a similar way. ☐

We apply now the preceding identities. Since $\mathrm{t}r(B\Gamma^k)=0$ for every $k\geq 1$ by 4.28, it follows from 4.29 and 4.31 that

$$\mathrm{t}r(BM_{2l}^k) = \mathrm{t}r(\Gamma M_{2l-1}^{k-1}) = \frac{k-2l+1}{k}\mathrm{t}r(M_{2l-1}^k) = \frac{k-2l+1}{k}\frac{k}{2l-1}\mathrm{t}r(BM_{2l-2}^{k-1}), k\geq 2l+1, \tag{4.33}$$

and so

$$0 = \mathrm{t}r(B\Gamma^k) = \mathrm{t}r(BM_2^{k+1}) = \cdots = \mathrm{t}r(BM_{2s}^{k+s}) = \cdots = \mathrm{t}r(BB^{2k}). \tag{4.34}$$

Hence

$$\mathrm{t}rB = \mathrm{t}r(B(\mathbb{I}_m + c^2B^2 + c^4B^4 + \cdots)) = \mathrm{t}r(B(\mathbb{I}_m - c^2B^2)^{-1}), c \text{ small enough.} \tag{4.35}$$

Since the mapping $\mathbb{R}\to M_m(\mathbb{R}), c\mapsto (\mathbb{I}_m - c^2B^2)^{-1} = (\mathbb{I}_m - cB)^{-1}(\mathbb{I}_m + cB)^{-1}$ is analytic, it follows that

$$\mathrm{t}rB = \mathrm{t}r(B(\mathbb{I}_m + c^2B^2 + c^4B^4 + \cdots)) = \mathrm{t}r(B(\mathbb{I}_m - c^2B^2)^{-1}), c\in\mathbb{R}.$$

Since

$$\mathrm{t}r(B(\mathbb{I}_m - c^2B^2)^{-1}) = \sum_{\lambda\in\sigma(B)} \frac{\lambda}{1-c^2\lambda^2}, c\in\mathbb{R},$$

we see that either $\mathrm{t}rB=0$, if B is nilpotent, or that $\mathrm{t}r(B) = \lim_{c\to\infty}\sum_{\lambda\in\sigma(B)}\frac{\lambda}{1-c^2\lambda^2} = 0$. ☐

Let us continue with the proof of Proposition 4.2. In the case we are just considering we have that $\mathfrak{p}_1\cap\mathfrak{p}_2 = \mathfrak{p}_1'\cap\mathfrak{p}_2$ and $[X_2, \mathfrak{p}_1\cap\mathfrak{p}_2]\subset[\mathfrak{p}_1,\mathfrak{p}_2]$. Let $\mathfrak{k} := \mathbb{R}X_2\oplus\mathfrak{p}_1\cap\mathfrak{p}_2$ and $K := \exp(\mathfrak{k})$. This allows us to identify the space P_2/K with $P_2^0/P_1\cap P_2$. Hence using again 2.3 and 2.14 and the preceding lemma, we obtain for $\varphi\in C_0^\infty(G/P_1,\chi_f), g\in G$ and $I(g) = I_{2,1'}\circ I_{1',1}\varphi(g)$ that

$$\begin{aligned} I(g) &= \oint_{P_2/P_2\cap P_1'} (I_{1',1}\varphi)(gp_2)\Delta_{P_2,G}^{-1/2}(p_2)\chi_f(p_2)d\nu(p_2) \\ &= \oint_{P_2/K}\oint_{K/P_1\cap P_2} (I_{1',1}\varphi)(gp_2k)\Delta_{P_2,G}^{-1/2}(p_2k)\Delta_{K,P_2}^{-1}(k)\chi_f(p_2)d\nu(k)d\nu(p_2) \end{aligned}$$

$$
\begin{aligned}
&= \oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\times \\
&\times \int_{\mathbb{R}} (I_{1',1}\varphi)(gp_2\mathrm{exp}(xX_2))e^{-\frac{x}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_2}X_2}e^{-x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/\mathfrak{k}}X_2}dx \\
&= \oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\times \\
&\times \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(gp_2\mathrm{exp}(xX_2)\mathrm{exp}(yY))dye^{-\frac{x}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_2}X_2}e^{-x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/\mathfrak{k}}X_2}dx \\
&= \oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\int_{\mathbb{R}} e^{-\frac{x}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_2}X_2}e^{-x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/\mathfrak{k}}X_2}dx\times \\
&\times \int_{\mathbb{R}} \varphi(gp_2\mathrm{exp}((x+ay)X_2)\mathrm{exp}(-ayX_1)\mathrm{exp}(-\frac{1}{2}ay^2Z))dx \\
&= \oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\int_{\mathbb{R}} e^{-\frac{x}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_2}X_2}e^{-x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/\mathfrak{k}}X_2}dx\times \\
&\times \int_{\mathbb{R}} e^{\frac{i}{2}ay^2}e^{-\frac{1}{2}ay\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_1}X_1}\varphi(gp_2\mathrm{exp}((x+ay)X_2)dy \\
&= \frac{1}{|a|}\oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\int_{\mathbb{R}} e^{-\frac{x}{2}\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_2}X_2}e^{-x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/\mathfrak{k}}X_2}dx\times \\
&\times \int_{\mathbb{R}} e^{\frac{i}{2a}(y-x)^2}e^{-\frac{1}{2}(y-x)\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_1}X_1}\varphi(gp_2\mathrm{exp}(yX_2))dy \\
&= \frac{1}{|a|}\oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\times \\
&\times \int_{\mathbb{R}} dx\int_{\mathbb{R}} e^{\frac{i}{2a}(y-x)^2}e^{-\frac{1}{2}y\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_1}X_1}\varphi(gp_2\mathrm{exp}(yX_2))dy \\
&= \sqrt{\frac{2\pi}{|a|}}e^{\frac{i\pi}{4}\mathrm{sgn}(a)}\oint_{P_2/K} \chi_f(p_2)\Delta_{P_2,G}^{-1/2}(p_2)d\nu(p_2)\times \\
&\times \int_{\mathbb{R}} e^{-\frac{1}{2}x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}_2}X_2}e^{-x\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}_2/\mathfrak{k}}X_2}\varphi(gp_2\mathrm{exp}(xX_2))dx \\
&= e^{\frac{i\pi}{4}\mathrm{sgn}(a)}\oint_{P_2/P_1\cap P_2} \chi_f(p)\Delta_{P_2,G}^{-1/2}(p)\varphi(gp)d\nu(p) \\
&= e^{\frac{i\pi}{4}\mathrm{sgn}(a)}(I_{2,1}\varphi)(g),
\end{aligned}
$$

which achieves the proof in this case.
As for the cases where $\dim(\mathfrak{a}) = 3$, they are treated in a similar way. It suffices to adopt the calculations made in [1]. ☐

In combining Proposition 3.4 and Proposition 4.2, we obtain the main upshot:

Theorem 4.6. *Let* $G = exp(\mathfrak{g})$ *be an exponential solvable Lie group. Let* $f \in \mathfrak{g}^*$ *and let* $\mathfrak{p}_1, \mathfrak{p}_2$ *be two Puk-polarizations at* f*. Let* $\pi_1 := \pi_{f,\mathfrak{p}_1}, \pi_2 := \pi_{f,\mathfrak{p}_2}$*. The mapping* $I_{\mathfrak{p}_2,\mathfrak{p}_1}$ *extends to a unitary intertwining operator for the representations* π_1 *and* π_2*.*

5. The Composition formula and the Maslov index

We still deal in this section with the case of an exponential Lie group $G = \exp(\mathfrak{g})$ and let as before $f \in \mathfrak{g}^*$. Consider three Lagrangian subspaces W_j $(1 \leq j \leq 3)$ of $\mathfrak{g}$ for the bilinear form B_f and define, following Kashiwara, a quadratic form Q on $W_1 \times W_2 \times W_3$ by the formula

$$Q(X_1, X_2, X_3) = f([X_1, X_2]) + f([X_2, X_3]) + f([X_3, X_1]).$$

The index of this quadratic form is called the Maslov index of the spaces W_j and is denoted by $\tau(W_1, W_2, W_3)$. We begin by recording some the main properties of this index.

Lemma 5.1. *([10], [11]) Let us write for simplicity τ_{ijk} instead of $\tau(W_i, W_j, W_k)$. Then we have*

1. $\tau_{123} = -\tau_{213} = -\tau_{132}$*;*
2. $\tau_{234} - \tau_{134} + \tau_{124} - \tau_{123} = 0$*;*
3. *If $\mathfrak{p}$ is an isotropic subspace of $\mathfrak{g}$ containing $\mathfrak{g}(f)$ and if W is a Lagrangian subspace of $\mathfrak{g}$, then $W^{\mathfrak{p}} = \left(W \cap \mathfrak{p}^f\right) + \mathfrak{p}$ is also Lagrangian. Furthermore, if $\mathfrak{p}$ is contained in $W_1 \cap W_2 + W_2 \cap W_3 + W_3 \cap W_1$, then we have*
$$\tau_{123} = \tau(W_1^{\mathfrak{p}}, W_2^{\mathfrak{p}}, W_3^{\mathfrak{p}}).$$

The spaces $W_i^{\mathfrak{p}}, i = 1, 2, 3$ will be replaced by Puk-polarizations $\mathfrak{p}_i, i = 1, 2, 3$ at $f \in \mathfrak{g}^*$ in $\mathfrak{g}$. In Théorème 6.1 of [1], the formula

$$I_{\mathfrak{p}_2,\mathfrak{p}_1} = e^{(i\pi/4)\tau(\mathfrak{p}_2,\mathfrak{p}_0,\mathfrak{p}_1)} I_{\mathfrak{p}_2,\mathfrak{p}_0} \circ I_{\mathfrak{p}_0,\mathfrak{p}_1} = T_{\mathfrak{p}_2,\mathfrak{p}_1}$$

was obtained in the general case for a special construction of the intertwining operators, making use of any polarization $\mathfrak{p}_0$ at f of Vergne type. Therefore we have by Théorème 5.3 in [1] that

Theorem 5.2. *Let $\mathfrak{p}_i (1 \leq i \leq 3)$ be three Puk-polarizations at $f \in \mathfrak{g}^*$ in $\mathfrak{g}$. If the operators $I_{\mathfrak{p}_i\mathfrak{p}_j}$ are isometric (which requires their normalization if necessary), then*

$$I_{\mathfrak{p}_1\mathfrak{p}_3} \circ I_{\mathfrak{p}_3\mathfrak{p}_2} \circ I_{\mathfrak{p}_2\mathfrak{p}_1} = e^{(i\pi/4)\tau(\mathfrak{p}_3,\mathfrak{p}_2,\mathfrak{p}_1)} \mathbb{I}.$$

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