Estimate of the L^p-Fourier Transform Norm on Strong $*$-Regular Exponential Solvable Lie Groups

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Abstract

We study the L^p-Fourier transform for a special class of exponential Lie groups, the strong $*$-regular exponential Lie groups. Moreover, we provide an estimate of its norm using the orbit method.

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1 Introduction

Let G be a separable locally compact unimodular group of type I and $\widehat{G}$ its unitary dual, the set of equivalence classes of irreducible unitary representations, endowed with the Mackey-Borel structure. Let dg be the Haar measure on G. The operator valued Fourier transform on G maps each $\varphi \in L^1(G)$ to the field $\mathcal{F}(\varphi) = (\pi(\varphi))_{\pi\in\widehat{G}}$ of bounded operators on $\widehat{G}$, where $\pi(\varphi)$ is defined by:

$$\pi(\varphi) = \int_G \varphi(g)\pi(g)dg.$$

Let μ be the Plancherel measure on $\widehat{G}$, which is uniquely determined by the abstract Plancherel formula: for $\varphi \in L^1(G) \cap L^2(G)$,

$$\int_G |\varphi(g)|^2 dg = \int_{\widehat{G}} \operatorname{tr}((\pi(\varphi)^*\pi(\varphi))d\mu(\pi).$$

The Hausdorff-Young inequality (see [16]), proved for separable locally compact unimodular group of type I, is the assertion

$$\Big(\int_{\widehat{G}} \|\pi(\varphi)\|_{C_q}^q d\mu(\pi)\Big)^{\frac{1}{q}} \leq \Big(\int_G |\varphi(g)|^p dg\Big)^{\frac{1}{p}}, \tag{1.1}$$

where $\varphi \in L^1(G) \cap L^p(G),\ 1 < p \leq 2,\ q$ is the conjugate of $p,$ i.e., $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\|\pi(\varphi)\|_{C_q}^q = \operatorname{tr}\Big((\pi(\varphi)^*\pi(\varphi))^{\frac{q}{2}}\Big).$$

For a $\mu-$measurable field of bounded operators F on $\widehat{G}$, let

$$\|F\|_q = \Big(\int_{\widehat{G}} \|F(\pi)\|_{C_q}^q d\mu(\pi)\Big)^{\frac{1}{q}}.$$

Thus we can write (1.1) as

$$\|\mathcal{F}(\varphi)\|_q \leq \|\varphi\|_p.$$

Hence the map $\varphi \mapsto \mathcal{F}\varphi$ from $L^1(G) \cap L^p(G)$ to $L^q(\widehat{G})$ extends to a continuous operator $\mathcal{F}^p : L^p(G) \to L^q(\widehat{G})$ and its operator norm

$$\|\mathcal{F}^p(G)\| = \sup_{\|\varphi\|_p \leq 1} \|\mathcal{F}(\varphi)\|_q \leq 1.$$

It has been proved in [10], that $\|\mathcal{F}^p(G)\| = 1$ if and only if G contains a compact open subgroup. The aim of the present paper is to get an estimate of the norm $\|\mathcal{F}^p(G)\|$ for a certain class of solvable Lie groups. Beckner [3] treated the case of the abelian group $G = \mathbb{R}^n$ and proved that $\|\mathcal{F}^p(\mathbb{R}^n)\| = A_p^n$, where $A_p = \left(\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\right)^{\frac{1}{2}}$. Many results have been obtained for several kinds of groups (see [1],[3],[10],[11],[13],[20]). Recently, the following result concerning general nilpotent Lie groups has been shown in [2]:

Theorem 1.1. *Let G be a connected and simply connected nilpotent Lie group and m the maximal dimension of the coadjoint orbits. Then for $1 < p \leq 2$,*

$$\|\mathcal{F}^p(G)\| \leq A_p^{\frac{2dimG-m}{2}}. \tag{1.2}$$

For the case of non-unimodular locally compact group of type I, there exists a field of non-zero positive self-adjoint operators $(K_\pi)_{\pi\in\widehat{G}}$ and a measure μ on $\widehat{G}$ such that for $\varphi \in L^1(G) \cap L^2(G)$ and for $\mu-$almost all $\pi \in \widehat{G}$ the operator $\pi(\varphi)K_\pi^{\frac{-1}{2}}$ extends to a Hilbert-Schmidt operator on the space $\mathcal{H}_\pi$ of π and the operator $K_\pi^{\frac{-1}{2}}\pi(\varphi^* * \varphi)K_\pi^{\frac{-1}{2}}$ is trace class. Then the Plancherel formula reads (see [8]):

$$\|\varphi\|_2^2 = \int_{\widehat{G}} \mathrm{tr}\,\left(K_\pi^{\frac{-1}{2}}\pi(\varphi^* * \varphi)K_\pi^{\frac{-1}{2}}\right)d\mu(\pi).$$

By taking as a definition of the Fourier transform of φ, the operator field

$$\mathcal{F}(\varphi) = \left(\pi(\varphi)K_\pi^{\frac{-1}{q}}\right)_{\pi\in\widehat{G}},$$

Terp [23] and Führ [12] extend the Hausdorff-Young theorem to this case. Eymard and Terp [9] and Russo [21] obtained a more precise estimate of the norm $\|\mathcal{F}^p(G)\|$ for the group of affine transformations of the real line, the so-called "ax+b"-group. An extension of these results to a certain class of completely solvable Lie groups was given by Inoue. It concerns the class of connected and simply connected Lie groups $G = \exp \mathfrak{g}$, where $\mathfrak{g}$ is a normal $j-$algebra which is characterized by the fact that G is an affine automorphism group acting simply and transitively on a Siegel domain of type II (see [13]). In this paper, we shall show that for Boidol's group, a certain four dimensional exponential Lie group which is not $*$-regular, the known methods do not apply. In this paper we look therefore at the class of the strong $*$-regular exponential Lie groups (see definition 1 below), for which a sample of Boidol's group does not appear during the induction procedure. Our main result is the following:

Theorem 1.2. *Let G be an arbitrary exponential solvable Lie group satisfying the strong $*$-regularity condition. Let m be the maximal dimension of coadjoint orbits.*
Let $1 < p \leq 2$ and q the conjugate of p. Then for all $\varphi \in L^1(G) \cap L^p(G)$ and μ-almost all $\pi \in \widehat{G}$, the operator $\pi^p(\varphi) := \pi(\varphi)K_\pi^{\frac{-1}{q}}$ is bounded, its extension is of class C_q and we have the following inequality:

$$\Big(\int_{\widehat{G}} \|\pi^p(\varphi)\|_{C_q}^q d\mu(\pi)\Big)^{\frac{1}{q}} \leq A_p^{\frac{2\dim G - m}{2}} \|\varphi\|_p.$$

2 Preliminaries

2.1 Some notations and basic facts

We recall first some basic notions and facts. For further details, see [4]. Let G be a connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$, i.e., a connected, simply connected Lie group such that the exponential mapping:

$$\exp \ : \ \mathfrak{g} \longrightarrow G$$

is a diffeomorphism of $\mathfrak{g}$ into G. We denote by log its inverse mapping. Let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_\mathfrak{g}$, i.e.,

$$\mathrm{ad}_\mathfrak{g}(X)Y = \mathrm{ad}(X)Y = [X,Y], \ \forall \ X, \ Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G, i.e.,

$$\mathrm{Ad}_G(g)Y = \mathrm{Ad}(g)Y = \mathrm{e}^{\mathrm{ad}(X)}Y, \ g = \exp X \in G, \ Y \in \mathfrak{g},$$

and on $\mathfrak{g}^*$ by the coadjoint representation Ad_G^*, i.e.,

$$< \mathrm{Ad}_G^*(g)l, X >=< g \cdot l, X >=< l, \mathrm{Ad}(g^{-1})X >, \ g \in G, \ \ l \in \mathfrak{g}^*, \ X \in \mathfrak{g}.$$

The set $G \cdot l = \{g \cdot l, \ g \in G\} =: \Omega_l$ is called the G-orbit of l, we denote by $\mathfrak{g}^*/G$ the space of coadjoint orbits. Let $\mathfrak{g}(l) = \{X \in \mathfrak{g}, \ < l, [X, \mathfrak{g}] >= \{0\} \ \}$ be the stabilizer of $l \in \mathfrak{g}^*$ in $\mathfrak{g}$, it's also the Lie algebra of $G_l = \{g \in G, \ g \cdot l = l \ \}$. For $\mathfrak{p} \subset \mathfrak{g}$, define $\mathfrak{p}^\perp = \{f \in \mathfrak{g}^*, \ f_{|\mathfrak{p}} = 0\}$, the annihilator of $\mathfrak{p}$ in $\mathfrak{g}^*$. A subspace $\mathfrak{b}(l) \subset \mathfrak{g}$ is called a polarization for $l \in \mathfrak{g}^*$, if $\mathfrak{b}(l)$ is a maximal dimensional isotropic subalgebra related to the skew-symmetric bilinear form B_l defined by:

$$B_l(X,Y) =< l, [X,Y] >, \ X, \ Y \in \mathfrak{g}.$$

Moreover, a polarization $\mathfrak{b}(l)$ for l satisfies Pukanszky's condition or is a Pukanszky polarization, if

$$l + \mathfrak{b}(l)^\perp = \mathrm{Ad}^*(B(l))l = B(l) \cdot l$$

where $B(l) = \exp \mathfrak{b}(l)$. So we can consider the unitary character $\chi_l = \chi$ of $B(l)$ associated to l defined by:

$$\chi(\exp \mathrm{X}) = \mathrm{e}^{-2\pi i<l,X>},\ X \in \mathfrak{b}(l).$$

We say that the coadjoint orbit Ω_l of $l \in \mathfrak{g}^*$ is saturated with respect to a one codimensional ideal $\mathfrak{g}_0 = \text{Lie } G_0$ in $\mathfrak{g}$, if $\mathfrak{g}(l) \subset \mathfrak{g}_0$. In such case, we have that $G \cdot l = G \cdot l + \mathfrak{g}_0^\perp$ and

$$\dim(G_0 \cdot l_0) = \dim(G \cdot l) - 2,\ l_0 = l_{|\mathfrak{g}_0}. \tag{2.1}$$

Let dg be a left Haar measure on G and Δ_G the modular function of G, which is defined, for all $x \in G$ and f in $C_c(G)$, i.e., the set of continuous functions on G with compact support, by the relation:

$$\int_G f(gx^{-1})\ dg = \Delta_G(x) \int_G f(g)\ dg.$$

It's well known that, for $x \in G$:

$$\Delta_G(x) = |\det\ \mathrm{Ad}(x)|^{-1} = \exp(-\mathrm{tr}\ \mathrm{ad}(\log x)).$$

Let H be a closed subgroup with corresponding Lie algebra $\mathfrak{h}$. We denote by $\Delta_{H,G}$ the positive character of H defined by:

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}.$$

So, we have:

$$\Delta_{H,G}(\mathrm{X}) = \exp(\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}} X),\ X \in \mathfrak{h}.$$

It's clear that, if H is a normal subgroup of G, then $\Delta_{H,G}(h) = 1,\ \forall h \in H$.

2.2 Induced representations

Let $K(G, H)$ be the space of continuous and compactly supported modulo H functions $F :\ G \longrightarrow \mathbb{C}$ such that:

$$F(gh) = \Delta_{G,H}(h)^{-1} F(g),\ g \in G, h \in H.$$

G acts on this space by left translation. It is shown in [4] that, up to a multiplicative scalar, there exists a unique non-negative G-invariant linear form on $K(G, H)$. It's usually denoted by $\nu_{G,H}$ or more simply ν and so we have:

$$\nu_{G,H}(F) = \oint_{G/H} F(g)\ d\nu_{G,H}(\dot{g}),\ \forall F \in K(G, H).$$

It's clear that if $\Delta_G = \Delta_H$ on H, then $\nu_{G,H}$ is a G-invariant measure on the homogeneous space G/H and $K(G, H) = C_c(G/H)$.

Let π be a unitary representation of H in the Hilbert space $\mathcal{H}_\pi$. We consider the space:

$$K_\pi(G,H) = \Big\{ F : G \longrightarrow \mathcal{H}_\pi, \text{continuous and with compact support modulo } H \\ \text{such that : } F(gb) = \Delta_{G,H}(h)^{-\frac{1}{2}}\ \rho(h^{-1})\big(\xi(g)\big), \forall (g,h) \in G \times H \Big\}.$$

If F is in $K_\pi(G,H)$, the mapping $g \longmapsto \|F(g)\|^2_{\mathcal{H}_\pi}$ belongs to $K(G,H)$. This relation allows us to define an L^2-norm on $K_\pi(G,H)$ in the following way:

$$\|F\|_2 = \Big(\oint_{G/H} \|F(g)\|^2_{\mathcal{H}_\pi} d\nu(\dot{g}) \Big)^{\frac{1}{2}}.$$

We define the induced representation $\mathrm{Ind}_H^G \pi$ of G as the left regular representation of G on the completion $L^2(G/H,\pi)$ of $K_\pi(G,H)$ with respect to the norm $\|.\|_2$ defined above, i.e.,

$$(\mathrm{Ind}_H^G \pi)(x)(\xi)(y) = \xi(x^{-1}y),\ \forall x,y \in G,\ \xi \in L^2(G/H,\pi).$$

2.3 Orbit theory

The unitary dual $\widehat{G}$ of G can be parametrized via the Kirillov-Bernat-Vergne orbit method. Let l be in $\mathfrak{g}^*$. We take a polarization $\mathfrak{b}$ for l satisfying Pukanszky's condition. For such a polarization, we define $\pi_{l,\mathfrak{b}}$ by:

$$\pi_l = \pi_{l,\mathfrak{b}} = \mathrm{Ind}_B^G \chi_l, \quad (B = \exp \mathfrak{b}).$$

Then $\pi_{l,\mathfrak{b}}$ is an irreducible representation of G and its equivalence class $[\pi_{l,\mathfrak{b}}]$ depends only on the coadjoint orbit of l. Every irreducible representation π is equivalent to an induced representation $\pi_{l,\mathfrak{b}}$ from a character χ_l of a Pukanszky polarization. Moreover, the following mapping, called the Kirillov-Bernat-Pukanszky-Vergne mapping

$$\begin{array}{rccc} \mathcal{K}: & \mathfrak{g}^*/G & \longrightarrow & \widehat{G} \\ & G\cdot l & \longmapsto & [\pi_{l,\mathfrak{b}}] =: \pi_{G\cdot l} \end{array}$$

is a homeomorphism. For details see [17].

2.4 Localized Plancherel formula

2.4.1 The Plancherel formula for exponential Lie groups was obtained by Duflo and Rais in [8]. Let $Z = \exp \mathfrak{z}$ be the center of G. Let $A = \exp \mathfrak{a}$ be a closed connected subgroup of Z and χ_ψ be the unitary character of A associated to a fixed $\psi \in \mathfrak{a}^*$. Let

$$\mathfrak{g}^*_\psi = \{ l \in \mathfrak{g}^* \ : \ l_{|\mathfrak{a}} = \psi \} \text{ and} \tag{2.2}$$

$$\widehat{G}_{\chi_\psi} = \{ \pi \in \widehat{G} \ : \ \pi_{|A} = \chi_\psi . \mathrm{Id} \}.$$

It follows from [17] that the orbit space $\mathfrak{g}_\psi^*/G$ is homeomorphic to $\widehat{G}_{\chi_\psi}$ via the Kirillov-mapping. For $1 \leq p < +\infty$, the space $L^p(G/A, \chi_\psi)$ is defined as the set of all measurable functions $\varphi : G \to \mathbb{C}$ such that $\varphi(ga) = \overline{\chi_\psi(a)}\varphi(g)$ for all $g \in G$ and all $a \in A$ and

$$\|\varphi\|^p_{L^p(G/A,\chi_\psi)} = \int_{G/A} |\varphi(g)|^p dg < \infty.$$

For $p = 1$ we obtain a $*$-Banach algebra. The convolution here is defined for φ and φ' in $L^1(G/A, \chi_\psi)$ by:

$$\varphi * \varphi'(g) = \int_{G/A} \varphi(u)\varphi'(u^{-1}g)du, \ g \in G/A$$

and the involution * by:

$$f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}, \quad x \in G, f \in L^1(G/A, \chi_\psi).$$

Of course, the space $\widehat{G}_{\chi_\psi}$ is also the dual space of the algebra $L^1(G/A, \chi_\psi)$.
We denote by $\Omega_\psi \in \mathfrak{g}_\psi^*/G$ a coadjoint orbit and by $\pi_{\Omega_\psi} \in \widehat{G}_{\chi_\psi}$ the corresponding representation of G.
By [19], there exists a non zero rational function ξ on $\mathfrak{g}^*$ such that:

$$\xi(x \cdot l) = \Delta_G(x^{-1})\xi(l), \ \text{for all } x \in G \text{ and } l \in \mathfrak{g}^*. \tag{2.3}$$

Fix one such function ξ. There is a unique measure $\mu_{\xi,\psi}$ on $\mathfrak{g}_\psi^*/G$ such that, for all Borel function ϕ on $\mathfrak{g}^*$, we have by [8]:

$$\int_{\mathfrak{g}_\psi^*} \phi(l)|\xi(l)|dl = \int_{\mathfrak{g}_\psi^*/G} \int_{\Omega_\psi} \phi(l)d\beta_{\Omega_\psi}(l)d\mu_{\xi,\psi}(\Omega_\psi), \tag{2.4}$$

where $d\beta_{\Omega_\psi}$ is the canonical measure on Ω_ψ. Then $d\mu_{\xi,\psi}$ is called the localized Plancherel measure on $\mathfrak{g}_\psi^*/G \simeq \widehat{G}_{\chi_\psi}$.
We recall the fact that, if π is an irreducible representation of G in a Hilbert space $\mathcal{H}_\pi$, then by [7], there exists a unique self-adjoint and positive operator K_π in $\mathcal{H}_\pi$ which is semi-invariant with weight Δ_G^{-1}, which means that

$$\pi(g)K_\pi\pi(g)^{-1} = \Delta_G^{-1}(g)K_\pi, \ g \in G.$$

In the case where $\pi = \pi_{l,\mathfrak{b}}$ and $\Delta_{G|B} \equiv 1$, we have that K_π is nothing else but the operator of multiplication by the function $\tilde{\xi}$, where $\tilde{\xi}$ is defined by: $\tilde{\xi}(x) = \xi(x \cdot l), \ x \in G$ (see [8]). Then, for every $\phi \in C_c^\infty(G)$, for almost all $\Omega_\psi \in \mathfrak{g}_\psi^*/G$, the operator $K_\pi^{\frac{-1}{2}} \pi_{\Omega_\psi}(\phi) K_\pi^{\frac{-1}{2}}$ is trace class and

$$\mathrm{tr}(K_\pi^{\frac{-1}{2}} \pi_{\Omega_\psi}(\phi) K_\pi^{\frac{-1}{2}}) = \int_{\Omega_\psi} (\Gamma \cdot (\phi \circ \exp))^\wedge(l)|\xi(l)|^{-1} d\beta_{\Omega_\psi}(l),$$

where Γ is a positive $\mathrm{Ad}(G)-$invariant function on $\mathfrak{g}$, which does not depend on ξ.

By [7] and [8] the localized Plancherel formula reads:

$$\|\phi\|_2^2 = \int_{\mathfrak{g}^*_\psi/G} \operatorname{tr}(K_\pi^{\frac{-1}{2}} \pi_{\Omega_\psi}(\phi^* * \phi) K_\pi^{\frac{-1}{2}}) d\mu_{\xi,\psi}(\Omega_\psi). \tag{2.5}$$

On the other hand, we obtain a decomposition of the Plancherel measure: for a measurable function F on $\widehat{G}$, we have:

$$\int_{\mathfrak{g}^*/G} F(\pi_\Omega) d\mu(\Omega) = \int_{\mathfrak{a}^*} \int_{\mathfrak{g}^*_\psi/G} F(\pi_{\Omega_\psi}) d\mu_{\xi,\psi}(\Omega_\psi) d\psi.$$

Let $\mathfrak{a}_0$ be the kernel of ψ and $A_0 = \exp \mathfrak{a}_0$. Let $P : G \longrightarrow G/A_0$, be the canonical projection and $\bar{\chi}_{\bar{\psi}}$ the character of A/A_0 associated to $\bar{\psi} \in (\mathfrak{a}/\mathfrak{a}_0)^*$, defined by the formula: $\bar{\chi}_{\bar{\psi}} \circ P_{|A} = \chi_\psi$. So we have for $1 \leq p < +\infty$, that:

$$L^p(G/A, \psi) = L^p((G/A_0)/(A/A_0), \overline{\psi}) \tag{2.6}$$

and $A' = A/A_0$ is a central subgroup of $G' = \exp \mathfrak{g}' = G/A_0$. We remark that $L^q(\widehat{G}_{\chi_\psi})$ is isometrically isomorphic to $L^q(\widehat{G'}_{\chi_{\bar{\psi}}})$ by means of the localized Plancherel formula (2.5), where q is the conjugate of p.

2.4.2 Suppose now that G is not unimodular. Let $G_0 = \exp \mathfrak{g}_0$ be the kernel of the modular function Δ_G. Then $\mathfrak{g}_0$ is a codimension one ideal in $\mathfrak{g}$ and there exists a dense open subset $\mathfrak{g}^*_{\text{gen}}$ of $\mathfrak{g}^*$ such that the coadjoint orbit of an element in $\mathfrak{g}^*_{\text{gen}}$ is of maximal dimension and saturated with respect to $\mathfrak{g}_0$. We call the elements of $\mathfrak{g}^*_{\text{gen}}$ generic, or in general position. Let $X \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}X$. Let μ_0 be the Plancherel measure of $\widehat{G}_0$. The group $G/G_0 \simeq \exp \mathbb{R}X$ acts by conjugation on $\widehat{G}_0$. We need to compute the Plancherel measure μ_0 of $\widehat{G}_0$ from μ and the action of G/G_0 on $\widehat{G}_0$.
Let $p^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}_0^*$ be the canonical projection and write l_0 for the restriction to $\mathfrak{g}_0$ of an element $l \in \mathfrak{g}^*$. If l is generic, then $\mathfrak{g}(l)$ is an ideal of codimension one in $\mathfrak{g}_0(l_0)$ and for its orbit Ω_l we have that:

$$p^*(\Omega_l) = \bigcup_{t \in \mathbb{R}} \mathrm{Ad}^*(\exp tX) \Omega^0_{l_0},$$

where $\Omega^0_{l_0}$ is the G_0-orbit of l_0.
For $t \in \mathbb{R}$ and $(\pi_0, \mathcal{H}_{\pi_0}) \in \widehat{G}_0$, we define the representation $\exp(tX) \cdot \pi_0$ on $\mathcal{H}_{\pi_0}$ by:

$$\exp(tX) \cdot \pi_0(g_0) = \pi_0(\exp(-tX) g_0 \exp(tX)) =: \pi_0^t(g_0), \quad g_0 \in G_0.$$

For generic $l \in \mathfrak{g}^*$, the stabilizer $\mathfrak{g}(l)$ is nilpotent. Hence for these forms l, we have that:

$$\Delta_G(s) = \Delta_{G(l)}(s) = 1, \text{ for all } s \in G(l) \tag{2.7}$$

(see [4], chap. II). This shows that $G(l) \subset G_0$ and so the co-adjoint orbits of all the l's are saturated with respect to $\mathfrak{g}_0$. In particular every Pukanszky polarisation $\mathfrak{p}$ for l_0 is also a Pukanszky polarization for $l \in \mathfrak{g}^*_{\text{gen}}$.

Let again $l \in \mathfrak{g}^*_{\rm gen}$ and let $a \in \mathfrak{g}_0^{\perp}$. There exists $t \in G_0(l_0) \subset G_0$ such that $l+a = \mathrm{Ad}^*(t)l$ and so by (2.7)

$$\xi(l+a) = \xi(\mathrm{Ad}^*(t)l) = \Delta_G(t^{-1})\xi(l) = \xi(l). \tag{2.8}$$

We can thus define a function $\check{\xi}$ on the generic elements in $\mathfrak{g}_0^*$ by letting:

$$\check{\xi}(l_0) := \xi(l),\ l \in \mathfrak{g}^*. \tag{2.9}$$

This function $\check{\xi}$ is rational (since such is ξ) and by (2.3)

$$\check{\xi}(x \cdot l_0) = \Delta_G(x^{-1})\check{\xi}(l_0), \tag{2.10}$$

for all $x \in G$ and $l_0 \in \mathfrak{g}_0^*$. In particular, the function $\check{\xi}$ is G_0-invariant. Define a positive Borel function ξ_0 on $\widehat{G}_0$ by:

$$\xi_0(\pi_{l_0}) := \check{\xi}(l_0)$$

and the Borel subset U of $\widehat{G}_0$ by:

$$U := \{\pi_0 \in \widehat{G}_0, \quad \xi_0(\pi_0) = 1\}.$$

Then we have for $\pi_0 \in U$ that:

$$\xi_0(\exp(tX)\cdot\pi_0) = \check{\xi}(\exp(tX)\cdot l_0) = \Delta_G^{-1}(\exp(tX)).$$

By Duflo and Moore (see [7], Theorem 6), the G-invariant Borel subset U of $\widehat{G}_0$ has the property that $V = \mathrm{Ind}U = \{\pi = \mathrm{Ind}_{G_0}^G \pi_0,\ \pi_0 \in U\}$ is a measurable subset of $\widehat{G}$ with $\mu(\widehat{G} - V) = 0$ and for every measurable function ϕ on U we have that:

$$\int_{\widehat{G}_0} \phi(\pi_0)d\mu_0(\pi_0) = \int_V \int_{\Omega_{\pi_0}} \phi(\exp(tX)\cdot\pi_0)\xi_0(\exp(tX)\cdot\pi_0)dtd\mu(\pi) \tag{2.11}$$

where for $\pi_0 \in U$, $\Omega_{\pi_0} = \exp(\mathbb{R}X)\cdot\pi_0$.

2.4.3 We suppose now that G is unimodular and $\mathfrak{g}_0$ an ideal of codimension one of $\mathfrak{g}$ which contains the center $\mathfrak{z}$ of $\mathfrak{g}$. We can now assume that the function ξ in (2.3) is equal to 1 and we can omit it in the formulas. Let $X \in \mathfrak{g} \setminus \mathfrak{g}_0$. The closed subgroup $G_0 = \exp\mathfrak{g}_0$ is unimodular and normal in G. By Mackey's theory (see [15]), there exists a Borel subset $U \subset \widehat{G}_0$ such that $W = \mathrm{Ind}U = \{\mathrm{Ind}_{G_0}^G\pi_0\ :\ \pi_0 \in U\}$ is a Borel subset of $\widehat{G}$ and $\mu(\widehat{G} - W) = 0$. Moreover, we can suppose that U meets each orbit $\exp(\mathbb{R}X)\cdot\pi_0$, $\pi_0 \in U$, only in the point π_0 (see [18], Theorem 3.2).
We obtain in this way a decomposition of the localized Plancherel measure μ_χ^0 of $(\widehat{G_0})_\chi$: for a subspace $\mathfrak{a}$ of $\mathfrak{z}$, for every $\chi \in \widehat{A}$ and every Borel function ϕ on $(\widehat{G_0})_\chi$:

$$\int_{(\widehat{G_0})_\chi} \phi(\pi_0)d\mu_\chi^0(\pi_0) = \int_{\widehat{G}_\chi}\int_{\mathbb{R}} \phi(\exp(tX)\cdot\pi_0)dtd\mu_\chi(\pi). \tag{2.12}$$

2.5 Hausdorff-Young type's inequality for integral operators

This inequality is due to Fournier and Russo [11]. Let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on $\mathcal{H}$, and $\mathcal{X}$ be a σ-finite measure space. Let $L^2(\mathcal{X},\mathcal{H})$ be the Hilbert space of square integrable $\mathcal{H}$-valued functions on $\mathcal{X}$ and K an integral operator on $L^2(\mathcal{X},\mathcal{H})$ with an operator-valued kernel k defined as:

$$K\eta(x) = \int_{\mathcal{X}} k(x,y)\eta(y)dy,$$

for all $\eta \in L^2(\mathcal{X},\mathcal{H})$, and almost all $x \in \mathcal{X}$. Let $k^*(x,y) = \overline{k(y,x)},\ x,y \in \mathcal{X}$.
For p,q in $[1,+\infty[$, let us define:

$$\|k\|_{p,q} = \Big(\int_{\mathcal{X}}(\int_{\mathcal{X}} \|k(x,y)\|^p_{C_p}dx)^{\frac{q}{p}}dy\Big)^{\frac{1}{q}}.$$

If $1 < p \leq 2$, q is the conjugate of p and if $\|k\|_{p,q}$ and $\|k^*\|_{p,q}$ are finite, then the integral operator K belongs to $C_q(L^2(\mathcal{X},\mathcal{H}))$ and

$$\|K\|_{C_q} \leq \|k\|^{\frac{1}{2}}_{p,q}\|k^*\|^{\frac{1}{2}}_{p,q}. \tag{2.13}$$

3 Estimate of the L^p-Fourier transform norm

3.1 Induction by an ideal of codimension one

The proof of the following lemma can be adapted from the proof of Theorem 3 in [2] (page 517).

Lemma 3.1. *Let G be an exponential solvable unimodular Lie group. Let $A = \exp\mathfrak{a}$ be a subgroup of the center Z of G and let $\chi = \chi_\psi$, $\psi \in \mathfrak{a}^*$, be a unitary character of A. Let $\mathfrak{g}_0 = Lie(G_0)$ be an ideal of codimension one of $\mathfrak{g}$ such that:*
i) almost all G-orbits are saturated with respect to $\mathfrak{g}_0$
ii) for $1 < p \leq 2$,

$$\Big(\int_{(\widehat{G_0})_\chi} \|\pi_0(\varphi_0)\|^q_{C_q}d\mu^0_\chi(\pi_0)\Big)^{\frac{1}{q}} \leq A_p^{\frac{2\dim(G_0/A)-m^0_\psi}{2}}\|\varphi_0\|_{L^p(G_0/A,\chi)},$$

where $\varphi_0 \in C_c(G_0/A)$ and $m^0_\psi = \sup\Big\{\dim(G_0\cdot l_0)\ ;\ l_0 \in \mathfrak{g}^*_{0,\psi}\Big\}$. Then for $1 < p \leq 2$,

$$\Big(\int_{\widehat{G}_\chi} \|\pi(\varphi)\|^q_{C_q}d\mu_\chi(\pi)\Big)^{\frac{1}{q}} \leq A_p^{\frac{2\dim(G/A)-m_\psi}{2}}\|\varphi\|_{L^p(G/A,\chi)},$$

where $\varphi \in C_c(G/A)$ and $m_\psi = \sup\Big\{\dim(G\cdot l)\ ;\ l \in \mathfrak{g}^*_\psi\Big\}$.

3.2 The unimodular case

We suppose now that G is unimodular. In this case we can take $\xi \equiv 1$. Thus $K_\pi = \mathrm{Id}$ and the localized Plancherel formula reduces to

$$\|\phi\|_2^2 = \int_{\mathfrak{g}^*_\psi/G} \mathrm{tr}(\pi_{\Omega_\psi}(\phi^* * \phi))d\mu_\psi(\Omega_\psi).$$

Definition 1. Let G be an exponential solvable Lie group and $\mathfrak{n}$ the nil-radical of $\mathfrak{g}$ (which contains $[\mathfrak{g},\mathfrak{g}]$). For $l \in \mathfrak{g}^*$, let

$$\mathfrak{g}(l_{|\mathfrak{n}}) = \{X \in \mathfrak{g} \ : \ l([X,\mathfrak{n}]) = \{0\}\},$$

$$\mathfrak{m}(l) = \mathfrak{g}(l) + \mathfrak{n} \subset \mathfrak{d}(l) = \mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n}$$

and

$$\mathfrak{m}(l)^\infty = \bigcap_{k\geq 0} \mathcal{C}^k(\mathfrak{m}(l)), \quad \mathfrak{d}(l)^\infty = \bigcap_{k\geq 0} \mathcal{C}^k(\mathfrak{d}(l)),$$

where $\mathcal{C}^k(\mathfrak{d}(l)) = [\mathfrak{d}(l), \mathcal{C}^{k-1}(\mathfrak{d}(l))]$ is the k-th term in the descending central series of $\mathfrak{d}(l)$ and similarly for $\mathfrak{m}(l)^\infty$.

$*$-Regularity (see [5]): we say that an element $l \in \mathfrak{g}^*$ is $*$-regular if

$$< l, \mathfrak{m}(l)^\infty >= \{0\}.$$

Strong $*$-Regularity : we say that an element in $\mathfrak{g}^*$ is strong $*$-regular if

$$< l, \mathfrak{d}(l)^\infty >= \{0\}.$$

An exponential Lie group $G = \exp \mathfrak{g}$ is called strong $*$-regular, if there exists a Zariski open subset in $\mathfrak{g}^*$ such that every element in this subset is strong $*$-regular.

Remarks.

(a) Every nilpotent Lie group is strong $*$-regular. All connected simply connected exponential Lie groups G with $\dim G \leq 4$ are strong $*$-regular, except the group $G_{4,9}(0)$, known as Boidol's group (see [5]). This particular example will be studied later in Section 4.2.

(b) We can find groups $G = \exp \mathfrak{g}$ such that every generic element in $\mathfrak{g}^*$ is $*$-regular, but which are not strong $*$-regular. An example is given by the group G whose corresponding Lie algebra $\mathfrak{g}$ is given by the basis $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$ and the following brackets:

$$[Z_1, Z_3] = -Z_3,\ [Z_1, Z_4] = Z_4,\ [Z_3, Z_4] = Z_7,$$

$$[Z_2, Z_5] = -Z_5,\ [Z_2, Z_6] = Z_6,\ [Z_5, Z_6] = Z_7,\ [Z_1, Z_2] = Z_7.$$

We have: $\mathfrak{n} = \mathrm{span} < Z_3,\ Z_4,\ Z_5,\ Z_6,\ Z_7 >$. Take $l \in \mathfrak{g}^\star$ in general position, then in particular $l(Z_1) \neq 0,\ l(Z_2) \neq 0$ and $l(Z_7) \neq 0$. It follows that:

$\mathfrak{g}(l) =< Z_7 >$, so $(\mathfrak{g}(l) + \mathfrak{n})^\infty = \{0\}$, which means that l is $*$-regular.

But $\mathfrak{d}(l) = \mathfrak{g}$ so $\mathfrak{d}(l)^\infty = \mathfrak{n}$, which means that G is not strong $*$-regular.

(c) The set $\mathfrak{d}(l)^\infty$ is the smallest ideal in $\mathfrak{d}(l)$ such that $\mathfrak{d}(l)/\mathfrak{d}(l)^\infty$ is nilpotent.

(d) Let $\mathfrak{n}$ be the nil-radical of a Lie algebra $\mathfrak{g}$. Let $\mathfrak{b}$ be an ideal of $\mathfrak{g}$. The nil-radical $\mathfrak{m}$ of $\mathfrak{b}$ is characteristic, hence $\mathfrak{m}$ is also a (nilpotent) ideal of $\mathfrak{g}$ and so $\mathfrak{m} \subset \mathfrak{n}$. On the other hand $\mathfrak{n} \cap \mathfrak{b}$ is a nilpotent ideal of $\mathfrak{b}$, and so is contained in $\mathfrak{m}$. This shows that $\mathfrak{m} = \mathfrak{b} \cap \mathfrak{n}$.

As our main theorem is localized, we need a localized version of the strong $*$-regularity.

Definition 2. Let $A = \exp \mathfrak{a}$ be a closed connected subgroup of the center Z of G and $\psi \in \mathfrak{a}^*$. We define:

Strong ψ-$*$-Regularity : we say that G is strong ψ-$*$-regular if there exists a Zariski open subset in $\mathfrak{g}^*_\psi$ such that:

$$< l, \mathfrak{d}(l)^\infty >= \{0\}, \text{ for every } l \text{ in this subset.}$$

We are going to prove the following:

Proposition 3.1. *Let G be a strong ψ-$*$-regular exponential solvable unimodular Lie group. Then, for $1 < p \leq 2$,*

$$\Big(\int_{\mathfrak{g}^*_\psi/G} \|\pi_{\Omega_\psi}(\varphi)\|^q_{C_q} d\mu_\psi(\Omega_\psi). \Big)^{\frac{1}{q}} \leq A_p^{\frac{2\dim G - m_\psi}{2}} \|\varphi\|_{L^p(G/A,\chi_\psi)},$$

where $\varphi \in C_c(G/A)$ and $m_\psi = \sup\Big\{ \ \dim(G \cdot l) \ ; \ l \in \mathfrak{g}^*_\psi \Big\}$.

Proof. We proceed by induction on $\dim G + \dim(G/A) =: \delta(G, A)$. If $\delta(G, A) = 1$, then $G = A = \mathbb{R}$ and we have the result by W. Beckner [3], as we are in the abelian situation. If $\delta(G, A) > 1$, let $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$, $\mathfrak{a}_0$ the kernel of ψ and $A_0 = \exp \mathfrak{a}_0$. By [17] page 34, we have to treat the following cases:

Case 1. $\dim \mathfrak{z} = 0$.

Subcase 1.1. : there exists a non-zero $Y \in \mathfrak{g}$ and a non-zero real linear form α on $\mathfrak{g}$ such that:

$$[X, Y] = \alpha(X)Y, \ \forall X \in \mathfrak{g}.$$

Subcase 1.2. : there exists a non-zero $Y_1, \ Y_2 \in \mathfrak{g}, \ \theta \in \mathbb{R} \setminus \{0\}$ and a non-zero real linear form α on $\mathfrak{g}$ such that:

$$[X, Y_1] = \alpha(X)(Y_1 - \theta Y_2),$$

$$[X, Y_2] = \alpha(X)(\theta Y_1 + Y_2), \ \forall X \in \mathfrak{g}.$$

Case 2. $\dim \mathfrak{z} > 0$.

Subcase 2.1. : $\dim \mathfrak{a}_0 \geq 1$.

Subcase 2.2. : $\dim \mathfrak{a}_0 = 0$ and $\dim \mathfrak{z} \geq 2$.

Without loss of generality, we can assume that $\psi_{|\mathfrak{z}} \neq 0$.

Subcase 2.3. : $\dim \mathfrak{a}_0 \geq 0$ and $\dim \mathfrak{z} = 1$, i.e., $\mathfrak{z} = \mathbb{R}Z$.

We can suppose that: $\psi(Z) = 1$.

Subcase 2.3.1. : there exists a non-zero $Y \in \mathfrak{g}$ and two real linear forms, α, β, on $\mathfrak{g}$ such that:

$$[X, Y] = \alpha(X)Y + \beta(X)Z, \ \forall X \in \mathfrak{g}.$$

Subcase 2.3.2. : there exists non-zero $Y_1, \ Y_2 \in \mathfrak{g}, \ \theta \in \mathbb{R} \setminus \{0\}$ and three non-zero real linearly independent linear forms, α, β_1, β_2, on $\mathfrak{g}$ such that:

$$[X, Y_1] = \alpha(X)(Y_1 - \theta Y_2) + \beta_1(X)Z,$$

$$[X, Y_2] = \alpha(X)(\theta Y_1 + Y_2) + \beta_2(X)Z, \ \forall X \in \mathfrak{g}.$$

We can now begin the proof.

Case 1. $\dim \mathfrak{z} = 0$.

Subcase 1.1. There exists a non zero vector $Y \in \mathfrak{g}$ such that for all $X \in \mathfrak{g}$,

$$[X, Y] = \alpha(X)Y,$$

where α is a non-zero real linear form on $\mathfrak{g}$. Let

$$\mathfrak{g}_0 = \ker \alpha = \ \{X \in \mathfrak{g} \ : \ [X, Y] = 0\}.$$

Then $\mathfrak{g}_0$ is an ideal of codimension one in $\mathfrak{g}$, which is strong ψ-$*$-regular. In fact, $\mathfrak{n}$ is contained in $\mathfrak{g}_0$, since for every element U of $\mathfrak{g}$, the number $\alpha(U)$ is in the spectrum of $\mathrm{ad}(U)$. Let $l_0 \in (\mathfrak{g}_0^*)_\psi$ and l be an extension of l_0 to $\mathfrak{g}^*$. So we have:

$$< l_0, (\mathfrak{g}_0(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty >=< l, (\mathfrak{g}_0(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty > \subset < l, \mathfrak{d}(l)^\infty > .$$

It follows that $< l_0, (\mathfrak{g}_0(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty >= \{0\}$ for almost all $l_0 \in \mathfrak{g}_0^*$ as $< l, \mathfrak{d}(l)^\infty >$ does. On the other hand, if $l(Y) \neq 0$ then $\mathrm{Ad}^*(\exp \mathbb{R}Y)(l) = l + \mathfrak{g}_0^\perp$. This implies that almost every G-orbit is saturated with respect to $\mathfrak{g}_0$. Whence, we have the result in this case, by Lemma 3.1.

Subcase 1.2. There exists non-zero $Y_1, Y_2 \in \mathfrak{g}, \ \theta \in \mathbb{R} \setminus \{0\}$ and a non-zero real linear form α on $\mathfrak{g}$ such that for all $X \in \mathfrak{g}$,

$$[X, Y_1] = \alpha(X)(Y_1 - \theta Y_2),$$

$$[X, Y_2] = \alpha(X)(\theta Y_1 + Y_2).$$

So, $[X, Y_1 + iY_2] = \alpha(X)(1 + i\theta)(Y_1 + iY_2) = \alpha'(X)(Y_1 + iY_2)$ with $\alpha' = (1 + i\theta)\alpha$ a non-zero complex valued linear form.

Let $\mathfrak{g}_0 = \ker \alpha = \ker \alpha'$ and $G_0 = \exp \mathfrak{g}_0$. Then $\mathfrak{g}_0$ is an ideal of codimension one of $\mathfrak{g}$ and so G_0 is a unimodular normal subgroup of G. Moreover $\mathfrak{g}_0$ satisfies the strong ψ-$*$-regularity since, as in the case 1.1, $\mathfrak{n} \subset \mathfrak{g}_0$. To settle this case, we have to prove the saturation of generic orbits with respect to $\mathfrak{g}_0$. Let $\mathfrak{m} = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$. If $l_{|\mathfrak{m}} \neq 0$, then

$$\mathrm{Ad}^*(\exp(\mathfrak{m}))l = l + \mathfrak{g}_0^{\perp}.$$

In fact, let $X \in \mathfrak{g}$ such that: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}X$ and $uY_1 + vY_2 \in \mathfrak{m}$. Thus we have:

$$[X, uY_1 + vY_2] = u[X, Y_1] + v[X, Y_2] = \alpha(X)((u + v\theta)Y_1 + (v - u\theta)Y_2).$$

We can choose $u, v \in \mathbb{R}$ such that $[X, uY_1 + vY_2] \neq 0$. Hence

$$\begin{aligned} \mathrm{Ad}^*(\exp(uY_1 + vY_2))(l)(X) &= l(\mathrm{e}^{\mathrm{ad}(-uY_1 - vY_2)}X) \\ &= l(X + [-uY_1 - vY_2, X]) \\ &= l(X) + \alpha(X)l((u + v\theta)Y_1 + (v - u\theta)Y_2). \end{aligned}$$

As $l_{|\mathfrak{m}} \neq 0$, it follows that $\mathrm{Ad}^*(\exp(\mathbb{R}Y_1 + \mathbb{R}Y_2))l = l + \mathfrak{g}_0^{\perp}$. Finally, we have the result using Lemma 3.1.

Case 2. suppose that $\dim \mathfrak{z} \geq 1$.
Let as before $\mathfrak{a}_0$ be the kernel of ψ and $A_0 = \exp \mathfrak{a}_0$. The following two subcases can be treated as in [2]*:*

Subcase 2.1. $\dim \mathfrak{a}_0 \geq 1$

and

Subcase 2.2. We suppose that $\dim \mathfrak{a}_0 = 0$ and $\dim \mathfrak{z} \geq 2$.

The next subcase is

Subcase 2.3. We suppose that $\dim \mathfrak{a}_0 = 0$ and $\dim \mathfrak{z} = 1$. Let $\mathfrak{z} = \mathbb{R}Z$. We can assume that $\psi(Z) = 1$.

Subcase 2.3.1. there exists a non zero $Y \in \mathfrak{g}$ and two real linear forms, α, β, on $\mathfrak{g}$ such that for all $X \in \mathfrak{g}$,

$$[X, Y] = \alpha(X)Y + \beta(X)Z.$$

We have the following subcases:

A. α and β are linearly independent.

Let

$$\mathfrak{g}_\alpha = \ker \alpha = \{X \in \mathfrak{g} \ : \ [X, Y] = \beta(X)Z\}.$$

The subspace $\mathfrak{g}_\alpha$ is an ideal of codimension one of $\mathfrak{g}$ and the associated subgroup $G_\alpha = \exp \mathfrak{g}_\alpha$ is a unimodular normal subgroup of G. It also satisfies the strong ψ-$*$-regularity. Indeed, it is clear that $\mathfrak{n} \subset \mathfrak{g}_\alpha$. So, as $\mathfrak{g}_\alpha(l_{|\mathfrak{n}}) \subset \mathfrak{g}(l_{|\mathfrak{n}})$, we have that $(\mathfrak{g}_\alpha(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty \subset \mathfrak{d}(l)^\infty$. Hence

$$< l_{|\mathfrak{g}_\alpha}, (\mathfrak{g}_\alpha(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty >= \{0\}.$$

On the other hand, the generic $G-$orbits are saturated with respect to $\mathfrak{g}_\alpha$, i.e., $\mathfrak{g}(l) \subset \mathfrak{g}_\alpha$. In fact, let $T \in \mathfrak{g}(l)$. Put

$$\mathfrak{g}_\beta = \ker \beta = \{X \in \mathfrak{g} \ : \ [X, Y] = \ \alpha(X)Y\}.$$

As α and β are linearly independent, we have the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = (\mathfrak{g}_\alpha \cap \mathfrak{g}_\beta) \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \quad \text{with} \quad X_1 \in \mathfrak{g}_\alpha \setminus \mathfrak{g}_\beta \text{ and } X_2 \in \mathfrak{g}_\beta \setminus \mathfrak{g}_\alpha.$$

As $[X_1, Y] = \beta(X_1)Z$ and $[X_2, Y] = \alpha(X_2)Y$, we can suppose $\alpha(X_2) = \beta(X_1) = 1$. If $T \in \mathfrak{g}$, there exists $\gamma,\ \delta \in \mathbb{R}$ and $U_0 \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\beta$ such that $T = \gamma X_1 + \delta X_2 + U_0$. We have $[T, Y] = \gamma Z + \delta Y$ and we have to show that $\delta = 0$. Otherwise, we can show that $Z \in \mathfrak{d}(l)^\infty$. We have first that $Z \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} + \mathfrak{g}(l_{|\mathfrak{n}}) = \mathcal{C}^0(\mathfrak{d}(l))$. As $T \in \mathfrak{g}(l) \subset \mathfrak{g}(l_{|\mathfrak{n}})$, we have that $[T, Y] \in \mathcal{C}^1(\mathfrak{d}(l))$. This implies that $[X_1, [T, Y]] = \delta Z \in \mathcal{C}^1(\mathfrak{d}(l))$ and then $Z \in \mathcal{C}^1(\mathfrak{d}(l))$. Since $[T, Y] = \gamma Z + \delta Y \in \mathcal{C}^1(\mathfrak{d}(l))$ and $Z \in \mathcal{C}^1(\mathfrak{d}(l))$, we obtain $Y \in \mathcal{C}^1(\mathfrak{d}(l))$. Let k be a positive integer. As $\mathrm{ad}^k(T)(Y) \in \mathcal{C}^k(\mathfrak{d}(l))$, we have that

$$[X_1, \mathrm{ad}^k(T)(Y)] = \delta^{k+1} Z \in \mathcal{C}^k(\mathfrak{d}(l)).$$

Hence $Z \in \mathcal{C}^k(\mathfrak{d}(l))$ for all $k \geqslant 0$, and so $Z \in \mathfrak{d}(l)^\infty$. The strong $*$-regularity implies that $l(Z) = 0$, which contradicts the fact that l is generic. Hence $\delta = 0$, T is in $\mathfrak{g}_\beta$ and $G \cdot l$ is saturated with respect to $\mathfrak{g}_\alpha$. Lemma 3.1 allows us to settle this case.

B. We suppose that $\alpha = 0$ and $\beta \neq 0$.
For all $X \in \mathfrak{g}$, we can write that

$$[X, Y] = \beta(X)Z.$$

The subspace $\mathfrak{g}_\beta = \{X \in \mathfrak{g} \ : \ [X, Y] = 0\} = C(Y)$ is an ideal of codimension one in $\mathfrak{g}$, where $C(Y)$ denotes the centralizer of Y in $\mathfrak{g}$ and $G_\beta = \exp \mathfrak{g}_\beta$ is a unimodular normal subgroup of G. Let $\mathfrak{n}_\beta$ be the nil-radical of $\mathfrak{g}_\beta$ and, by Remark(d) in 3.2, we have that $\mathfrak{n}_\beta = \mathfrak{n} \cap \mathfrak{g}_\beta$.

B1. Suppose that $\mathfrak{n} \subset \mathfrak{g}_\beta$, which means that $\mathfrak{n}_\beta = \mathfrak{n} \cap \mathfrak{g}_\beta = \mathfrak{n}$. Take $\tilde{l} \in \mathfrak{g}_\beta^*$ and l any extension of $\tilde{l}$ to $\mathfrak{g}^*$, then

$$\mathfrak{g}_\beta(\tilde{l}_{|\mathfrak{n}_\beta}) + \mathfrak{n}_\beta \subset \mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n},$$

so

$$\left(\mathfrak{g}_\beta(\tilde{l}_{|\mathfrak{n}_\beta}) + \mathfrak{n}_\beta\right)^\infty \subset \mathfrak{d}(l)^\infty.$$

This proves that $\mathfrak{g}_\beta$ has the strong ψ-$*$-regularity and it is not hard to see that the generic orbits are saturated with respect to $\mathfrak{g}_\beta$, since $\mathrm{Ad}^*(\exp \mathbb{R}Y)(l) = l + \mathfrak{g}_\beta^\perp$. We can apply again the Lemma 3.1.

B2. Suppose that $\mathfrak{n} \not\subset \mathfrak{g}_\beta$. We have that $\mathfrak{n}_\beta = \mathfrak{n} \cap \mathfrak{g}_\beta$. Let $\tilde{l} \in \mathfrak{g}_\beta^*$ and l any extension of $\tilde{l}$ to $\mathfrak{g}^*$. So we have

$$\mathfrak{g}_\beta(\tilde{l}_{|\mathfrak{n}_\beta}) \subset \mathfrak{g}(l_{|\mathfrak{n}}) + \mathbb{R}Y.$$

In fact, since $\beta \neq 0$, there is an element X of $\mathfrak{n}$ such that $[Y, X] = Z$. Let U be an element of $\mathfrak{g}_\beta(l_{|\mathfrak{n}_\beta})$. If $l([U, X]) = 0$, then $U \in \mathfrak{g}(l_{|\mathfrak{n}})$. Otherwise, let $l([U, X]) = \delta \neq 0$. For $\lambda \in \mathbb{R}$,

$$l([U + \lambda Y, X]) = l([U, X]) + \lambda l([Y, X]) = \delta + \lambda l(Z).$$

So we can choose λ such that $l([U + \lambda Y, X]) = 0$. Since Y is an element of $\mathfrak{g}_\beta$, it follows that $U + \lambda Y \in \mathfrak{g}(l_{|\mathfrak{n}}) + \mathbb{R}Y$, hence

$$U \in \mathfrak{g}(l_{|\mathfrak{n}}) + \mathbb{R}Y.$$

This implies that, for $U, U' \in \mathfrak{g}_\beta(\tilde{l}_{|\mathfrak{n}_\beta})$ and $N, N' \in \mathfrak{n}_\beta$, we can choose λ and λ' in $\mathbb{R}$ such that:

$$[U + N, U' + N'] = [U + \lambda Y + N, U' + \lambda' Y + N'] \in [\mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n}, \mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n}] = C^1(\mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n}).$$

It follows immediately that:

$$\big(\mathfrak{g}_\beta(l_{|\mathfrak{n}_\beta}) + \mathfrak{n}_\beta\big)^\infty \subset \mathfrak{d}(l)^\infty.$$

So the strong ψ-$*$-regularity of $\mathfrak{g}_\beta$ is satisfied. As the saturation of the generic orbits is clear, Lemma 3.1 gives us the result.

C. α, β are linearly dependent and $\alpha \neq 0$.

There exists $\delta \in \mathbb{R}$ such that $\beta = \delta\alpha$. Then, for all $X \in \mathfrak{g}$, $[X, Y] = \alpha(X)(Y + \delta Z)$, whence $[X, Y + \delta Z] = \alpha(X)(Y + \delta Z)$. Let

$$\mathfrak{g}_0 = \{A \in \mathfrak{g} \ : \ [A, Y + \delta Z] = 0\} = \ker \alpha.$$

And we are in the same situation as in subcase (1.1).

Subcase 2.3.2. There exists non zero $Y_1,\ Y_2 \in \mathfrak{g}$, $\theta \in \mathbb{R} \setminus \{0\}$ and three non-zero real linearly independent linear forms α, β_1, β_2, on $\mathfrak{g}$ such that: for all $X \in \mathfrak{g}$,

$$[X, Y_1] = \alpha(X)(Y_1 - \theta Y_2) + \beta_1(X)Z,$$

$$[X, Y_2] = \alpha(X)(\theta Y_1 + Y_2) + \beta_2(X)Z.$$

Let $\beta = \beta_1 + i\beta_2$ and $\alpha' = (1 + i\theta)\alpha$. Then we have:

$$[X, Y_1 + iY_2] = \alpha'(X)(Y_1 + iY_2) + \beta(X)Z.$$

On the other hand, we can easily construct $X_1, X_2, S \in \mathfrak{g}$ such that: $\alpha(S) = 1$, $\beta(S) = 0$, i.e.

$$[S, Y_1] = Y_1 - \theta Y_2,$$

$$[S, Y_2] = \theta Y_1 + Y_2.$$

Hence, $[S, Y_1 + iY_2] = Y_1 + iY_2$ and

$$\alpha(X_1) = \alpha(X_2) = 0,\ \beta(X_1) = \beta(X_2) = 1,$$

$$[X_j, Y_k] = \delta_{jk} Z, \quad 1 \leq j, k \leq 2.$$

Let

$$\mathfrak{g}_\alpha = \text{ Ker } \alpha = \{X \in \mathfrak{g} \ : \ [X, Y_1 + iY_2] = \beta(X) Z\}$$

and $G_\alpha = \exp \mathfrak{g}_\alpha$. The subspace $\mathfrak{g}_\alpha$ is an ideal of codimension one in $\mathfrak{g}$ and G_α is a unimodular normal subgroup of G. As $\mathfrak{n} \subset \mathfrak{g}_\alpha$, the strong ψ-$*$-regularity is satisfied. Moreover, the generic orbits are saturated with respect to $\mathfrak{g}_\alpha$. Indeed, let $T \in \mathfrak{g}(l)$, we must show that $T \in \mathfrak{g}_\alpha$. Since $l(Z) \neq 0$, we can assume that $l(Y_1) = l(Y_2) = 0$. Let

$$\mathfrak{g}_\beta = \text{ Ker } \beta = \{X \in \mathfrak{g} \ : \ [X, Y_1 + iY_2] = \alpha'(X)(Y_1 + iY_2)\}.$$

It follows that

$$\mathfrak{g} = (\mathfrak{g}_\alpha \cap \mathfrak{g}_\beta) \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}S.$$

Then there exists $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ and $U_0 \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\beta$ such that $T = \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 S + U_0$. As $T \in \mathfrak{g}(l)$, we have:

$$l([T, Y_1]) = l([\gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 S + U_0, Y_1]) = \gamma_1 l(Z) + \gamma_3 l(Y_1 - \theta Y_2) = 0$$

and

$$l([T, Y_2]) = l([\gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 S + U_0, Y_2]) = \gamma_2 l(Z) + \gamma_3 l(\theta Y_1 + Y_2) = 0.$$

Recall that $l(Y_1) = l(Y_2) = 0$ and $l(Z) \neq 0$, so $\gamma_1 = \gamma_2 = 0$. Hence, $T = U_0 + \gamma_3 S$. We claim now that $\gamma_3 = 0$. Otherwise, we show that $Z \in \mathfrak{d}(l)^\infty$ which is a contradiction. We have first that $Y_1, Y_2 \in [\mathfrak{g}, \mathfrak{g}] \subset \mathcal{C}^0(\mathfrak{d}(l))$, since $[S, Y_1] = Y_1 - \theta Y_2$ and $[S, Y_2] = \theta Y_1 + Y_2$. Now, suppose that $Y_1, Y_2 \in \mathcal{C}^k(\mathfrak{d}(l))$. Thus,

$$[T, Y_1] = [U_0 + \gamma_3 S, Y_1] = \gamma_3 (Y_1 - \theta Y_2) \in \mathcal{C}^{k+1}(\mathfrak{d}(l))$$

and

$$[T, Y_2] = [U_0 + \gamma_3 S, Y_2] = \gamma_3 (\theta Y_1 + Y_2) \in \mathcal{C}^{k+1}(\mathfrak{d}(l)),$$

This implies that $Y_1, Y_2 \in \mathcal{C}^{k+1}(\mathfrak{d}(l))$. We obtain inductively that:

$$Y_1, Y_2 \in \mathcal{C}^k(\mathfrak{d}(l)), \quad \text{for all} \quad k \geq 0.$$

Hence, one deduces that:

$$[X_1, Y_1] = Z \in \mathcal{C}^k(\mathfrak{d}(l)), \quad \text{for all} \quad k \geq 0$$

and that $Z \in \mathfrak{d}(l)^\infty$. Finally, by Lemma 3.1, we have the result.

□

Taking $A = \{e\}$ in proposition 3.1, we obtain the following corollary:

Corollary 3.1. *Let G be a strong $*$-regular exponential solvable unimodular Lie group. Then, for $1 < p \leq 2$,*

$$\Big(\int_{\widehat{G}} \|\pi(\varphi)\|^q_{C_q} d\mu(\pi) \Big)^{\frac{1}{q}} \leq A_p^{\frac{2 \dim G - m}{2}} \|\varphi\|_p,$$

where $\varphi \in C_c(G)$ and m is the maximal dimension of coadjoint orbits.

3.3 The main theorem

Theorem 3.1. *Let G be an exponential solvable Lie group satisfying the strong $*$-regularity condition. Let m be the maximal dimension of coadjoint orbits. Let $1 < p \leq 2$ and q the conjugate of p. Then, for all $\varphi \in L^1(G) \cap L^p(G)$ and μ-almost all $\pi \in \widehat{G}$, the operator $\pi^p(\varphi) := \pi(\varphi) K_\pi^{\frac{-1}{q}}$ is bounded, its extension is of class C_q and satisfies the following inequality:*

$$\Big(\int_{\widehat{G}} \| \pi^p(\varphi) \|_{C_q}^q d\mu(\pi) \Big)^{\frac{1}{q}} \leq A_p^{\frac{2 \dim G - m}{2}} \| \varphi \|_p.$$

Proof. Let G be a strong $*$-regular exponential solvable non-unimodular Lie group, with Lie algebra $\mathfrak{g}$. Let G_0 be the kernel of the modular function. Then G_0 is a closed unimodular subgroup of G. As generically, $\mathfrak{g}(l)$ is nilpotent, we can assume that for all $\pi \in V$, we can write $\pi = \mathrm{ind}_{G_0}^G \pi_0$ for some $\pi_0 \in U$ (see section 2.4 for notations).
We are going to prove an elementary lemma which will be of interest in the sequel of the proof.

Lemma 3.2. *Let $\pi \in V$ and $\pi = \mathrm{Ind}_{G_0}^G \pi_0$. Then for $\varphi \in L^1(G) \cap L^p(G)$, the integral operator $\pi(\varphi) K_\pi^{\frac{-1}{q}}$ is realized on $L^2(\mathbb{R}, \mathcal{H}_{\pi_0})$ with operator valued kernel $k_{\pi(\varphi)}$ defined by:*

$$\pi(\varphi) K_\pi^{\frac{-1}{q}} \eta(s) = \int_{\mathbb{R}} k_{\pi(\varphi)}(s,t) \eta(t) dt,$$

where $k_{\pi(\varphi)}(s,t) = \pi_0^t(\varphi^{s-t}) \Delta_G^{\frac{1}{q}}(\exp tX)$ and $\varphi^{s-t}(g_0) = \varphi(\exp(s-t)X g_0)$.

Proof. We use the notations of section 2.4.2. Let $g = \exp(tX) g_0$, $\eta \in L^2(\mathbb{R}, \mathcal{H}_{\pi_0})$, $t \in \mathbb{R}$ and $g_0 \in G_0$. We have that:

$$\begin{aligned}
\pi(\exp(tX) g_0) K_\pi^{\frac{-1}{q}} \eta(s) &= K_\pi^{\frac{-1}{q}} \eta(g_0^{-1} \exp(s-t)X) \\
&= \Delta_G^{\frac{1}{q}}(\exp(s-t)X) \eta(g_0^{-1} \exp(s-t)X) \\
&= \Delta_G^{\frac{1}{q}}(\exp(s-t)X) \eta(\exp(s-t)X \exp(t-s)X g_0^{-1} \exp(s-t)X) \\
&= \Delta_G^{\frac{1}{q}}(\exp(s-t)X) \pi_0(\exp((t-s)X) g_0 \exp((s-t)X)) \eta(s-t) \\
&= \Delta_G^{\frac{1}{q}}(\exp(s-t)X) \pi_0^{s-t}(g_0) \eta(s-t).
\end{aligned}$$

On the other hand, we get:

$$\pi(\varphi) K_\pi^{\frac{-1}{q}} \eta(s) = \int_G \varphi(g) \pi(g) K_\pi^{\frac{-1}{q}} \eta(s) dg$$

$$= \int_{\mathbb{R}} \int_{G_0} \varphi(\exp(tX) g_0) \pi(\exp(tX) g_0) K_\pi^{\frac{-1}{q}} \eta(s) dg_0 dt$$

(using the computation above)

$$= \int_{\mathbb{R}} \int_{G_0} \varphi(\exp(tX) g_0) \pi_0^{s-t}(g_0) \eta(s-t) \Delta_G^{\frac{1}{q}}(\exp(s-t)X) dg_0 dt$$

$$= \int_{\mathbb{R}} \int_{G_0} \varphi(\exp(s-t)Xg_0)\pi_0^t(g_0)\eta(t)\Delta_G^{\frac{1}{q}}(\exp tX)dg_0dt$$

$$= \int_{\mathbb{R}} \pi_0^t(\varphi^{s-t})\Delta_G^{\frac{1}{q}}(\exp tX)\eta(t)dt.$$

□

We shall use the inequality (2.13) to get an estimate of the norm $\|\pi(\varphi)K_\pi^{\frac{-1}{q}}\|_{C_q}$. We start by computing the norm $\|k_{\pi(\varphi)}\|_{p,q}$:

$$\|k_{\pi(\varphi)}\|_{p,q} = \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\|k_{\pi(\varphi)}(s,t)\|_{C_q}^p ds\Big)^{\frac{q}{p}}dt\Big)^{\frac{1}{q}}$$

$$= \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\|\pi_0^t(\varphi^{s-t})\Delta_G^{\frac{1}{q}}(\exp tX)\|_{C_q}^p ds\Big)^{\frac{q}{p}}dt\Big)^{\frac{1}{q}}$$

$$= \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\|\pi_0^t(\varphi^{s})\|_{C_q}^p ds\Big)^{\frac{q}{p}}\Delta_G(\exp tX)dt\Big)^{\frac{1}{q}}.$$

It follows that:

$$\int_{\widehat{G}}\|k_{\pi(\varphi)}\|_{p,q}^q\, d\mu(\pi) = \int_{\widehat{G}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\|\pi_0^t(\varphi^s)\|_{C_q}^p ds\Big)^{\frac{q}{p}}\Delta_G(\exp tX)dtd\mu(\pi)$$

(we use the generalized Minkowski inequality for the measures $\Delta_G(\exp tX)dtd\mu$ and ds)

$$\leq \Big(\int_{\mathbb{R}}\Big(\int_{\widehat{G}}\int_{\mathbb{R}}\|\pi_0^t(\varphi^s)\|_{C_q}^q\Delta_G(\exp tX)dtd\mu(\pi)\Big)^{\frac{p}{q}}ds\Big)^{\frac{q}{p}}$$

$$\leq \Big(\int_{\mathbb{R}}\Big(\int_{\widehat{G}}\int_{\mathbb{R}}\|\pi_0^t(\varphi^s)\|_{C_q}^q\xi_0(\exp tX\cdot\pi_0)dtd\mu(\pi)\Big)^{\frac{p}{q}}ds\Big)^{\frac{q}{p}}$$

$$= \Big(\int_{\mathbb{R}}\Big(\int_{\widehat{G}_0}\|\pi_0(\varphi^s)\|_{C_q}^q d\mu_0(\pi_0)\Big)^{\frac{p}{q}}ds\Big)^{\frac{q}{p}}$$

(by the measure disintegration formula (2.11))

$$\leq \Big(\int_{\mathbb{R}} A_p^{p\frac{2\dim G_0-(m-2)}{2}}\|\varphi^s\|_p^p ds\Big)^{\frac{q}{p}}$$

(by using Proposition 3.1 for G_0).

Hence,

$$\int_{\widehat{G}}\|k_{\pi(\varphi)}\|_{p,q}^q\, d\mu(\pi) \leq A_p^{q\frac{2\dim G-m}{2}}\|\varphi\|_p^q.$$

Since the norm $\|k_{\pi(\varphi)}\|_{p,q}$ is finite for μ-almost all π, it follows from [11], that for μ-almost all $\pi \in \widehat{G}$, $\pi(\varphi)K_\pi^{\frac{-1}{q}}$ is bounded and has a C_q-class extension on $\mathcal{H}_\pi$.
Finally, by Remark 2.3.2 in [13], we have that $(\pi(\varphi)K_\pi^{\frac{-1}{q}})^* = \pi(\varphi^{*(p)})K_\pi^{\frac{-1}{q}}$, where $\varphi^{*(p)}(g) = \Delta_G^{\frac{-1}{p}}(g)\overline{\varphi(g^{-1})}$. Hence, using (2.13) we obtain:

$$\int_{\widehat{G}}\|\pi(\varphi)\|_{C_q}^q\, d\mu(\pi) \leq \int_{\widehat{G}}\|k_{\pi(\varphi)}\|_{p,q}^{\frac{q}{2}}\|k_{\pi(\varphi^{*(p)})}\|_{p,q}^{\frac{q}{2}}d\mu(\pi)$$

$$\leq \Big(\int_{\widehat{G}} \|k_{\pi(\varphi)}\|_{p,q}^q d\mu(\pi)\Big)^{\frac{1}{2}}\Big(\int_{\widehat{G}} \|k_{\pi(\varphi^{*(p)})}\|_{p,q}^q d\mu(\pi)\Big)^{\frac{1}{2}}$$

$$\leq A_p^{q\frac{2\dim G - m}{2}} \|\varphi\|_p^{\frac{q}{2}} \ \|\varphi^{*(p)}\|_p^{\frac{q}{2}}$$

$$= A_p^{q\frac{2\dim G - m}{2}} \|\varphi\|_p^q.$$

Remark. The previous theorem asserts that the mapping $\varphi \mapsto \pi(\varphi)K_\pi^{\frac{-1}{q}}$ extends to a continuous operator $\mathcal{F}^p : L^p(G) \to L^q(\widehat{G})$ and its norm satisfies $\|\mathcal{F}^p(G)\| \leq A_p^{\frac{2\dim G - m}{2}}$.

□

4 Examples

4.1 A strong $*$-regular example

Let consider the following Lie algebra $\mathfrak{g} = \text{span} < A, X, Y, Z, B, V >$ with non-vanishing brackets:

$$[A, B] = V, \ [A, X] = -X, \ [A, Y] = Y, \ [X, Y] = Z$$

and with dual basis $\{A^*, X^*, Y^*, Z^*, B^*, V^*\}$.
We identify the exponential group associated to $\mathfrak{g}$ with $\mathbb{R}^6$ equipped with the multiplication:

$$(a, x, y, z, b, v)\cdot(t', x', y', z', b', v') = (a+a', x'+x\mathrm{e}^{a'}, y'+y\mathrm{e}^{-a'}, z+z'-x'y\mathrm{e}^{-a'}, b+b', v+v'-ba')$$

$$(a, x, y, z, b, v)^{-1} = (-a, \ -x\mathrm{e}^{-a}, -y\mathrm{e}^{a}, -z - xy, -b, -v - ba).$$

The Haar measure on G is given by $dadxdydzdbdv$ and G is unimodular. The group G is strong $*$-regular. Let $l \in \mathfrak{g}^*$ in general position, i.e., $l(V) \neq 0$ and $l(Z) \neq 0$. Then the coadjoint orbit of l contains an element l' of the form $l' = \lambda Z^* + \mu V^*$ with $(\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^*$. The nil-radical $\mathfrak{n}$ of $\mathfrak{g}$ is generated by $\{X, Y, Z, B, V\}$ and we have that $\mathfrak{g}(l) = \text{span} < V, Z >, \ \mathfrak{g}(l_{|\mathfrak{n}}) = \text{span} < B, V, Z >, \ \mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n} = \mathfrak{n}$ and then $\mathfrak{d}(l)^\infty = \{0\}$. It follows that $< l, \mathfrak{d}(l)^\infty >= \{0\}$.
The subalgebra $\mathfrak{b}(l) = \text{span} < Y, Z, B, V >$ is a Pukanszky polarization of $l = \lambda Z^* + \mu V^*, \ (\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^*$ and $B(l) = \exp \mathfrak{b}(l)$. Then $B(l)$ is unimodular and so $\Delta_{B(l),G} \equiv 1$.
Let χ_l be the unitary character of $B(l)$ defined by: for all $A \in \mathfrak{b}(l)$,

$$\chi_l(\exp A) = \mathrm{e}^{-2\pi i l(A)}.$$

The induced representation $\pi_l = \pi_{\lambda,\mu} = \mathrm{Ind}_{B(l)}^G \chi_l$ of G is irreducible. The Plancherel measure on $\widehat{G}$ is identified to the measure $|\lambda||\mu| d\lambda d\mu$ on $\mathbb{R}^* \times \mathbb{R}^*$.
So, for $\varphi \in L^1(G) \cap L^2(G)$, the Plancherel formula reads:

$$\int_{\mathbb{R}^*}\int_{\mathbb{R}^*} \|\pi_{\lambda,\mu}(\varphi)\|_{C_2}^2 |\lambda||\mu| d\lambda d\mu = \|\varphi\|_{L^2(\mathbb{R}^6)}^2.$$

We have, for $\xi \in \mathcal{H}_{\pi_{\lambda,\mu}}$ and $\varphi \in C_c(G)$, that:

$$\pi_{\lambda,\mu}(\varphi)\xi(g) = \int_{\mathbb{R}^6} \varphi(a,x,y,z,b,v)\pi_{\lambda,\mu}(a,x,y,z,b,v)\xi(g)dtdxdydz$$

If we identify $G/B(l)$ with $\mathbb{R}^2$, then the kernel k_φ of the operator $\pi_{\lambda,\mu}(\varphi)$ is given by the expression:

$$k_\varphi((a,x),(a',x')) = \mathrm{e}^{-a'}\widehat{\varphi}\,(a-a',(x-x')\mathrm{e}^{-a'},\cdot,\cdot,\cdot,\cdot)(-x'\mathrm{e}^{-a'}\lambda,\lambda,a'\mu,\mu\,),\ (a,x),(a',x') \in \mathbb{R}^2,$$

where $\widehat{\varphi}$ is the partial Fourier transform in the last four variables. By Proposition 3.1, we have that:

$$\int_{\mathbb{R}^2} \|k_\varphi\|_{p,q}^q |\lambda||\mu| d\lambda d\mu \leq A_p^{4q}\ \|\varphi\|_p^q.$$

4.2 Boidol's group

In this example we have a group which is not strong $*$-regular. We shall see that the methods used in this paper cannot give an estimate of the L^p-Fourier transform norm. Let $\mathfrak{g} = \mathrm{span} < T, X, Y, Z >$ with the brackets:

$$[X,Y] = Z,\ [T,Y] = Y,\ [T,X] = -X.$$

Let $\{T^*, X^*, Y^*, Z^*\}$ be the dual basis of $\mathfrak{g}$. We identify the group $G = \exp\mathfrak{g}$ with $\mathbb{R}^4$ and we have the multiplication:

$$(t,x,y,z)\cdot(t',x',y',z') = (t+t', x'+x\mathrm{e}^{t'}, y'+y\mathrm{e}^{-t'}, z+z'-yx'\mathrm{e}^{-t'})$$

and the inverse:

$$(t,x,y,z)^{-1} = (-t, -x\mathrm{e}^{-t}, -y\mathrm{e}^{t}, -xy-z).$$

The Haar measure in G is given by $dtdxdydz$ and the modular function Δ_G is equal to one. The G-orbits in general position are parametrized by the $\lambda Z^* + \mu T^*$, $(\lambda,\mu) \in \mathbb{R}^* \times \mathbb{R}$. The Plancherel measure on $\widehat{G}$ is identified with the measure $|\lambda| d\lambda d\mu$ on $\mathbb{R}^* \times \mathbb{R}$. So, for $\varphi \in L^1(G) \cap L^2(G)$, the Plancherel formula reads:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^*} \|\pi_{\lambda Z^*+\mu T^*}(\varphi)\|_{C_2}^2 |\lambda| d\lambda d\mu = \|\varphi\|_{L^2(\mathbb{R}^4)}^2.$$

The group G fails to be strong $*$-regular. In fact, if we take $l = \lambda Z^* + \mu T^*$ $(\lambda \neq 0)$, then

$$\mathfrak{g}(l_{|\mathfrak{n}}) = \mathrm{span} < Z, T >,\ \mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n} = \mathfrak{g}\ \text{ and }\ (\mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty = \mathfrak{n}.$$

This implies that $< l, (\mathfrak{g}(l_{|\mathfrak{n}}) + \mathfrak{n})^\infty > \neq \{0\}$. The subalgebra $\mathfrak{b}(l) = \mathrm{span} < T, Y, Z >$ is a Pukanszky polarization for $l = \lambda Z^* + \mu T^*,\ (\lambda,\mu) \in \mathbb{R}^* \times \mathbb{R}$ and $B(l) = \exp\mathfrak{b}(l)$ is non-unimodular:

$$\Delta_{B(l)}(t,y,z) = \Delta_{B(l),G}(t,y,z) = \mathrm{e}^{-t},\ \text{for all } t,y,z \in \mathbb{R}.$$

Let χ_l be the character of $B(l)$ defined by:

$$\chi_l(\exp A) = \mathrm{e}^{-2i\pi l(A)},\ A \in \mathfrak{b}(l).$$

The induced representation $\pi_l = \pi_{\lambda,\mu} = \mathrm{Ind}_{B(l)}^G \chi_l$, of G is irreducible and acts on $L^2(\mathbb{R})$ by: for all $\xi \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$,

$$\pi_{\lambda,\mu}(t,x,y,z)\xi(a) = \mathrm{e}^{2i\pi(\lambda(xy+z-aye^t)+\mu t)}\mathrm{e}^{\frac{t}{2}}\xi(a\mathrm{e}^t - x).$$

For $\varphi \in C_c(G)$, the kernel k_φ of the operator $\pi_{\lambda,\mu}(\varphi)$ is given by:

$$k_\varphi(x,a) = \int_{\mathbb{R}^3} \varphi(t, a\mathrm{e}^t - x, y, z)\mathrm{e}^{\frac{t}{2}}\mathrm{e}^{2i\pi(\lambda(z-xy)+\mu t)}dtdydz,\ x,a \in \mathbb{R}.$$

As $(k_\varphi)^* = k_{\varphi^*}$, we have:

$$\int_{\mathbb{R}^2} \|\pi_{\lambda,\mu}(\varphi)\|_{C_q}^q |\lambda| d\lambda d\mu \le \int_{\mathbb{R}^2} \|k_\varphi\|_{p,q}^{\frac{q}{2}}\ \|k_{\varphi^*}\|_{p,q}^{\frac{q}{2}} |\lambda| d\lambda d\mu$$

$$\le \Big(\int_{\mathbb{R}^2} \|k_\varphi\|_{p,q}^q |\lambda| d\lambda d\mu\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^2} \|k_{\varphi^*}\|_{p,q}^q |\lambda| d\lambda d\mu\Big)^{\frac{1}{2}}.$$

Since the dimension of the generic orbits is equal to two, then we like to get that:

$$\int_{\mathbb{R}^2} \|k_\varphi\|_{p,q}^q |\lambda| d\lambda d\mu \le A_p^{3q}\|\varphi\|_p^q.$$

But

$$\int_{\mathbb{R}^2} \|k_\varphi\|_{p,q}^q |\lambda| d\lambda d\mu = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |k_\varphi(a,x)|^p da\Big)^{\frac{q}{p}} dx |\lambda| d\lambda d\mu$$

$$= \int_{\mathbb{R}^3} \Big(\int_{\mathbb{R}} |\int_{\mathbb{R}^3} \varphi(t, a\mathrm{e}^t - x, y, z)\mathrm{e}^{\frac{t}{2}}\mathrm{e}^{2i\pi(\lambda(z-xy)+\mu t)}dtdydz|^p da\Big)^{\frac{q}{p}} dx|\lambda| d\lambda d\mu$$

$$= \int_{\mathbb{R}^3} \Big(\int_{\mathbb{R}} |\int_{\mathbb{R}} \widehat{\varphi}(t, a\mathrm{e}^t - x, \cdot, \cdot)(\lambda, -\lambda x)\mathrm{e}^{\frac{t}{2}}\mathrm{e}^{2i\pi\mu t}dt|^p da\Big)^{\frac{q}{p}} dx|\lambda| d\lambda d\mu$$

(we consider the function $\zeta_{a,x}(t) = \mathrm{e}^{\frac{t}{2}}\widehat{\varphi}(t, a\mathrm{e}^t - x, \cdot, \cdot)(\lambda, -\lambda x))$

$$= \int_{\mathbb{R}^3} \Big(\int_{\mathbb{R}} |\widehat{\zeta}_{a,x}(\mu)|^p da\Big)^{\frac{q}{p}} dx|\lambda| d\lambda d\mu$$

$$\le \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |\widehat{\zeta}_{a,x}(\mu)|^q d\mu\Big)^{\frac{p}{q}} da\Big)^{\frac{q}{p}} dx|\lambda| d\lambda$$

(by the generalized Minkowski's inequality)

$$\le A_p^q \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\zeta_{a,x}(t)|^p dtda\Big)^{\frac{q}{p}} dx|\lambda| d\lambda$$

(by Beckner's result for $\mathbb{R}$)

$$= A_p^q \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\widehat{\varphi}(t, a\mathrm{e}^t - x, \cdot, \cdot)(\lambda, -\lambda x)|^p \mathrm{e}^{\frac{pt}{2}} dtda\Big)^{\frac{q}{p}} dx|\lambda| d\lambda$$

$$= A_p^q \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\widehat{\varphi}(t,a,\cdot,\cdot)(\lambda,-\lambda x)|^p \mathrm{e}^{(\frac{p}{2}-1)t} dtda \Big)^{\frac{q}{p}} dx|\lambda| d\lambda$$

$$\leq A_p^q \Big(\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} |\widehat{\varphi}(t,a,\cdot,\cdot)(\lambda,-\lambda x)|^q dx|\lambda| d\lambda \Big)^{\frac{p}{q}} \mathrm{e}^{(\frac{p}{2}-1)t} dtda \Big)^{\frac{q}{p}}$$

using the generalized Minkowski's inequality.
This shows that

$$\int_{\mathbb{R}^2} \|k_\varphi\|_{p,q}^q |\lambda| d\lambda d\mu \leq A_p^{3q} \Big(\int_{\mathbb{R}^4} |\varphi(t,a,y,z)|^p \mathrm{e}^{(\frac{p}{2}-1)t} dtdadydz \Big)^{\frac{q}{p}}$$

by Beckner's result for $\mathbb{R}^2$. But the right hand of this inequality is clearly different from $A_p^{3q}\|\varphi\|_p^q$.

Remark. If we view the Boidol group as a semi-direct product of the Heisenberg group $H = \exp(\mathrm{span} < X, Y, Z >)$ and the abelian group $\exp(\mathbb{R}T)$, then by Theorem 2 in [14], we have for the particular case when $p = \dfrac{2k}{2k-1}$, k an integer ≥ 2,

$$\|\mathcal{F}^p(G)\| \leq \|\mathcal{F}^p(H)\| \|\mathcal{F}^p(\mathbb{R})\| = A_p^3.$$

References

[1] Babenko, K. I.: An inequality in the theory of Fourier integrals, Izv. Akad. Nauk SSSR Ser. Mat. **25**, 531-542(1961); English translation., Amer. Math. Soc. Transl., (2) **44**, 115-128(1965)

[2] Baklouti, A., Ludwig, J. and Smaoui, K.: Estimate of the L^p-Fourier transform norm on nilpotent Lie groups, J. Funct. Anal., **199**, 508-520(2003)

[3] Beckner, W.: Inequalities in Fourier analysis, Ann. of Math., **102**, 159-182(1975)

[4] Bernat, M.P. and Al.: Représentations des groupes de Lie résolubles, Dunod, Paris, 1972

[5] Boidol, J.: $*$-Regularity of Exponential Lie groups. Inventiones Math., **56**, 231-238(1980)

[6] Corwin, L. and Greenleaf, P.: Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples. Cambridge Studies in Advanced Mathematics, 1990

[7] Duflo, M. and Moore, C.C.: On the regular representation of a non-unimodular locally compact group, J. Funct. Anal., **21**, 209-243(1976)

[8] Duflo, M. and Rais, M.: Sur l'analyse harmonique sur les groupes de Lie résolubles, Ann. Sci. École Norm. Sup., (4) **9**, 107-144(1976)

[9] Eymard, P. et Terp, M.: La transformation de Fourier et son inverse sur le groupe des $ax+b$ d'un corps local, Lecture notes in Mathematics, Springer, **739**, 207-278(1979)

[10] Fournier, J.J.F.: Sharpness in Young's inequality for convolution, *Pacific J. Math.*, **72**, 383-397(1977)

[11] Fournier, J.J.F. and Russo, B.: Abstract interpolation and operator-valued kernels, J. London Math. Soc., **16**, 283-289(1977)

[12] Führ, H.: Hausdorff-Young inequalities for non-unimodular groups, preprint

[13] Inoue, J.: L^p-Fourier transforms on nilpotent Lie groups and solvable Lie groups acting on Siegel domains, Pacific J. Math., **155**, 2, 295-318(1992)

[14] Klein, A. and Russo, B.: Sharp inequalities for Weyl operators and Heisenberg groups, Math. Ann. Springer, **235**, 175-194(1978)

[15] Kleppner, A. and Lipsman, R.L.: The Plancherel formula for group extensions, I and II, Ann. Sci. Ecole Norm. Sup., **5**, 459-516(1972); ibid. **6**, 103-132(1973)

[16] Kunze, R.A.: L^p-Fourier transforms on locally compact unimodular groups, Trans. Amer. Math. Soc., **89**, 519-540(1958)

[17] Leptin, H. and Ludwig, J.: Unitary representation theory of exponential Lie groups, de Gruyter Expositions in Mathematics, vol. **18**, Walter de Gruyter & Co., Berlin, 1994

[18] Mackey, G.: Borel structure in groups and their duals, Trans. Amer. Math. Soc., **85**, 134-165(1957)

[19] Pukanszky, L.: Unitary representations of solvable Lie groups, Ann. Scient. Ec. Norm. Sup., **4**, 457-608(1971)

[20] Russo, B.: The norm of the L^p-Fourier transform, II, Canad. J. Math, **28**, 1121-1131(1976)

[21] Russo, B.: On the Hausdorff-Young theorem for integral operators, Pacific J. Math., **68**, 241-253(1977)

[22] Russo, B.: The norm of the L^p-Fourier transform on unimodular groups, Trans. Amer. Math. Soc., **192**, 293-305(1974)

[23] Terp, M.: L^p-Fourier transformation on non-unimodular locally compact groups, preprint