

# THE PALEY-WIENER THEOREM FOR CERTAIN NILPOTENT LIE GROUPS

J. Ludwig\* and C. Molitor-Braun\*

ABSTRACT. We generalize the classical Paley-Wiener theorem to special types of connected, simply connected, nilpotent Lie groups: First we consider nilpotent Lie groups whose Lie algebra admits an ideal which is a polarization for a dense subset of generic linear forms on the Lie algebra. Then we consider nilpotent Lie groups such that the co-adjoint orbits of all the elements of a dense subset of the dual of the Lie algebra  $\mathfrak{g}^*$  are flat.

## INTRODUCTION

One important theorem in classical harmonic analysis is the Paley-Wiener theorem. It may be stated as follows:

**Theorem 0.1.** *a) If  $f \in \mathcal{S}(\mathbb{R}^n)$  and  $\text{supp} \hat{f} \subset rB$ , then  $f$  may be extended to an entire function on  $\mathbb{C}^n$  by the formula*

$$(0.1) \quad f(z) := \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i z \cdot t} dt.$$

*Moreover, there are constants  $\gamma_N < +\infty$  such that*

$$(0.2) \quad |f(z)| \leq \gamma_N (1 + \|z\|)^{-N} e^{2\pi r \|Imz\|}, \quad z \in \mathbb{C}^n, N = 0, 1, 2, \dots$$

*b) Conversely, if an entire function  $f$  satisfies the conditions (0.2), then  $f|_{\mathbb{R}^n} \in \mathcal{S}(\mathbb{R}^n)$ ,  $\text{supp} \widehat{f|_{\mathbb{R}^n}} \subset rB$  and (0.1) holds.*

(see [Ru])

An immediate consequence of this result is of course the fact that if  $f$  is a  $\mathcal{C}^\infty$ -function such that  $\text{supp} \hat{f} \subset rB$  and if  $f \equiv 0$  on a non-trivial open set, then  $f$  is identically 0, as this is true for holomorphic functions. Some authors have generalized this last property to non-abelian groups, and among others to nilpotent Lie groups ([Ar-Lu], [Fü], [Ga], [Ga1], [Ka-La-Sch]). A result of this type is often called a weak Paley-Wiener theorem. But almost nothing seems to be known about a generalization of the Paley-Wiener theorem itself to nilpotent Lie groups, except for some very special results for the Heisenberg group ([Th]) or for step 2 nilpotent Lie groups ([Th1]). It is the aim of this paper to give first results concerning a generalization of the whole Paley-Wiener theorem to nilpotent Lie groups. We consider the classes of connected, simply connected, nilpotent Lie groups which satisfy one of the following three conditions:

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(i) There exists a fixed ideal  $\mathfrak{a}$  in  $\mathfrak{g}$  such that  $\mathfrak{a}$  is isotropic for every  $l$  in a dense subset of  $\mathfrak{g}^*$  and such that the co-adjoint orbits of these  $l$ 's are saturated with respect to  $\mathfrak{a}$ .

(ii) There exists a fixed ideal  $\mathfrak{a}$  in  $\mathfrak{g}$  which is a polarization for every  $l$  in a dense subset of  $\mathfrak{g}^*$ .

(iii) The co-adjoint orbits of all the elements in a dense subset of  $\mathfrak{g}^*$  are flat.

Under any one of these three hypotheses we characterize those  $C^\infty$ -functions  $f$  on the group  $G$ , which may be extended to an entire function on the complexified group  $G_{\mathbb{C}}$  satisfying growth conditions of the type (0.2). We give this characterization in terms of the operator kernels  $F(l, \cdot, \cdot)$  of the operators  $\pi_l(f)$ , where  $\pi_l = \text{ind}_{P(l)}^G \chi_l$  denotes the irreducible unitary representation induced from the character  $\chi_l(a) := e^{-2\pi i \langle l, \log a \rangle}$ ,  $a \in P(l) = \exp \mathfrak{p}(l)$  ( $\mathfrak{p}(l)$  being a polarization for  $l$ ). This is a good generalization of the classical theorem, as the map  $l \mapsto \pi_l(f)$ , or, equivalently,  $l \mapsto F(l, \cdot, \cdot)$ , may be considered as a generalization of the classical Fourier transform. In particular, we characterize those fields of operator kernels  $(F(l, \cdot, \cdot))_l$  which come from an entire function satisfying growth conditions similar to (0.2) and we deduce from this a necessary condition on the co-adjoint orbits. In case (i), our result is an abstract one. But under the stronger conditions (ii) or (iii), explicite computations show the exact relationship between the function  $f$  and the field of operator kernels  $F(l, \cdot, \cdot)$ . Examples of the type of groups studied in this paper are the Heisenberg group and the threadlike groups for instance.

## 1. ON REPRESENTATIONS AND OPERATOR KERNELS

**1.1. Irreducible unitary representations.** Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group obtained by endowing the nilpotent Lie algebra  $\mathfrak{g}$  with the Campbell-Baker-Hausdorff product. The irreducible unitary representations of the group  $G$  are obtained in the following way: Let  $l \in \mathfrak{g}^*$  and let  $\mathfrak{p} = \mathfrak{p}(l)$  be a polarization for  $l$  in  $\mathfrak{g}$  (i. e. a maximal subalgebra such that  $\langle l, [\mathfrak{p}, \mathfrak{p}] \rangle = 0$ ). Then  $\chi_l$  given by  $\chi_l(x) = e^{-2\pi i \langle l, \log x \rangle}$  defines a character on the subgroup  $P = \exp \mathfrak{p}$ . The induced representation  $\pi = \pi_{l, \mathfrak{p}} := \text{ind}_P^G \chi_l$  is irreducible and unitary. All the irreducible unitary representations may be obtained in this way (up to equivalence). Different polarizations for the same  $l$  give equivalent representations. If not otherwise specified, we will always assume that the polarization  $\mathfrak{p}(l)$  is the Vergne polarization with respect to a given fixed Jordan-Hölder basis. Two different linear forms  $l$  and  $l'$  give equivalent representations if and only if they belong to the same co-adjoint orbit. One denotes by  $\hat{G}$  (dual of  $G$ ) the space of equivalence classes of irreducible unitary representations of  $G$  equipped with the Fell topology. The space  $\hat{G}$  is isomorphic and homeomorphic to  $\mathfrak{g}^*/\text{Ad}^*G$ .

**1.2. Operator kernels.** Let now  $l \in \mathfrak{g}^*$  be fixed. Let  $\mathfrak{p}(l)$  be the Vergne polarization for  $l$  in  $\mathfrak{g}$  and  $P(l) = \exp \mathfrak{p}(l)$ . Let  $\pi_l := \text{ind}_{P(l)}^G \chi_l$  and let's denote by  $\mathcal{H}_l = L^2(G/P(l), \chi_l)$  the corresponding representation (Hilbert) space. For every  $f \in L^1(G)$ , the operator

$$\pi_l(f) := \int_G f(x) \pi_l(x) dx$$

is a kernel operator: It is of the form

$$\pi_l(f) \xi(x) = \int_{G/P(l)} F(l, x, y) \xi(y) dy,$$

where the operator kernel  $F$  is given by

$$F(l, x, y) = \int_{P(l)} f(x \cdot a \cdot y^{-1}) \chi_l(a) da.$$

One sees that  $F(l, x, y)$  depends only on  $\mathfrak{p}(l)$  and on the restriction  $l|_{\mathfrak{p}(l)}$  and one may hence write  $F(l|_{\mathfrak{p}(l)}, x, y)$ . The operator kernel satisfies the covariance relation

$$F(l, xa, ya') = \overline{\chi_l(a)} \chi_l(a') F(l, x, y), \quad \forall a, a' \in P(l).$$

If  $f \in L^1(G) \cap L^2(G)$  (for instance if  $f$  is a Schwartz function), then  $\pi_l(f)$  is a Hilbert-Schmidt operator and its Hilbert-Schmidt norm is given by

$$\|\pi_l(f)\|_{HS}^2 = \int_{G/P(l) \times G/P(l)} |F(l, s, t)|^2 ds dt = \|F_l\|_2^2.$$

**1.3. Explicite computation.** Let  $f \in \mathcal{S}(G)$  (Schwartz space) and  $l \in \mathfrak{g}^*$ . Let  $e$  be the unit element of the group  $G$ . The operator kernel of  $\pi_l(f)$  is then given by

$$\begin{aligned} F(l|_{\mathfrak{p}(l)}, x, y) &= \int_{P(l)} f(xay^{-1}) e^{-2\pi i \langle l, \log a \rangle} da \\ &= \int_{\text{Ad}(y)P(l)} f((xy^{-1})ae) e^{-2\pi i \langle l, \log(y^{-1}ay) \rangle} da \\ &= F(\text{Ad}^*(y)(l)|_{\mathfrak{p}(\text{Ad}^*(y)(l))}, xy^{-1}, e) \end{aligned}$$

with the appropriate covariance relations for  $F(l|_{\mathfrak{p}(l)}, \cdot, \cdot)$  and  $F(\text{Ad}^*(y)(l)|_{\mathfrak{p}(\text{Ad}^*(y)(l))}, \cdot, \cdot)$  respectively.

## 2. THE LIE GROUP, ITS ALGEBRA AND SATURATED ORBITS

**2.1. Parametrization of the Lie group.** Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be as previously. Let  $\mathfrak{a}$  be an ideal of  $\mathfrak{g}$  and  $\{Z_1, \dots, Z_n\}$  a Jordan-Hölder basis passing through  $\mathfrak{a}$ , i. e. such that

$$\mathfrak{a} = \langle Z_1, \dots, Z_r \rangle \quad \text{and} \quad \mathfrak{g} = \langle Z_1, \dots, Z_r, Z_{r+1}, \dots, Z_n \rangle.$$

Let's put  $\mathfrak{v} = \langle Z_{r+1}, \dots, Z_n \rangle$ . Hence the subspace  $\mathfrak{v}$  is such that  $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{a}$ . Because we use the Campbell-Baker-Hausdorff multiplication, the exponential map is the identity map on  $\mathfrak{g}$ . Hence we may make the following identifications

$$\begin{aligned} A &= \exp \mathfrak{a} \equiv \mathfrak{a} \\ V &= \exp \mathfrak{v} \equiv \mathfrak{v} \equiv \mathfrak{g}/\mathfrak{a} \equiv G/A \end{aligned}$$

Let

$$\begin{aligned} Q_{\mathfrak{a}, \mathfrak{v}} : G \equiv \mathfrak{v} \oplus \mathfrak{a} \equiv \mathfrak{v} \times \mathfrak{a} &\rightarrow G \\ v + a \equiv (v, a) &\mapsto \exp v \cdot_{CBH} \exp a = v \cdot_{CBH} a, \end{aligned}$$

i. e.

$$Q_{\mathfrak{a}, \mathfrak{v}}(v + a) = v + a + \frac{1}{2}[v, a] + \frac{1}{12}[v, [v, a]] + \dots$$

is a polynomial map given by the Campbell-Baker-Hausdorff formula. In the Jordan-Hölder basis  $\{Z_1, \dots, Z_n\}$  this map has the form

$$Q_{\mathfrak{a}, \mathfrak{v}}(v + a) = \sum_{j=r+1}^n v_j Z_j + \sum_{j=1}^r (a_j + P_j(v, a_{j+1}, \dots, a_r)) Z_j$$

where the  $P_j$ 's are polynomial functions, as  $\mathfrak{a}$  is an ideal in  $\mathfrak{g}$  and as  $\{Z_1, \dots, Z_n\}$  is a Jordan-Hölder basis. This form shows that  $Q_{\mathfrak{a}, \mathfrak{v}}$  is a bijection and that  $Q_{\mathfrak{a}, \mathfrak{v}}^{-1}$  is again polynomial, i. e. that  $Q_{\mathfrak{a}, \mathfrak{v}}$  is a bipolynomial diffeomorphism. The Jacobian matrices of  $Q_{\mathfrak{a}, \mathfrak{v}}$  and  $Q_{\mathfrak{a}, \mathfrak{v}}^{-1}$  are triangular with 1's on the diagonal. In particular, their determinant is equal to 1.

Similar observations may be made for the transition of coordinates of the first kind to coordinates of the second kind, and conversely. We hence have three coordinate systems to consider: coordinates of the first kind, coordinates of the second kind, a parametrization of the form  $v \cdot a$  with  $v \in V, a \in A$  where  $V$  and  $A$  are identified with  $\mathfrak{v}$  and with  $\mathfrak{a}$  respectively.

Because the transitions between these different coordinate systems are bipolynomial diffeomorphisms, a map  $f$  defined on  $G \equiv \mathfrak{g}$  is a Schwartz function if and only if it is a Schwartz function of the coordinates in any one of the three systems. We denote by  $\mathcal{S}(G) \equiv \mathcal{S}(\mathfrak{g})$  the algebra of these Schwartz functions. Moreover,  $f$  is Schwartz if and only if  $f \circ Q_{\mathfrak{a}, \mathfrak{v}}$  is Schwartz.

Let's complexify  $G$  by  $G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}} = \mathfrak{v}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$ . By a complexification of a function  $f$  we mean an extension of  $f$  to a function on  $G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}} = \mathfrak{v}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$ . Again, this may be done in any one of the three coordinate systems. Then the complexification of  $f$  is entire if and only if it is entire if expressed in any one of the three systems. Of course, the polynomial  $Q_{\mathfrak{a}, \mathfrak{v}}$  may also be extended to an entire function on  $G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}}$ . Similarly for  $Q_{\mathfrak{a}, \mathfrak{v}}^{-1}$ . Hence  $f$  is entire if and only if  $f \circ Q_{\mathfrak{a}, \mathfrak{v}}$  is entire.

For the rest of this paper we shall write the Campbell-Baker-Hausdorff product simply by  $g \cdot g'$  instead of  $g \cdot_{CBH} g'$ .

**2.2. Saturated orbits.** Let  $l \in \mathfrak{g}^*$  fixed and let  $\mathcal{O} = \mathcal{O}(l)$  be the co-adjoint orbit of  $l$ . Let's assume that there exists an ideal  $\mathfrak{a} = \mathfrak{a}(l)$  such that the orbit  $\mathcal{O}$  is saturated with respect to  $\mathfrak{a}$ , which means that

$$\mathcal{O} = \mathcal{O} + \mathfrak{a}^{\perp} \equiv \mathcal{O}|_{\mathfrak{a}} + \mathfrak{a}^{\perp},$$

where  $\mathfrak{a}^{\perp} = \{k \in \mathfrak{g}^* \mid \langle k, \mathfrak{a} \rangle \equiv 0\}$ . Let's also assume that the Jordan-Hölder basis  $\{Z_1, \dots, Z_n\}$  goes through  $\mathfrak{a}$ , i. e. that  $\mathfrak{a} = \langle Z_1, \dots, Z_r \rangle$ . As in (2.1), let  $\mathfrak{v} = \langle Z_{r+1}, \dots, Z_n \rangle$ . The definitions and arguments of (2.1) remain valid. Let's define different stabilizers:

$$\begin{aligned} \mathfrak{g}(l) &= \{X \in \mathfrak{g} \mid \langle l, [X, \mathfrak{g}] \rangle \equiv 0\} \\ G(l) &= \exp \mathfrak{g}(l) = \{g \in G \mid \text{Ad}^*(g)(l) = l\} \equiv \mathfrak{g}(l) \\ \mathfrak{g}(l|_{\mathfrak{a}}) &= \{X \in \mathfrak{g} \mid \langle l, [X, \mathfrak{a}] \rangle \equiv 0\} \\ G(l|_{\mathfrak{a}}) &= \exp \mathfrak{g}(l|_{\mathfrak{a}}) = \{g \in G \mid \text{Ad}^*(g)(l)|_{\mathfrak{a}} = l|_{\mathfrak{a}}\} \equiv \mathfrak{g}(l|_{\mathfrak{a}}) \end{aligned}$$

For the rest of this paper, let's make the following conventions: For any  $l' \in \mathfrak{g}^*$ , we write  $l'|_{\mathfrak{a}}$  for the element of  $\mathfrak{g}^*$  such that

$$\begin{cases} \langle l'|_{\mathfrak{a}}, Z_j \rangle = \langle l', Z_j \rangle & \text{if } 1 \leq j \leq r, & \text{i. e. if } Z_j \in \mathfrak{a} \\ \langle l'|_{\mathfrak{a}}, Z_j \rangle = 0 & \text{if } r+1 \leq j \leq n, \end{cases}$$

i. e. we identify  $\mathfrak{a}^*$  with a subspace of  $\mathfrak{g}^*$ . Similarly, we write  $\mathcal{O}(l) = \mathcal{O}(l)|_{\mathfrak{a}} + \mathfrak{a}^{\perp}$ , i. e. we identify  $\mathcal{O}(l)|_{\mathfrak{a}}$  with a subset of  $\mathfrak{g}^*$ . Then the orbit  $\mathcal{O} = \mathcal{O}(l)$  is given by

$$\begin{aligned} \mathcal{O}(l) &= \{\text{Ad}^*(g)(l) \mid g \in G/G(l)\} \\ &= \{\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q \mid g \in G/G(l|_{\mathfrak{a}}), q \in \mathfrak{a}^{\perp}\} \end{aligned}$$

and the map

$$\begin{aligned} G/G(l|_{\mathfrak{a}}) \times \mathfrak{a}^{\perp} &\rightarrow \mathcal{O} \\ (\dot{g}, q) &\mapsto \text{Ad}^*(g)(l)|_{\mathfrak{a}} + q \end{aligned}$$

is a parametrization of the orbit.

In fact, let's write  $G/G(l) = G/G(l|_{\mathfrak{a}}) \cdot G(l|_{\mathfrak{a}})/G(l)$ . For  $\dot{b} \in G(l|_{\mathfrak{a}})/G(l)$ ,  $\dot{g} \in G/G(l|_{\mathfrak{a}})$  and  $a \in \mathfrak{a}$  we have

$$\begin{aligned} \langle \text{Ad}^*(g)\text{Ad}^*(b)(l), a \rangle &= \langle \text{Ad}^*(b)(l), \text{Ad}(g^{-1})(a) \rangle \\ &= \langle l, \text{Ad}(g^{-1})(a) \rangle \\ &= \langle \text{Ad}^*(g)(l), a \rangle \end{aligned}$$

as  $\mathfrak{a}$  is an ideal and hence  $\text{Ad}(g^{-1})(a) \in \mathfrak{a}$ . As moreover  $\mathcal{O} = \mathcal{O}|_{\mathfrak{a}} + \mathfrak{a}^{\perp}$ , the map is onto. It is one-to-one because if  $\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q = \text{Ad}^*(g_1)(l)|_{\mathfrak{a}} + q_1$  with  $g, g_1 \in G/G(l|_{\mathfrak{a}})$  and  $q, q_1 \in \mathfrak{a}^{\perp}$ , then  $q = q_1$  and  $\text{Ad}^*(g_1^{-1}g)(l)|_{\mathfrak{a}} = l|_{\mathfrak{a}}$ , i. e.  $g = g_1 \text{ mod } G(l|_{\mathfrak{a}})$ . This is justified by the fact that if  $a \in \mathfrak{a}$ , then  $\text{Ad}(g_1)a \in \mathfrak{a}$  and so

$$\begin{aligned} \langle \text{Ad}^*(g)(l), a \rangle &= \langle \text{Ad}^*(g_1)(l), a \rangle \quad \forall a \in A \\ \Rightarrow \langle \text{Ad}^*(g)(l), \text{Ad}(g_1)a \rangle &= \langle \text{Ad}^*(g_1)(l), \text{Ad}(g_1)a \rangle \quad \forall a \in A \\ \Rightarrow \langle \text{Ad}^*(g_1^{-1}g)(l), a \rangle &= \langle l, a \rangle \quad \forall a \in A \end{aligned}$$

So we get a parametrization of the orbit. Moreover, the change of parametrization from the canonical orbit parametrization to this one is bipolynomial and the canonical measure on the orbit is given by the following proposition:

**Proposition 2.3.** *Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group. Let  $\mathfrak{a}$  be an ideal of  $\mathfrak{g}$  and  $l \in \mathfrak{g}^*$  such that the orbit  $\mathcal{O} = \mathcal{O}(l)$  is saturated with respect to  $\mathfrak{a}$ . Then the canonical measure  $\mu$  on the orbit is given (up to a constant) by*

$$\int_{\mathcal{O}} f(p) d\mu(p) = \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}^{\perp}} f(\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q) dq d\dot{g}$$

**Proof:** We know already that

$$\begin{aligned} G/G(l|_{\mathfrak{a}}) \times \mathfrak{a}^{\perp} &\rightarrow \mathcal{O} \\ (\dot{g}, q) &\mapsto \text{Ad}^*(g)(l)|_{\mathfrak{a}} + q \end{aligned}$$

is a parametrization of the orbit  $\mathcal{O}(l)$ . Because  $\mathfrak{a}$  is an ideal in  $\mathfrak{g}$  and because the Jordan-Hölder basis goes through  $\mathfrak{a}$ , it is easy to check that the measure on the right hand side is  $G$ -invariant. But one knows that the canonical measure on the orbit is (up to a constant) the unique  $G$ -invariant measure on the orbit. This proves the claim.  $\square$

**Remark 2.4.** Let's from now on assume that the Lebesgue measure  $dq$  on  $\mathfrak{a}^{\perp}$  and  $d\dot{g}$  on  $G/G(l|_{\mathfrak{a}})$  are normalized in such a way that the trace formula of section 2.5 holds.

**2.5.** Let's from now on add to the assumptions of (2.2) the hypothesis that the ideal  $\mathfrak{a}$  is abelian. Let  $f \in \mathcal{S}(G)$  and let's define its (adapted) Fourier transform on the orbit by

$$\begin{aligned}\tilde{f}(\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q) &:= \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q) \\ &= \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v \cdot a) e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, a \rangle} e^{-2\pi i \langle q, v \rangle} da dv\end{aligned}$$

Then the Plancherel formula and the abelian Fourier inversion theorem give, with  $\mathfrak{a}^\perp \equiv \mathfrak{v}^*$  and  $\exp = \text{id}$ ,  $f \circ \exp = f$ ,

$$\begin{aligned}\text{tr} \pi_l(f) &= \int_{\mathcal{O}} \widehat{f \circ \exp}(p) d\mu(p) \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}^\perp} \hat{f}(\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q) dq d\dot{g} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{v}^*} \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v + a) e^{-2\pi i \langle q, v \rangle} e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, a \rangle} dv dad q d\dot{g} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} f(a) e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, a \rangle} da d\dot{g}\end{aligned}$$

This computation makes of course sense, because if  $f$  is a Schwartz function, then  $\pi_l(f)$  is a traceclass operator. Similarly,

$$\begin{aligned}\|\pi_l(f)\|_{HS}^2 &= \text{tr} \pi_l(f * f^*) \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} f * f^*(a) e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, a \rangle} da d\dot{g} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} \int_G f(u) f^*(u^{-1}a) e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, a \rangle} du dad \dot{g} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v \cdot b) \overline{f(v \cdot (v^{-1}a^{-1}v) \cdot b)} \\ &\quad e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, a \rangle} db dv dad \dot{g} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v \cdot b) \overline{f(va^{-1}b)} \\ &\quad e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}}, va v^{-1} \rangle} db dv dad \dot{g} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v \cdot b) \overline{f(v \cdot (b - a))} \\ &\quad e^{-2\pi i \langle \text{Ad}^*(v^{-1}g)(l)|_{\mathfrak{a}}, a \rangle} db dv dad \dot{g} \\ &\quad \text{as } A \equiv \mathfrak{a} \text{ is abelian} \\ &= \int_{G/G(l|_{\mathfrak{a}})} \int_{\mathfrak{a}} \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v \cdot b) \overline{f(va^{-1})} \\ &\quad e^{-2\pi i \langle \text{Ad}^*(v^{-1}g)(l)|_{\mathfrak{a}}, a+b \rangle} db dv dad \dot{g}\end{aligned}$$

Let's now consider the function

$$v' \mapsto f((v + v') \cdot a^{-1}).$$

By the inverse Fourier transform,

$$f(va^{-1}) = \int_{\mathfrak{v}^*} \int_{\mathfrak{v}} f((v+v') \cdot a^{-1}) e^{-2\pi i \langle q, v' \rangle} dv' dq.$$

Hence, by left invariance of the measure  $d\dot{g}$ ,

$$\begin{aligned} \|\pi_l(f)\|_{HS}^2 &= \int_{G/G(l)_\mathfrak{a}} \int_{\mathfrak{a}} \int_{\mathfrak{v}} \int_{\mathfrak{a}} \int_{\mathfrak{v}^*} \int_{\mathfrak{v}} f(v \cdot b) \overline{f((v+v') \cdot a^{-1})} e^{2\pi i \langle q, v' \rangle} \\ &\quad e^{-2\pi i \langle \text{Ad}^*(v^{-1}g)(l)|_{\mathfrak{a}, \mathfrak{a}+b} \rangle} dv' dq db dv d\dot{g} \\ &= \int_{G/G(l)_\mathfrak{a}} \int_{\mathfrak{a}} \int_{\mathfrak{v}} \int_{\mathfrak{a}} \int_{\mathfrak{v}^*} \int_{\mathfrak{v}} f(v \cdot b) \overline{f(v'a^{-1})} e^{2\pi i \langle q, v' \rangle} \\ &\quad e^{-2\pi i \langle q, v \rangle} e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}, \mathfrak{a}+b} \rangle} dv' dq db dv d\dot{g} \\ &= \int_{G/G(l)_\mathfrak{a}} \int_{\mathfrak{a}^\perp \equiv \mathfrak{v}^*} \left[ \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(vb) e^{-2\pi i \langle q, v \rangle} e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}, b} \rangle} db dv \right] \\ &\quad \overline{\left[ \int_{\mathfrak{v}} \int_{\mathfrak{a}} f(v'a) e^{-2\pi i \langle q, v' \rangle} e^{-2\pi i \langle \text{Ad}^*(g)(l)|_{\mathfrak{a}, a} \rangle} da dv' \right]} d\dot{g} \\ &= \int_{\mathcal{O}} |\tilde{f}(p)|^2 d\mu(p) \end{aligned}$$

**2.6. Remarks.** a) The computations of (2.5) show that

$$\pi_l(f) = 0 \Leftrightarrow \tilde{f}|_{\mathcal{O}(l)} = \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}|_{\mathcal{O}(l)} \equiv 0.$$

b) There exists a similar formula for arbitrary connected, simply connected, nilpotent Lie groups, if we replace the usual Fourier transform by the adapted Fourier transform  $\hat{f}^a$ :

$$\|\pi_{l_0}(f)\|_{HS}^2 = \int_{\mathcal{O}(l_0)} |\hat{f}^a|_{\mathcal{O}(l_0)}(l)|^2 d\mu(l)$$

(See [Ar-Cor], [Lu], [Lu-Za] for the adapted Fourier transform)

### 3. A FIRST APPROACH TO THE PALEY-WIENER THEOREM

**3.1.** Let  $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{a}$  and  $G = \exp \mathfrak{g} \cong \mathfrak{g}$  with all the assumptions of the sections (2.1). With respect to the fixed Jordan-Hölder basis (see section (2.1)), a function  $f$  on  $G \cong \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{a}$  (with coordinates of the type  $v \cdot a$  for instance) may be considered as a function on  $\mathbb{R}^{n-r} \times \mathbb{R}^r \cong \mathbb{R}^n$  and the classical Paley-Wiener theorem may be applied to study the functions  $f$  that can be extended to entire functions (with growth conditions as previously) on  $\mathbb{C}^n \cong G_{\mathbb{C}} \cong \mathfrak{v}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$ . But in this setting, the classical Fourier transform has no real group theoretical meaning. So our aim is to establish conditions in terms of representations, co-adjoint orbits, fields of operator kernels, ...

**3.2. A condition on the co-adjoint orbits.** A necessary condition for the extension of  $f$  to an entire function with the given growth conditions, can be easily derived from the previous arguments:

**Proposition 3.3.** *Let  $G = \exp \mathfrak{g} \equiv \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{a}$  where  $\mathfrak{a}$  is an abelian ideal of  $\mathfrak{g}$  and  $Q_{\mathfrak{a}, \mathfrak{v}} : G \rightarrow G$  be as previously. If  $f : G \rightarrow \mathbb{C}$  can be extended to an entire function on  $G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}}$  such that*

$$|f \circ Q_{\mathfrak{a}, \mathfrak{v}}(z)| \leq \gamma_N (1 + \|z\|)^{-N} e^{2\pi r \|\Im m z\|}, \quad z \in G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}}, N = 0, 1, 2, \dots$$

and if  $l \in \mathfrak{g}^*$  is such that the orbit  $\mathcal{O}(l)$  of  $l$  is saturated with respect to  $\mathfrak{a}$ , then  $\pi_l(f) \neq 0$  implies that the co-adjoint orbit of  $l$  intersects the ball  $rB$ , i. e.  $\mathcal{O}(l) \cap rB \neq \emptyset$ .

**Proof:** By the classical Paley-Wiener theorem applied to  $f \circ Q_{\mathfrak{a}, \mathfrak{v}}$ ,  $\text{supp} \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}} \subset rB$ , as

$$|f \circ Q_{\mathfrak{a}, \mathfrak{v}}(z)| \leq \gamma_N (1 + \|z\|)^{-N} e^{2\pi r \|\Im m z\|}, \quad z \in G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}} \equiv \mathbb{C}^N, N = 0, 1, 2, \dots$$

One then applies the results of (2.5) and in particular the formula

$$\|\pi_l(f)\|_{HS}^2 = \int_{\mathcal{O}(l)} |\widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(p)|_{\mathcal{O}(l)}^2 d\mu(p).$$

□

It is obvious that this condition cannot be sufficient. As a matter of fact, it only says something about different orbits, but nothing about the behavior on each orbit.

**3.4. A condition on fields of operator kernels.** Let's now consider the following question: Given an appropriate field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  on  $L^2(\mathbb{R}^{n-r})$  ( $\mathfrak{g}_{gen}^*$  being a dense subset of generic orbits of fixed dimension  $2r$ ), when does this field come from an entire function  $f$  on  $G_{\mathbb{C}}$  with the growth conditions of proposition (3.3), in the sense that  $\pi_l(f|_{G_{\mathbb{R}}})$  has  $F(l, \cdot, \cdot)$  as an operator kernel for all  $l \in \mathfrak{g}_{gen}^*$ , if the corresponding polarizations  $\mathfrak{p}(l)$  are the Vergne polarizations with respect to a fixed Jordan-Hölder basis.

If  $F(l, \cdot, \cdot)$  is the operator kernel of  $\pi_l(f)$  with  $f \in \mathcal{S}(G)$ , then  $F$  (extended to  $\mathfrak{g}_{gen}^* \times G \times G$ ) belongs to  $\mathcal{S}(G/P(l) \times G/P(l), \chi_l)$ , i. e.  $F$  satisfies the covariance relation

$$F(l, x \cdot a, y \cdot a') = \overline{\chi_l(a)} \chi_l(a') F(l, x, y) \quad \forall a, a' \in P(l) \equiv \mathfrak{p}(l)$$

and  $F$  is Schwartz on  $G/P(l) \times G/P(l)$ .

We know already that  $F(l, \cdot, \cdot)$  must only depend on  $\mathfrak{p}(l)$  and on  $l|_{\mathfrak{p}(l)}$ . But there is a more general compatibility relation that  $F$  has to satisfy. In fact, in order to reflect correctly the unitary equivalence between  $\pi_l$  and  $\pi_{(\text{Ad}^* g)(l)}$  and its incidence on the corresponding operator kernels, the function  $F$  (considered as a function on  $\mathfrak{g}_{gen}^* \times G \times G$ ) has to satisfy

$$F(\text{Ad}^*(g)(l), x, y) = F(l, x \cdot g, y \cdot g) \quad \forall g, x, y \in G.$$

In order to be able to answer the question raised at the beginning of this paragraph, we will have to make more restrictive assumptions on the group and its algebra. This will be done in the following sections.

#### 4. GROUPS AND ALGEBRAS WITH ORBITS SATURATED WITH RESPECT TO A FIXED ISOTROPIC IDEAL

**4.1. Further assumptions on the algebra and the group.** Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group. Let  $\mathfrak{a}$  be a fixed ideal of  $\mathfrak{g}$  and  $\{Z_1, \dots, Z_n\}$  a Jordan-Hölder basis through  $\mathfrak{a}$ . Let  $\mathfrak{v}$  be as in (2.1). For the rest of this section, let's assume that for every  $l$  in a dense subset  $\mathfrak{g}_{gen}^*$  of  $\mathfrak{g}_{Puk}^*$  (set of generic elements in the sense of Pukanszky),  $\langle l, [\mathfrak{a}, \mathfrak{a}] \rangle \equiv 0$ . This implies that for every  $l \in \mathfrak{g}_{gen}^*$ ,  $\mathfrak{a} \subset \mathfrak{p}(l)$ , where  $\mathfrak{p}(l)$  denotes the Vergne polarization associated to  $l$ . In fact, if we write  $\mathfrak{g}_k = \langle Z_1, \dots, Z_k \rangle$  and  $l_k = l|_{\mathfrak{g}_k}$  for every  $k \in \{1, \dots, n\}$ , then

$$\langle l_r, [\mathfrak{g}_r, \mathfrak{g}_r] \rangle = \langle l, [\mathfrak{a}, \mathfrak{a}] \rangle = 0$$

and  $\mathfrak{g}_r(l_r) = \mathfrak{g}_r = \mathfrak{a}$ . So  $\mathfrak{a} \subset \mathfrak{p}(l) = \sum_{k=1}^n \mathfrak{g}_k(l_k)$ . In particular, as  $\mathfrak{g}_{Puk}^*$  and  $\mathfrak{g}_{gen}^*$  are dense in  $\mathfrak{g}^*$ , the ideal  $\mathfrak{a}$  has to be abelian. We may of course take  $\mathfrak{g}_{gen}^*$  to be  $G$ -invariant, as  $\mathfrak{a}$  is an ideal. Let's also assume that for every  $l$  in  $\mathfrak{g}_{gen}^*$ , the orbit  $\mathcal{O}(l)$  is saturated with respect to  $\mathfrak{a}$ , i. e.  $\mathcal{O}(l) = \mathcal{O}(l)|_{\mathfrak{a}} + \mathfrak{a}^\perp$ . The computations of (2.5) remain of course valid. In particular,

$$\|\pi_l(f)\|_{HS}^2 = \int_{\mathcal{O}} |\tilde{f}(p)|^2 d\mu(p)$$

implies that

$$\pi_l(f) = 0 \Leftrightarrow \tilde{f} \equiv 0 \text{ on } \mathcal{O}(l).$$

**4.2.** Let now  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be a field of operator kernels. Let  $\mathcal{O} = \mathcal{O}(l_0)$ ,  $l_0 \in \mathfrak{g}_{gen}^*$ , and let's consider  $(F(l, \cdot, \cdot))_{l \in \mathcal{O}}$ . By a result of Howe ([Ho]), there exists  $f_{\mathcal{O}} \in \mathcal{S}(G)$  such that  $\pi_{l_0}(f_{\mathcal{O}})$  admits  $F(l_0, \cdot, \cdot)$  as an operator kernel. Because of the compatibility relations of our operator field,  $\pi_l(f_{\mathcal{O}})$  admits  $F(l, \cdot, \cdot)$  as an operator kernel for all  $l$  in  $\mathcal{O} = \mathcal{O}(l_0)$ . The function  $f_{\mathcal{O}}$  is uniquely determined modulo  $\ker \pi_{l_0} = \ker \pi_l$ ,  $l \in \mathcal{O}$ , as, if  $k$  is another such function,

$$\|\pi_l(f - k)\|_{HS}^2 = \|F(l, \cdot, \cdot) - F(l, \cdot, \cdot)\|_2^2 = 0.$$

To this function  $f_{\mathcal{O}}$  let's now associate its adapted Fourier transform  $\tilde{f}_{\mathcal{O}}$  defined on the orbit  $\mathcal{O}$  by

$$\tilde{f}_{\mathcal{O}}(l) := f_{\mathcal{O}} \circ \widehat{Q_{\mathfrak{a}, \mathfrak{v}}}(l), \quad \forall l \in \mathcal{O}.$$

Let's notice that  $\tilde{f}_{\mathcal{O}}$  is uniquely determined by  $(F(l, \cdot, \cdot))_{l \in \mathcal{O}}$  as

$$\|F(l, \cdot, \cdot)\|_2^2 = \|\pi_l(f_{\mathcal{O}})\|_{HS}^2 = \int_{\mathcal{O}} |\tilde{f}_{\mathcal{O}}(p)|^2 d\mu(p), \quad \text{by (2.5).}$$

Hence, to our field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  we associate a unique function  $\tilde{F} : \mathfrak{g}_{gen}^* \rightarrow \mathbb{C}$  in the following way:

For  $l \in \mathfrak{g}_{gen}^*$ , there exists a unique orbit  $\mathcal{O}$  passing through  $l$ . We then define  $\tilde{F}(l) := \tilde{f}_{\mathcal{O}}(l) = f_{\mathcal{O}} \circ \widehat{Q_{\mathfrak{a}, \mathfrak{v}}}(l)$ .

**4.3. Definition.** We say that  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a  $C^\infty$ -field of operator kernels, if  $\tilde{F}$  is  $C^\infty$  on  $\mathfrak{g}_{gen}^*$  and if it may be extended to a  $C^\infty$  function on all of  $\mathfrak{g}^*$ . We say that  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a Schwartz field of operator kernels, if  $\tilde{F}$  is Schwartz on  $\mathfrak{g}_{gen}^*$  and if it may be extended to a Schwartz function on all of  $\mathfrak{g}^*$ .

We say that  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a  $C^\infty$ -field with compact support, if  $\tilde{F}$  may be extended to a  $C^\infty$  function with compact support in  $\mathfrak{g}^*$ .

We say that  $G = (\mathfrak{g}, \cdot_{CBH})$  satisfies the *ideal condition* if there exists a non trivial ideal  $\mathfrak{a}$  of  $\mathfrak{g}$  and a  $G$ -invariant dense subset  $\mathfrak{g}_{gen}^*$  of  $\mathfrak{g}^*$  such that  $\langle l, [\mathfrak{a}, \mathfrak{a}] \rangle \equiv 0$  for every  $l \in \mathfrak{g}_{gen}^*$  and such that the orbit of  $l$  is saturated with respect to  $\mathfrak{a}$ , i. e.  $\mathcal{O}(l) = \mathcal{O}(l)|_{\mathfrak{a}} + \mathfrak{a}^\perp$  for all  $l \in \mathfrak{g}_{gen}^*$ .

If  $G = (\mathfrak{g}, \cdot_{CBH})$  satisfies the ideal condition, we take a Jordan-Hölder basis  $\{Z_1, \dots, Z_n\}$  passing through  $\mathfrak{a}$ , i. e. such that  $\mathfrak{a} = \langle Z_1, \dots, Z_r \rangle$  and we put  $\mathfrak{v} = \langle Z_{r+1}, \dots, Z_n \rangle$ .

We also assume that the Jordan-Hölder basis  $\{Z_1, \dots, Z_n\}$  may be chosen such that  $\mathfrak{g}_{gen}^* \subset \mathfrak{g}_{Puk}^*$  (the set of Pukanszky generic elements of  $\mathfrak{g}^*$  with respect to this basis). To every  $l \in \mathfrak{g}_{gen}^*$  we associate its Vergne polarization  $\mathfrak{p}(l)$  with respect to the basis  $\{Z_1, \dots, Z_n\}$  and we write  $\pi_l$  for the induced representation  $\text{ind}_{P(l)}^G \chi_l$ ,  $P(l) = \exp \mathfrak{p}(l)$ . We then have the following result:

**Theorem 4.4.** *Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group. Let's assume that  $G$  satisfies the ideal condition. We have the following results:*

(i) *Let  $f \in \mathcal{S}(G)$  and  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  the field of operator kernels of the  $\pi_l(f)$ 's and  $\tilde{F}$  the corresponding function defined on  $\mathfrak{g}_{gen}^*$ . Then  $\tilde{F}$  may be extended to a Schwartz function on all of  $\mathfrak{g}^*$  and  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a Schwartz field of operator kernels.*

(ii) *For each  $\tilde{K} \in \mathcal{S}(\mathfrak{g}^*)$ , there exists a unique function  $f \in \mathcal{S}(G)$  such that the associated field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  and the corresponding function  $\tilde{F}$  satisfy  $\tilde{F} = \tilde{K}$  on  $\mathfrak{g}_{gen}^*$ . In particular, the field of operator kernels is a Schwartz field.*

(iii) *For every Schwartz field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$ , there exists a unique function  $f \in \mathcal{S}(G)$  such that  $F(l, \cdot, \cdot)$  is the operator kernel of  $\pi_l(f)$  for every  $l \in \mathfrak{g}_{gen}^*$ .*

**Proof:** (i) In this case we may of course take  $f_{\mathcal{O}} = f$  for every orbit  $\mathcal{O}$ , in the construction of (4.2). Hence

$$\tilde{F}(l) = \tilde{f}(l) = \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(l) \quad \forall l \in \mathfrak{g}_{gen}^*.$$

As  $f \circ Q_{\mathfrak{a}, \mathfrak{v}}$  is a Schwartz function on  $G \equiv \mathfrak{g}$ ,  $\widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(l)$  is a Schwartz function on all of  $\mathfrak{g}^*$ , which proves (i).

(ii) Let  $\tilde{K} \in \mathcal{S}(\mathfrak{g}^*)$  and  $k \in \mathcal{S}(\mathfrak{g})$  the (abelian) inverse Fourier transform of  $\tilde{K}$ . We put  $f = k \circ Q_{\mathfrak{a}, \mathfrak{v}}^{-1}$ . Then  $f \in \mathcal{S}(G) \equiv \mathcal{S}(\mathfrak{g})$ , as  $Q_{\mathfrak{a}, \mathfrak{v}}^{-1}$  is polynomial, and  $\widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}} = \hat{k} = \tilde{K}$ . It remains to show that  $\tilde{F} = \tilde{K}$  on  $\mathfrak{g}_{gen}^*$ . In fact, by (i),

$$\tilde{F}(l) = \tilde{f}(l) = \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(l) = \tilde{K}(l), \quad \forall l \in \mathfrak{g}_{gen}^*.$$

As  $\mathfrak{g}_{gen}^*$  is dense in  $\mathfrak{g}^*$  and as  $Q_{\mathfrak{a}, \mathfrak{v}}^{-1}$  is a bijection, this relation shows the uniqueness of the function  $f$ . The corresponding field of operator kernels is Schwartz, as  $\tilde{K} \in \mathcal{S}(\mathfrak{g}^*)$ .

(iii) Let  $\tilde{F} \in \mathcal{S}(\mathfrak{g}^*)$  be the function on  $\mathfrak{g}^*$  associated to the given field of operator kernels. By (ii), there exists  $f \in \mathcal{S}(G)$  such that

$$\tilde{f}(l) = \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(l) = \tilde{F}(l), \quad \forall l \in \mathfrak{g}_{gen}^*.$$

Let  $(F'(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be the field of operator kernels of the operators  $\pi_l(f)$ . Of course, if we denote  $\tilde{F}'$  for the associated function on  $\mathfrak{g}_{gen}^*$ , then

$$\tilde{F}'(l) = \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(l) = \tilde{F}(l), \quad \forall l \in \mathfrak{g}_{gen}^*.$$

Moreover, for every  $l_0 \in \mathfrak{g}_{gen}^*$  and  $\mathcal{O} = \mathcal{O}(l_0)$ ,

$$\|F'(l_0, \cdot, \cdot) - F(l_0, \cdot, \cdot)\|_2^2 = \int_{\mathcal{O}} |\tilde{F}'(l) - \tilde{F}(l)|^2 d\mu(l) = 0$$

and  $F'(l_0, \cdot, \cdot) = F(l_0, \cdot, \cdot)$ , i. e.  $\pi_l(f)$  has  $F(l, \cdot, \cdot)$  as operator kernel for every  $l \in \mathfrak{g}_{gen}^*$ . The function  $f$  is unique, because if  $f, f'$  are two such functions, then

$$\|\pi_l(f) - \pi_l(f')\|_2^2 = \|F(l, \cdot, \cdot) - F(l, \cdot, \cdot)\|_{HS}^2 = 0 \quad \forall l \in \mathfrak{g}_{gen}^*.$$

□

**4.5.** One may then ask the question of a Paley-Wiener theorem for  $\tilde{F}$ , resp.  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$ : When does the function  $\tilde{F}$  on  $\mathfrak{g}^*$ , resp. the field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  come from a (unique) entire function  $f$  with appropriate growth conditions in the sense that  $\pi_l(f|_{G_{\mathbb{R}}})$  admits  $F(l, \cdot, \cdot)$  as an operator kernel?

We have the following result:

**Theorem 4.6.** *Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group. Let's assume that  $G$  satisfies the ideal condition. Let  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be a field of operator kernels. Let  $\mathfrak{a}, \mathfrak{v}, Q_{\mathfrak{a}, \mathfrak{v}}$  be as previously. Then the following are equivalent:*

(i) *There exists a (unique) Schwartz function  $f \in \mathcal{S}(G)$  such that  $\pi_l(f)$  has  $F(l, \cdot, \cdot)$  as an operator kernel for every  $l \in \mathfrak{g}_{gen}^*$ . The function  $f$  may be extended to an entire function on  $G_{\mathbb{C}}$ , also denoted by  $f$ , and it satisfies the following growth conditions: There are constants  $\gamma_N$  such that*

$$|f \circ Q_{\mathfrak{a}, \mathfrak{v}}(z)| \leq \gamma_N (1 + \|z\|)^{-N} e^{2\pi r \|3mz\|}, \quad \forall z \in G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}}, \forall N \in \mathbb{N}$$

(ii)  *$(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a  $\mathcal{C}^\infty$ -field with compact support. The support of the field is contained in the closed ball  $rB$  of  $\mathfrak{g}^*$  of radius  $r$  centered at the origin.*

*If both conditions (i) and (ii) are satisfied and if  $l \in \mathfrak{g}_{gen}^*$  such that  $\pi_l(f) \neq 0$ , then the co-adjoint orbit of  $l$  intersects  $rB$ .*

**Proof:** This is a consequence of the classical Paley-Wiener theorem (0.1).

(i)  $\Rightarrow$  (ii) By theorem (4.4), we know that the field  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  has to be Schwartz, hence  $\mathcal{C}^\infty$ . By the classical Paley-Wiener theorem,  $\text{supp } \tilde{F} = \text{supp } \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}} \subset rB$ .

(ii)  $\Rightarrow$  (i) Let's assume that  $\tilde{F}$  is a  $\mathcal{C}^\infty$ -function on all of  $\mathfrak{g}^*$  and that  $\text{supp } \tilde{F} \subset rB$ . In particular,  $\tilde{F} \in \mathcal{S}(\mathfrak{g}^*)$ . By theorem (4.4), there exists a (unique) function  $f \in \mathcal{S}(G)$  such that  $\pi_l(f)$  has  $F(l, \cdot, \cdot)$  as an operator kernel and  $\widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(l) = \tilde{F}(l)$  for all  $l \in \mathfrak{g}_{gen}^*$ , even for all  $l \in \mathfrak{g}^*$  by density. Then the classical Paley-Wiener theorem implies that  $f$  may be extended to an entire function on  $G_{\mathbb{C}}$  with the correct growth conditions for  $f \circ Q_{\mathfrak{a}, \mathfrak{v}}(z)$ . □

**4.7. Remarks.** a) Given the operator field  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  in this general setting, the existence of the functions  $f$  and  $\tilde{F}$  are obtained via the existence of a retract at each fixed  $l_0 \in \mathfrak{g}_{gen}^*$ . The result is hence an abstract one, without a precise formula or algorithm to compute  $f$ ,  $\tilde{F}$  or to determine the condition on the operator field. This will be done under more restrictive assumptions, as explained in the following sections.

b) There exists a general retract theorem ([Lu-Mo-Sc]) giving the existence of a Schwartz function  $f$  associated to a given operator field under suitable conditions on the field. But this result is not necessary for the previous arguments.

## 5. GROUPS AND ALGEBRAS WITH ORBITS SATURATED WITH RESPECT TO A FIXED (IDEAL-)POLARIZATION

**5.1.** Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group. Let  $\mathfrak{a}$  be an ideal of  $\mathfrak{g}$  and  $\{Z_1, \dots, Z_n\}$  a Jordan-Hölder basis passing through  $\mathfrak{a}$ . Let's take all the conventions and notations of (2.1). Let's furthermore assume that the fixed ideal  $\mathfrak{a}$  is a polarization for every  $l \in \mathfrak{g}_{gen}^*$ . In particular, the ideal  $\mathfrak{a}$  has to be abelian, as  $[\mathfrak{a}, \mathfrak{a}]$  is annihilated by a dense subset of  $\mathfrak{g}^*$ . As the Jordan-Hölder basis goes through  $\mathfrak{a}$  by assumption,  $\mathfrak{a}$  is the Vergne polarization with respect to this basis. In fact, by (4.1)  $\mathfrak{a}$  is contained in the Vergne polarization  $\mathfrak{p}(l)$  for every  $l \in \mathfrak{g}_{gen}^*$  and has to be equal to it, by maximality of the polarization  $\mathfrak{a}$ .

As  $\mathfrak{a} = \mathfrak{p}(l)$  is a polarization for every  $l \in \mathfrak{g}_{gen}^*$ ,  $\mathfrak{g}(l) \subset \mathfrak{a}$  and  $r+1, \dots, n$  have to be Pukanszky jump indices for all  $l \in \mathfrak{g}_{gen}^*$ . This means that the co-adjoint orbit of  $l$  is saturated in the directions  $Z_{r+1}^*, \dots, Z_n^*$ , i. e.

$$\mathcal{O} = \mathcal{O}(l) = \mathcal{O}(l)|_{\mathfrak{a}} + \mathfrak{v}^* = \mathcal{O}(l)|_{\mathfrak{a}} + \mathfrak{a}^\perp.$$

So all the assumptions of section 4 are satisfied.

Examples of algebras (and groups) satisfying all the previous hypotheses are for instance the Heisenberg algebra and the threadlike algebras.

**5.2.** Under the additional hypotheses of this section,  $f$  and  $\tilde{F}$  may be computed explicitly from the operator field  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$ . In fact, in the computations of (1.3),  $\mathfrak{p}(l) = \mathfrak{p}(\text{Ad}^*(y)(l)) = \mathfrak{a} \equiv A$  for all  $l \in \mathfrak{g}_{gen}^*$  and all  $y$ , i. e.

$$F(l|_{\mathfrak{a}}, x, y) = F(\text{Ad}^*(y)(l)|_{\mathfrak{a}}, xy^{-1}, e), \quad \forall x, y \in G$$

with the appropriate covariance relations for  $F(l|_{\mathfrak{a}}, \cdot, \cdot)$  and  $F(\text{Ad}^*(y)(l)|_{\mathfrak{a}}, \cdot, \cdot)$  respectively. One has

$$\mathfrak{g}(l|_{\mathfrak{a}}) = \{X \in \mathfrak{g} \mid \langle l, [X, \mathfrak{a}] \rangle \equiv 0\} = \mathfrak{a} = \mathfrak{p}(l) \quad \text{and} \quad G(l|_{\mathfrak{a}}) = A = P(l).$$

In fact, if  $X_0 \in \mathfrak{g}$  is such that  $\langle l, [X_0, \mathfrak{a}] \rangle \equiv 0$ , then  $X_0 \in \mathfrak{a}$ , by maximality of  $\mathfrak{a}$ , as  $\mathbb{R}X_0 + \mathfrak{a}$  is also isotropic with respect to  $l$ .

Moreover, for  $f \in \mathcal{S}(G)$ ,  $l \in \mathfrak{g}_{gen}^*$ , let  $F(l, \cdot, \cdot)$  be the operator kernel of  $\pi_l(f)$ . We have  $F(l, \cdot, \cdot) \in \mathcal{S}(G/P(l) \times G/P(l), \chi_l)$ , where  $G/P(l) = G/A \equiv V \equiv (\mathfrak{v}, \cdot_{CBH})$ , i. e. we assume that in  $F(l, x, y)$  the variables  $x, y$  are expressed by their coordinates

of the first kind in  $\mathfrak{v} \equiv G/P(l)$ . One has:

$$\begin{aligned}
\tilde{f}(\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q) &= \widehat{f \circ Q_{\mathfrak{a}, \mathfrak{v}}}(\text{Ad}^*(g)(l)|_{\mathfrak{a}} + q) \\
&= \int_{\mathfrak{v}} e^{-2\pi i \langle q, v \rangle} \int_{\mathfrak{a}} e^{-2\pi i \langle \text{Ad}^*(g)(l), a \rangle} f \circ Q_{\mathfrak{a}, \mathfrak{v}}(v + a) da dv \\
&= \int_{\mathfrak{v}} e^{-2\pi i \langle q, v \rangle} \int_{\mathfrak{a}} e^{-2\pi i \langle \text{Ad}^*(g)(l), a \rangle} f(v \cdot a) da dv \\
&= \int_{\mathfrak{v}} e^{-2\pi i \langle q, v \rangle} F(\text{Ad}^*(g)(l)|_{\mathfrak{a}}, v, e) dv \\
&= \hat{F}^2(\text{Ad}^*(g)(l)|_{\mathfrak{a}}, q, e)
\end{aligned}$$

for  $q \in \mathfrak{v}^* \equiv \mathfrak{a}^\perp$  and with  $G/G(l|_{\mathfrak{a}}) = G/A \equiv \mathfrak{v}$ . Here  $\hat{F}^2$  denotes the Fourier transform in the second variable. In particular, if  $\tilde{F}$  denotes the function defined in (4.2), then

$$\tilde{F}(p + q) = \hat{F}^2(p, q, e) \quad \forall p \in \mathfrak{a}^*, \forall q \in \mathfrak{v}^* \equiv \mathfrak{a}^\perp \text{ such that } p + q \in \mathfrak{g}_{gen}^*$$

i. e. there exists a direct formula to compute  $\tilde{F}$  from the field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*} = (F(l|_{\mathfrak{a}}, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$ , if this field is expressed by coordinates of the first kind in  $G/P(l) \equiv \mathfrak{v}$ . Let's adopt the following definition:

**5.3. Definition.** We say that  $G = (\mathfrak{g}, \cdot_{CBH})$  satisfies the *ideal-polarization condition* if there exists an ideal  $\mathfrak{a}$  of  $\mathfrak{g}$  which is an ideal for every  $l$  in a  $G$ -invariant dense subset  $\mathfrak{g}_{gen}^*$  of  $\mathfrak{g}^*$ .

If  $G$  satisfies the ideal-polarization condition, we take a Jordan-Hölder basis  $\{Z_1, \dots, Z_n\}$  passing through  $\mathfrak{a}$  and we also assume that  $\mathfrak{g}_{gen}^* \subset \mathfrak{g}_{Puk}^*$  (the set of Pukanszky generic elements with respect to this basis). We have of course the results of (5.1) and (5.2). The theorem on the retract then takes the following form.

**Theorem 5.4.** *Let  $(G, \cdot) = (\mathfrak{g}, \cdot_{CBH})$ , be a connected, simply connected, nilpotent Lie group obtained by endowing the Lie algebra  $\mathfrak{g}$  with the Campbell-Baker-Hausdorff product. Let's assume that  $G$  satisfies the ideal-polarization condition. Let  $\mathfrak{a}$  and  $\mathfrak{v}$  be as previously. Let  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be a field of operator kernels, where the variables  $x, y$  in  $F(l, x, y)$  are expressed by their coordinates of the first kind in  $\mathfrak{v} \equiv G/P(l)$ . Then the following are equivalent:*

(i) *There exists  $f \in \mathcal{S}(G)$  such that  $\pi_l(f)$  has  $F(l, \cdot, \cdot) = F(l|_{\mathfrak{a}}, \cdot, \cdot)$  as operator kernel for all  $l \in \mathfrak{g}_{gen}^*$ .*

(ii) *The function  $\tilde{F}(p, x, e)$  is Schwartz on  $\mathfrak{a}_{gen}^* \times \mathfrak{v} \times \{e\}$  (where  $\mathfrak{a}_{gen}^* = \{l|_{\mathfrak{a}} \mid l \in \mathfrak{g}_{gen}^*\}$ ) and may be extended to a Schwartz function on all of  $\mathfrak{a}^* \times \mathfrak{v} \times \{e\}$ .*

**Proof:** We have

$$\tilde{f}(p + q) = \tilde{F}(p + q) = \hat{F}^2(p, q, e) \quad \forall p \in \mathfrak{a}^*, \forall q \in \mathfrak{v}^* \text{ s. t. } p + q \in \mathfrak{g}_{gen}^*.$$

Hence  $\tilde{F}$  may be extended to a Schwartz function on all of  $\mathfrak{g}^*$  if and only if  $\hat{F}^2(p, q, e)$  may be extended to a Schwartz function on all of  $\mathfrak{g}^* \equiv \mathfrak{a}^* \times \mathfrak{v}^* \times \{e\}$ , or, if and only if  $F(p, x, e)$  may be extended to a Schwartz function on all of  $\mathfrak{a}^* \times \mathfrak{v} \times \{e\}$ .  $\square$

As far as the Paley-Wiener theorem is concerned, we have the following precision: If (ii) of the general Paley-Wiener theorem (4.6) is satisfied, the entire function  $f$

is obtained by the computation

$$(5.1) \quad f \circ Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}}(z_1 + z_2) = \int_{\mathfrak{v}^*} \int_{\mathfrak{a}^*} \hat{F}^2(p, q, e) e^{2\pi i \langle q, z_1 \rangle} e^{2\pi i \langle p, z_2 \rangle} dp dq \quad \forall z_1 \in \mathfrak{v}_{\mathbb{C}}, \forall z_2 \in \mathfrak{a}_{\mathbb{C}}$$

and  $f$  is given by  $f = (f \circ Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}}) \circ (Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}})^{-1}$ , where  $Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}}$  denotes the complexified polynomial mapping  $Q_{\mathfrak{a}, \mathfrak{v}}$ . In fact, the right hand side of the equation (5.1) defines an entire function. Its restriction to  $G_{\mathbb{R}}$  has  $\hat{F}^2(\cdot, \cdot, e) = f \circ \widehat{Q_{\mathfrak{a}, \mathfrak{v}}}(\cdot)$  as a Fourier transform on  $\mathfrak{g}_{gen}^*$ , and hence on all of  $\mathfrak{g}^*$  if the field is  $\mathbb{C}^{\infty}$ . So both sides of the equation (5.1) are entire functions whose restrictions to  $G_{\mathbb{R}} \equiv \mathfrak{g}_{\mathbb{R}}$  coincide, which proves our claim. This characterizes completely the function  $f$ . We may summarize these results in the following theorem:

**Theorem 5.5.** *Let  $(G, \cdot) = (\mathfrak{g}, \cdot_{CBH})$ , be a connected, simply connected, nilpotent Lie group obtained by endowing the Lie algebra  $\mathfrak{g}$  with the Campbell-Baker-Hausdorff product. Let's assume that  $G$  satisfies the ideal-polarization condition. Let  $\mathfrak{a}, \mathfrak{v}, Q_{\mathfrak{a}, \mathfrak{v}}$  be as previously. Let  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be a field of operator kernels, where the variables  $x, y$  in  $F(l, x, y)$  are expressed by their coordinates of the first kind in  $\mathfrak{v} \equiv G/P(l)$ . Then the following are equivalent:*

(i) *There exists a (unique) Schwartz function  $f$  such that  $\pi_l(f)$  has  $F(l, \cdot, \cdot)$  as an operator kernel for every  $l \in \mathfrak{g}_{gen}^*$ . The function  $f$  may be extended to an entire function on  $G_{\mathbb{C}}$ , also denoted by  $f$ , and satisfies the following growth conditions: There are constants  $\gamma_N$  such that*

$$|f \circ Q_{\mathfrak{a}, \mathfrak{v}}(z)| \leq \gamma_N (1 + \|(z)\|)^{-N} e^{2\pi r \|\Im m(z)\|}, \quad \forall z \in \mathfrak{g}_{\mathbb{C}}, \forall N \in \mathbb{N}.$$

(ii)  *$\hat{F}^2(\cdot, \cdot, e)$  may be extended to a  $\mathcal{C}^{\infty}$ -function on all of  $\mathfrak{a}^* \times \mathfrak{v}^* \equiv \mathfrak{g}^*$  and  $\text{supp} \hat{F}^2(\cdot, \cdot, e) \subset rB$ , where  $rB$  is the closed ball of  $\mathfrak{g}^*$  of radius  $r$  and centered at the origin.*

*In case (ii) is satisfied, the function  $f$  of point (i) is given by*

$$f \circ Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}}(z_1 + z_2) = \int_{\mathfrak{v}^*} \int_{\mathfrak{a}^*} \hat{F}^2(p, q, e) e^{2\pi i \langle q, z_1 \rangle} e^{2\pi i \langle p, z_2 \rangle} dp dq, \quad \forall z_1 \in \mathfrak{v}_{\mathbb{C}}, \forall z_2 \in \mathfrak{a}_{\mathbb{C}}$$

*and  $f = (f \circ Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}}) \circ (Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}})^{-1}$ . Here  $Q_{\mathfrak{a}, \mathfrak{v}}^{\mathbb{C}}$  denotes the complexification of the map  $Q_{\mathfrak{a}, \mathfrak{v}}$ .*

*If both conditions (i) and (ii) are satisfied and if  $\pi_l(f) \neq 0$  for some  $l \in \mathfrak{g}_{gen}^*$ , then the co-adjoint orbit of  $l$  intersects  $rB$ .*

## 6. THE HEISENBERG GROUP

**6.1. Definition.** Let the Lie algebra  $\mathfrak{h}_n$  be defined by  $\mathfrak{h}_n = \langle X_1, \dots, X_n, Y_1, \dots, Y_n, Z \rangle$  with  $[X_i, Y_j] = \delta_{ij}Z$  (where  $\delta_{ij}$  is 1 if  $i = j$  and 0 otherwise). This nilpotent Lie algebra is called the Heisenberg Lie algebra. We may write

$$\mathfrak{h}_n = \mathfrak{v} \oplus \mathfrak{a}$$

with  $\mathfrak{v} = \langle X_1, \dots, X_n \rangle$  and  $\mathfrak{a} = \langle Y_1, \dots, Y_n, Z \rangle$ . Both  $\mathfrak{v}$  and  $\mathfrak{a}$  are abelian subalgebras and  $\mathfrak{a}$  is even an ideal in  $\mathfrak{h}_n$ . The generic elements of  $\mathfrak{h}_n^*$  are the linear forms  $l = \sum_{i=1}^n l_i X_i^* + \sum_{j=1}^n l'_j Y_j^* + \lambda Z^*$  with  $\lambda \neq 0$ . The abelian ideal  $\mathfrak{a}$  is a polarization for each generic  $l$ . So the hypotheses of theorem 5.5 are satisfied. In this case the group  $H_n = \exp \mathfrak{h}_n \equiv \mathfrak{v} \oplus \mathfrak{a} \equiv \mathfrak{v} \times \mathfrak{a}$  may even be written as a semi-direct product of  $V \equiv \mathfrak{v} \equiv \mathbb{R}^n$  with  $A \equiv \mathfrak{a} \equiv \mathbb{R}^{n+1}$ , as  $\mathfrak{v}$  is also a subalgebra in this case.

**6.2.** An orbit section  $\Sigma$  (of generic orbits) is given by

$$\Sigma = \{\lambda Z^* \mid \lambda \in \mathbb{R}^*\}.$$

For  $l_0 = \lambda Z^*$ ,  $\lambda \neq 0$ , in  $\Sigma$ , the co-adjoint orbit is given by

$$\mathcal{O}(l_0) = \left\{ \sum_{i=1}^n q_i X_i^* + \sum_{j=1}^n q'_j Y_j^* + \lambda Z^* \mid q_i, q'_j \in \mathbb{R} \right\}.$$

This is even more generally the case for the orbit  $\mathcal{O}(l)$  of an arbitrary element  $l = \sum_{i=1}^n l_i X_i^* + \sum_{j=1}^n l'_j Y_j^* + \lambda Z^*$ ,  $\lambda \neq 0$  of  $(\mathfrak{h}_n^*)_{gen}$ . Let  $l$  be such an element of  $(\mathfrak{h}_n^*)_{gen}$ . Let  $H_n = \exp \mathfrak{h}_n \equiv \mathfrak{v} \times \mathfrak{a}$ . Let  $f \in \mathcal{S}(H_n)$  and write  $f(v, a) := f(v + a)$ . This means that  $f$  is expressed in the coordinates of the first kind. The operator kernel of  $\pi_l(f)$  is then given by

$$\begin{aligned} F(l|_{\mathfrak{a}}, \sum_{i=1}^n u_i X_i, \sum_{j=1}^n v_j X_j) &= \int_{\mathbb{R}^{n+1}} f\left(\left(\sum_{i=1}^n u_i X_i, 0\right) \cdot \left(0, \sum_{j=1}^n a_j Y_j + bZ\right) \cdot \left(-\sum_{k=1}^n v_k X_k, 0\right)\right) \\ &\quad \cdot e^{-2\pi i(\sum_{j=1}^n l'_j a_j + \lambda b)} da_1 \dots da_n db \\ &= \hat{f}^2\left(\sum_{i=1}^n (u_i - v_i) X_i, \sum_{j=1}^n (l'_j - \lambda v_j) Y_j^* + \lambda Z^*\right) \end{aligned}$$

In particular, for  $l = \lambda Z^*$ ,  $\lambda \neq 0$ , in the orbit section  $\Sigma$ ,

$$F(l|_{\mathfrak{a}}, \sum_{i=1}^n u_i X_i, \sum_{j=1}^n v_j X_j) = \hat{f}^2\left(\sum_{i=1}^n (u_i - v_i) X_i, -\lambda \sum_{j=1}^n v_j Y_j^* + \lambda Z^*\right).$$

For arbitrary  $l = \sum_{i=1}^n l_i X_i^* + \sum_{j=1}^n l'_j Y_j^* + \lambda Z^* \in (\mathfrak{h}_n^*)_{gen}$ ,  $\lambda \neq 0$ , and for  $\sum_{j=1}^n v_j X_j = 0$ ,

$$F(l|_{\mathfrak{a}}, \sum_{i=1}^n u_i X_i, 0) = \hat{f}^2\left(\sum_{i=1}^n u_i X_i, \sum_{j=1}^n l'_j Y_j^* + \lambda Z^*\right)$$

and, for  $q = \sum_{j=1}^n q_j Y_j^* + r Z^* \in \mathfrak{a}^*$ ,  $r \neq 0$ , and  $p = \sum_{i=1}^n p_i X_i^*$ ,

$$\hat{F}^2(q, p, 0) = \hat{f}(p, q) \equiv \hat{f}(p + q),$$

which is the general result of (5.2) ( $Q_{\mathfrak{a}, \mathfrak{v}}$  doesn't appear in the formula as all the orbits are flat, see section 9.1).

## 7. THE THREADLIKE ALGEBRAS AND GROUPS

**7.1. Definition.** Let the Lie algebra  $\mathfrak{k}_n$  be defined by  $\mathfrak{k}_n = \langle Z_1, \dots, Z_n \rangle$  with  $[Z_n, Z_i] = Z_{i-1}$  for all  $i \in \{2, \dots, n-1\}$ . This nilpotent Lie algebra is called *threadlike*. We may write

$$\mathfrak{k}_n = \mathfrak{v} \oplus \mathfrak{a}$$

with  $\mathfrak{v} = \langle Z_n \rangle$  and  $\mathfrak{a} = \langle Z_1, \dots, Z_{n-1} \rangle$ . The generic elements of  $\mathfrak{k}_n^*$  are the linear forms  $l = \sum_{i=1}^n l_i Z_i^*$  such that  $l_1 \neq 0$ . The abelian ideal  $\mathfrak{a}$  is a polarization for each generic  $l$ . So the hypotheses of theorem 5.5 are satisfied. The group  $K_n = \exp \mathfrak{k}_n$  may even be realized as the semi-direct product of  $\mathbb{R} \equiv \langle Z_n \rangle = \mathfrak{v}$  with  $\mathfrak{a}$ , as  $\mathfrak{v}$  is a subalgebra of  $\mathfrak{k}_n$ .

**7.2.** Let's now take  $n = 4$ . In this case an orbit section  $\Sigma$  (of generic orbits) is given by

$$\Sigma = \{l_1 Z_1^* + l_3 Z_3^* \mid l_1 \in \mathbb{R}^*, l_3 \in \mathbb{R}\} \cong \mathbb{R}^* \times \mathbb{R}.$$

For  $l_0 = l_1 Z_1^* + l_3 Z_3^*$ ,  $l_1 \neq 0$ , in  $\Sigma$ , the co-adjoint orbit is given by

$$\mathcal{O}(l_0) = \{l_1 Z_1^* + (-tl_1)Z_2^* + (l_3 + \frac{1}{2}t^2 l_1)Z_3^* + pZ_4^* \mid (t, p) \in \mathbb{R}^2\}.$$

More generally, for  $l = l_1 Z_1^* + l_2 Z_2^* + l_3 Z_3^* + l_4 Z_4^*$ ,  $l_1 \neq 0$ , in  $(\mathfrak{k}_4^*)_{gen}$ , the co-adjoint orbit is

$$\mathcal{O}(l) = \{l_1 Z_1^* + (l_2 - tl_1)Z_2^* + (l_3 - tl_2 + \frac{1}{2}t^2 l_1)Z_3^* + pZ_4^* \mid (t, p) \in \mathbb{R}^2\}.$$

Let  $l = l_1 Z_1^* + l_2 Z_2^* + l_3 Z_3^* + l_4 Z_4^*$ ,  $l_1 \neq 0$ , be an arbitrary element of  $(\mathfrak{k}_4^*)_{gen}$ . Let  $f \in \mathcal{S}(K_n)$  and let's express  $f$  in the coordinates of the first kind. The operator kernel of  $\pi_l(f)$  is then given by

$$\begin{aligned} F(l|_{\mathfrak{a}}, u, v) &= \int_{\mathbb{R}^3} f((u, 0, 0, 0)(0, a_3, a_2, a_1)(-v, 0, 0, 0)) e^{-2\pi i(l_1 a_1 + l_2 a_2 + l_3 a_3)} da_1 da_2 da_3 \\ &= \int_{\mathbb{R}^3} f(u - v, a_3, a_2 + va_3, a_1 + va_2 + \frac{v^2}{2}a_3) e^{-2\pi i(l_1 a_1 + l_2 a_2 + l_3 a_3)} da_1 da_2 da_3 \\ &= \hat{f}^{3,2,1}(u - v, l_3 - vl_2 + \frac{1}{2}v^2 l_1, l_2 - vl_1, l_1), \end{aligned}$$

where we have numbered the coordinates 4, 3, 2, 1, and where  $\hat{f}^{3,2,1}$  denotes hence the partial Fourier transform with respect to the last three coordinates. In particular, for  $l = l_1 Z_1^* + l_3 Z_3^*$ ,  $l_1 \neq 0$ , in the orbit section  $\Sigma$ ,

$$F(l|_{\mathfrak{a}}, u, v) = \hat{f}^{3,2,1}(u - v, l_3 + \frac{1}{2}v^2 l_1, -vl_1, l_1).$$

For arbitrary  $l \in (\mathfrak{k}_4^*)_{gen}$  and  $v = 0$ ,  $F(l|_{\mathfrak{a}}, u, 0) = \hat{f}^{3,2,1}(u, l_3, l_2, l_1)$  and, for  $q = q_3 Z_3^* + q_2 Z_2^* + q_1 Z_1^* \in \mathfrak{a}^*$ ,  $q_1 \neq 0$ , and  $p \in \mathbb{R} \cong \mathbb{R}Z_4^* = \mathfrak{v}^*$ ,  $\hat{F}^2(q, p, 0) = \hat{f}(p + q)$ , which is the general result of (5.2).

## 8. ANOTHER EXAMPLE

Let  $\mathfrak{g} = \langle A, B, H, U, V, Z \rangle$  with the relations

$$[A, B] = H, \quad [A, H] = U, \quad [B, H] = V, \quad [A, V] = Z, \quad [B, U] = Z.$$

Let  $\mathfrak{a} = \langle H, U, V, Z \rangle$  and  $\mathfrak{v} = \langle A, B \rangle$ . Then  $\mathfrak{a}$  is an abelian ideal such that  $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{a}$ . With respect to the Jordan-Hölder basis  $\{A, B, H, U, V, Z\}$ , the generic elements of  $\mathfrak{g}^*$  form the set  $\mathfrak{g}_{gen}^* = \{l \in \mathfrak{g}^* \mid \langle l, Z \rangle \neq 0\}$ . Moreover, every  $l \in \mathfrak{g}_{gen}^*$  admits  $\mathfrak{a} = \langle H, U, V, Z \rangle$  as the Vergne polarization. The set of Pukanszky jump indices is  $\{1, 2, 4, 5\}$  and an orbit section of the generic elements is given by  $\Sigma = \mathbb{R}H^* + \mathbb{R}^*Z^*$ . The results of this paper may be applied.

## 9. A GENERALIZATION: ALGEBRAS WITH FLAT ORBITS

Let's recall the following result about flat orbits (see [Lu]):

**Theorem 9.1.** *Let  $G$  be a connected, simply connected nilpotent Lie group,  $\mathfrak{g}$  its Lie algebra. Let  $\mathcal{O} = \mathcal{O}(l_0)$  be a co-adjoint orbit in  $\mathfrak{g}^*$ ,  $\pi \equiv \pi_{l_0}$  the corresponding element in  $\hat{G}$ . The following statements are equivalent:*

- (i)  $\mathcal{O}$  is flat
- (ii) for every  $f \in \mathcal{S}(G)$ ,  $x \in G$ ,

$$\mathrm{tr}(\pi(x)\pi(f)) = \int_{\mathcal{O}} e^{-2\pi i \langle l, \log x \rangle} (f \circ \exp)^{\wedge}(l) d\mu_{\pi}(l)$$

- (iii) for every  $f \in L^1(G)$ ,  $\pi(f)$  is a Hilbert-Schmidt operator if and only if  $(f \circ \exp)^{\wedge}|_{\mathcal{O}}$  is in  $L^2(\mathcal{O}, d\mu_{\pi})$  and if  $\pi(f)$  is Hilbert-Schmidt, then

$$\|\pi(f)\|_{HS}^2 = \int_{\mathcal{O}} |(f \circ \exp)^{\wedge}(l)|^2 d\mu_{\pi}(l).$$

Moreover, if the orbit  $\mathcal{O} = \mathcal{O}(l_0)$  is flat, then the stabilizer  $\mathfrak{g}(l_0)$  is a commutative ideal of  $\mathfrak{g}^*$  and

$$\mathcal{O} = \mathcal{O}(l_0) = l_0 + \mathfrak{g}(l_0)^{\perp}.$$

If  $G = (\mathfrak{g}, \cdot_{CBH})$ , the exponential map  $\exp$  is the identity map and needs not be written in the previous formulas.

See ([Di]) for the commutativity of  $\mathfrak{g}(l_0)$ .

We then have the equivalent of proposition (3.3):

**Theorem 9.2.** *Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group. Let  $f : G \rightarrow \mathbb{C}$  be such that  $f = f \circ \exp$  may be extended to an entire function on  $\mathfrak{g}_{\mathbb{C}}$  such that*

$$|f(z)| = |f \circ \exp(z)| \leq \gamma_N (1 + \|z\|)^{-N} e^{\gamma \|Im z\|}, \quad z \in \mathfrak{g}_{\mathbb{C}}, \quad N = 0, 1, 2, \dots$$

Let  $l_0 \in \mathfrak{g}^*$  be such that its co-adjoint orbit  $\mathcal{O} = \mathcal{O}(l_0)$  is flat and  $\pi_{l_0}(f) \neq 0$ . Then  $\mathcal{O} \cap rB \neq \emptyset$ .

**Proof:** By theorem (9.1),

$$0 \neq \|\pi_{l_0}(f)\|_{HS}^2 = \int_{\mathcal{O}} |\widehat{f \circ \exp}(l)|^2 d\mu_{\pi}(l) = \int_{\mathcal{O}} |\hat{f}(l)|^2 d\mu_{\pi}(l)$$

and by the classical Paley-Wiener theorem (0.1),  $\mathrm{supp} \widehat{f \circ \exp} = \mathrm{supp} \hat{f} \subset rB$ . This proves the result.  $\square$

**9.3.** Let's take  $G = (\mathfrak{g}, \cdot_{CBH})$ . If  $l_0$  is such that  $\mathfrak{g}(l_0)$  is flat, we may hence take  $\mathfrak{a} = \mathfrak{a}(l_0) := \mathfrak{g}(l_0)$  in the computations of (2.5), as  $\mathfrak{g}(l_0)$  is an abelian ideal such that  $\mathcal{O}(l_0)$  is saturated with respect to  $\mathfrak{g}(l_0)$ . One has

$$\mathfrak{g}(\mathrm{Ad}^*(g)(l_0)) = \mathfrak{g}(l_0) = \mathfrak{a}(l_0).$$

In fact, for  $X \in \mathfrak{g}(l_0)$ ,  $Y, Z \in \mathfrak{g}$ ,

$$\langle \mathrm{Ad}^*(\exp Y)(l_0), [X, Z] \rangle = \sum_{k=0}^{\infty} (-1)^k \langle l_0, \mathrm{ad}^k(Y)[X, Z] \rangle = 0$$

as  $X, [X, Z], \mathrm{ad}^{(k-1)}(Y)[X, Z] \in \mathfrak{g}(l_0)$  for  $k \geq 1$ , i. e.  $X \in \mathfrak{g}(\mathrm{Ad}^*(\exp Y)(l_0))$ , for all  $Y \in \mathfrak{g}$  and  $\mathfrak{g}(\mathrm{Ad}^*(g)(l_0)) = \mathfrak{g}(l_0) = \mathfrak{a}(l_0)$  for all  $g \in G$ .

Then, for  $l \in \mathcal{O}(l_0) = \mathcal{O}(l)$ ,  $\mathfrak{a} = \mathfrak{g}(l) = \mathfrak{g}(l_0)$  and

$$\begin{aligned} \mathfrak{g}(l|_{\mathfrak{a}}) &= \mathfrak{g}, \\ G(l|_{\mathfrak{a}}) &= G, \\ G/G/l|_{\mathfrak{a}} &= \{e\}. \end{aligned}$$

Moreover

$$\mathrm{Ad}^*(g)(l)|_{\mathfrak{a}} = \mathrm{Ad}^*(g)(l)|_{\mathfrak{g}(l)} = l|_{\mathfrak{g}(l)}.$$

**9.4.** For any  $f \in \mathcal{S}(G)$  and  $l \in \mathcal{O}(l_0)$ , the previous computations (2.5) give:

$$\begin{aligned} \mathrm{tr}\pi_l(f) &= \int_{G/G(l|_{\mathfrak{a}})} \int_{G(l)} f(a) e^{-2\pi i \langle \mathrm{Ad}^*(g)(l)|_{\mathfrak{g}(l)}, a \rangle} da dj \\ &= \int_{G(l)} f(a) e^{-2\pi i \langle l|_{\mathfrak{g}(l)}, a \rangle} da \end{aligned}$$

and, by the Plancherel formula,

$$\begin{aligned} \mathrm{tr}(\pi_l(v^{-1})\pi_l(f)) &= \int_{G(l)} f(va) e^{-2\pi i \langle l|_{\mathfrak{g}(l)}, a \rangle} da \\ &= \int_{G(l)} f(v+a + \frac{1}{2}[v, a] + \frac{1}{12}[v, [v, a]] + \dots) e^{-2\pi i \langle l|_{\mathfrak{g}(l)}, a \rangle} da \\ &= \int_{G(l)} f(v+a) e^{-2\pi i \langle l|_{\mathfrak{g}(l)}, a \rangle} da \\ &= \int_{G/P(l)} F(l, v \cdot s, s) ds \end{aligned}$$

**9.5.** The orbit  $\mathcal{O}(l) = l + \mathfrak{g}(l)^\perp = l|_{\mathfrak{g}(l)} + \mathfrak{g}(l)^\perp$  is parameterized by  $\mathfrak{g}(l)^\perp \equiv \mathfrak{v}(l)^*$  through the map

$$q \in \mathfrak{g}(l)^\perp \mapsto l|_{\mathfrak{g}(l)} + q.$$

The (adapted) Fourier transform of  $f \in \mathcal{S}(G)$  on the flat orbit  $\mathcal{O} = \mathcal{O}(l)$  is given by

$$\begin{aligned} \tilde{f}(l|_{\mathfrak{g}(l)} + q) &:= f \circ \widehat{Q_{\mathfrak{g}(l), \mathfrak{v}(l)}}}(l|_{\mathfrak{g}(l)} + q) \\ &= \int_{\mathfrak{v}(l)} \int_{\mathfrak{g}(l)} f(v \cdot a) e^{-2\pi i \langle l|_{\mathfrak{g}(l)} + q, a + v \rangle} da dv \\ &= \int_{\mathfrak{v}(l)} \int_{\mathfrak{g}(l)} f(v+a + \frac{1}{2}[v, a] + \frac{1}{12}[v, [v, a]] + \dots) e^{-2\pi i \langle l|_{\mathfrak{g}(l)} + q, a + v \rangle} da dv \\ &= \int_{\mathfrak{v}(l)} \int_{\mathfrak{g}(l)} f(v+a) e^{-2\pi i \langle l|_{\mathfrak{g}(l)} + q, (a - \frac{1}{2}[v, a] - \frac{1}{12}[v, [v, a]] - \dots) + v \rangle} da dv \\ &= \int_{\mathfrak{v}(l)} \int_{\mathfrak{g}(l)} f(v+a) e^{-2\pi i \langle l|_{\mathfrak{g}(l)} + q, a + v \rangle} da dv \\ &= \hat{f}(l|_{\mathfrak{g}(l)} + q) \\ &= \int_{\mathfrak{v}(l)} \left[ \int_{\mathfrak{g}(l)} f(v+a) e^{-2\pi i \langle l|_{\mathfrak{g}(l)}, a \rangle} da \right] e^{-2\pi i \langle l|_{\mathfrak{g}(l)} + q, v \rangle} dv \\ &= \int_{\mathfrak{v}(l)} \left[ \int_{G/P(l)} F(l, v \cdot s, s) ds \right] e^{-2\pi i \langle l|_{\mathfrak{g}(l)} + q, v \rangle} dv \end{aligned}$$

So, if  $l = l|_{\mathfrak{g}(l)} + q$ , then, with  $\mathfrak{v}(l) \equiv G/G(l)$ ,

$$\hat{f}(l) = \tilde{f}(l) = \int_{G/G(l)} \left[ \int_{G/P(l)} F(l, v \cdot s, s) ds \right] e^{-2\pi i \langle l, v \rangle} dv$$

**9.6. Remarks.** a) The convergence of the last integral may be justified by the local Fourier inversion theorem: In fact, let's consider the space  $\mathcal{S}(G/G(l), \chi_l) \subset L^1(G/G(l), \chi_l)$  of functions satisfying the covariance relation

$$f(g \cdot a) = \overline{\chi_l(a)} f(g) \quad \forall a \in G(l)$$

and which are Schwartz on  $G/G(l)$ . The dual space of  $L^1(G/G(l), \chi_l)$  may be identified with

$$\hat{G}_{\chi_l} = \{\pi \in \hat{G} \mid \pi(a) = \chi_l(a) \text{Id}, \forall a \in G(l)\}.$$

But, in the present case,  $\hat{G}_{\chi_l} = \{\pi_l\}$ . On the other hand, if  $f \in \mathcal{S}(G)$ , the function  $f^\#$  defined by

$$f^\#(v) = \int_{\hat{G}_{\chi_l}} \text{tr}(\pi(v^{-1})\pi(f)) d\mu(\pi)$$

belongs to  $\mathcal{S}(G/G(l), \chi_l)$ . As  $\hat{G}_{\chi_l} = \{\pi_l\}$ , we have

$$f^\#(v) = \text{tr}(\pi_l(v^{-1})\pi_l(f)) = \int_{G/P(l)} F(l, v \cdot s, s) ds \in \mathcal{S}(G/G(l), \chi_l)$$

and the integral of (9.5) converges and is Schwartz, by the Fourier transform of a Schwartz function.

b) In the previous computations the measure on  $G/G(l)$  has been normalized such that the trace formula holds.

c) The integral

$$\int_{G/G(l)} \left[ \int_{G/P(l)} F(l, v \cdot s, s) ds \right] e^{-2\pi i \langle l, v \rangle} dv$$

is well defined. In fact, if  $v = v' \cdot a$  with  $a \in G(l) \equiv \mathfrak{g}(l)$ , then, as  $a \in \mathfrak{g}(l)$  and as  $\mathfrak{g}(l)$  is an ideal contained in  $\mathfrak{p}(l)$ ,

$$e^{-2\pi i \langle l, v' \cdot a \rangle} = e^{-2\pi i \langle l, v' \rangle} \cdot e^{-2\pi i \langle l, a \rangle}$$

and

$$F(l, v \cdot s, s) = F(l, v' \cdot a \cdot s, s) = e^{2\pi i \langle l, a \rangle} F(l, v' \cdot s, s)$$

So

$$F(l, v \cdot s, s) e^{-2\pi i \langle l, v \rangle} = F(l, v' \cdot s, s) e^{-2\pi i \langle l, v' \rangle}$$

and the integral over  $G/G(l)$  is well defined.

**9.7.** Let now  $\{Z_1, \dots, Z_n\}$  be a fixed Jordan-Hölder basis, let  $\mathfrak{g}_{Puk}^*$  be the set of Pukanszky generic elements of  $\mathfrak{g}^*$  (with respect to the given basis). Let  $\mathfrak{g}_{gen}^*$  be a dense,  $G$ -invariant subset of  $\mathfrak{g}_{Puk}^*$ . Let's assume that for all  $l \in \mathfrak{g}_{gen}^*$ , the orbit  $\mathcal{O}(l)$  is flat, i. e.  $\mathcal{O}(l) = l|_{\mathfrak{g}(l)} + \mathfrak{g}(l)^\perp$ . We may then make the same constructions as in section 4, even though the ideal  $\mathfrak{a} = \mathfrak{a}(l) := \mathfrak{g}(l)$ , the subspace  $\mathfrak{v}(l)$  and the corresponding Jordan-Hölder basis depend on  $l$  or, more precisely, on the orbit of  $l$ . For every  $l \in \mathfrak{g}_{gen}^*$ , the definition of  $\pi_l = \text{ind}_{P(l)}^G \chi_l$  and the corresponding Hilbert space  $\mathfrak{H}_{\pi_l}$  have fixed the invariant measure on  $G/P(l)$ . As pointed out previously, the measure on  $G/G(l)$  is normalized such that the trace formula holds for  $\pi_l$ .

Let's now be given a field of operator kernels  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$ . Let's fix an orbit  $\mathcal{O} = \mathcal{O}(l_0)$  with  $l_0 \in \mathfrak{g}_{gen}^*$ . By the result of Howe ([Ho]), there exists  $f_{\mathcal{O}} \in \mathcal{S}(G)$  such that  $\pi_{l_0}(f_{\mathcal{O}})$  has  $F(l_0, \cdot, \cdot)$  as operator kernel. By the compatibility relation we even have that  $\pi_l(f_{\mathcal{O}})$  has  $F(l, \cdot, \cdot)$  as operator kernel for every  $l \in \mathcal{O} = \mathcal{O}(l_0)$ . As previously, we define the (adapted) Fourier transform on the orbit by

$$\begin{aligned} \tilde{f}_{\mathcal{O}}(l_0|_{\mathfrak{g}(l_0)} + q) &:= \hat{f}_{\mathcal{O}}|_{\mathcal{O}}(l_0|_{\mathfrak{g}(l_0)} + q) \\ &= \int_{G/G(l_0)} \left[ \int_{G/P(l_0)} F(l_0, v \cdot s, s) d\dot{s} \right] e^{-2\pi i \langle l_0|_{\mathfrak{g}(l_0)} + q, v \rangle} d\dot{v} \end{aligned}$$

This function  $\tilde{f}_{\mathcal{O}}$  is unique as

$$\|F(l_0, \cdot, \cdot)\|_2^2 = \|\pi_{l_0}(f_{\mathcal{O}})\|_2^2 = \int_{\mathcal{O}} |\hat{f}_{\mathcal{O}}(l)|^2 d\mu(l) = \int_{G/G(l_0)} |\tilde{f}_{\mathcal{O}}(l_0|_{\mathfrak{g}(l_0)} + q)|^2 dq.$$

Hence, if  $f_{\mathcal{O}}$  and  $k_{\mathcal{O}}$  are such that  $\pi_{l_0}(f_{\mathcal{O}})$  and  $\pi_{l_0}(k_{\mathcal{O}})$  have both  $F(l_0, \cdot, \cdot)$  as operator kernel,  $\tilde{f}_{\mathcal{O}} = \tilde{k}_{\mathcal{O}}$  on the orbit  $\mathcal{O}$ .

Finally, let's define  $\tilde{F}$  on  $\mathfrak{g}_{gen}^*$  by: Given  $l \in \mathfrak{g}_{gen}^*$ , let  $\mathcal{O} = \mathcal{O}(l)$  be the co-adjoint orbit of  $l$ . We then put  $\tilde{F}(l) := \tilde{f}_{\mathcal{O}}(l)$ . This function  $\tilde{F}$  is uniquely determined by the field  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$ . It may be computed by the formula

$$\tilde{F}(l) = \int_{G/G(l)} \left[ \int_{G/P(l)} F(l, v \cdot s, s) d\dot{s} \right] e^{-2\pi i \langle l, v \rangle} d\dot{v}.$$

As in section 4, we give the following definitions:

**9.8. Definition.** We say that  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a  $\mathcal{C}^\infty$ -field of operator kernels, if  $\tilde{F}$  is  $\mathcal{C}^\infty$  on  $\mathfrak{g}_{gen}^*$  and if it may be extended to a  $\mathcal{C}^\infty$  function on all of  $\mathfrak{g}^*$ .

We say that  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a Schwartz field of operator kernels, if  $\tilde{F}$  is Schwartz on  $\mathfrak{g}_{gen}^*$  and if it may be extended to a Schwartz function on all of  $\mathfrak{g}^*$ .

We say that  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a  $\mathcal{C}^\infty$ -field with compact support, if  $\tilde{F}$  may be extended to a  $\mathcal{C}^\infty$  function with compact support in  $\mathfrak{g}^*$ .

We then have the following result:

**Theorem 9.9.** *Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group with a fixed Jordan-Hölder basis. Assume that there exists a  $G$ -invariant subset  $\mathfrak{g}_{gen}^*$  of  $\mathfrak{g}_{Puk}^*$  (Pukanszky generic elements of  $\mathfrak{g}^*$ ) such that for every  $l \in \mathfrak{g}_{gen}^*$ , the co-adjoint orbit  $\mathcal{O}(l)$  is flat, i. e.  $\mathcal{O}(l) = \mathcal{O}(l)|_{\mathfrak{g}(l)} + \mathfrak{g}(l)^\perp$ . The  $G$ -invariant measures on the quotients  $G/G(l)$  are normalized such that the trace formula holds. Let then  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be a field of operator kernels. The following are equivalent:*

- (i) *There exists  $f \in \mathcal{S}(G)$  such that  $\pi_l(f)$  has  $F(l, \cdot, \cdot) = F(l|_{\mathfrak{g}(l)}, \cdot, \cdot)$  as operator kernel for all  $l \in \mathfrak{g}_{gen}^*$ .*
- (ii) *The function*

$$\tilde{F}(l) = \int_{G/G(l)} \left[ \int_{G/P(l)} F(l, v \cdot s, s) d\dot{s} \right] e^{-2\pi i \langle l, v \rangle} d\dot{v}$$

*is Schwartz on  $\mathfrak{g}_{gen}^*$  and may be extended to a Schwartz function on all of  $\mathfrak{g}^*$ .*

**Proof:** (i)  $\Rightarrow$  (ii) This is true as  $\tilde{F}(l) = \hat{f}(l)$  for every  $l \in \mathfrak{g}_{gen}^*$  and as  $f \in \mathcal{S}(G) \equiv \mathcal{S}(\mathfrak{g})$ .

(ii)  $\Rightarrow$  (i) If  $\tilde{F} \in \mathcal{S}(\mathfrak{g}^*)$ , we put  $f$  to be the inverse Fourier transform of  $\tilde{F}$ . Then  $f \in \mathcal{S}(\mathfrak{g}) \equiv \mathcal{S}(G)$ . One has  $\tilde{f}(l) = \hat{f}(l) = \tilde{F}(l)$  for all  $l \in \mathfrak{g}_{gen}^*$ . Let  $(F'(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be the field of operator kernels corresponding to the  $\pi_l(f)$ 's and  $\tilde{F}'$  the associated function on  $\mathfrak{g}^*$ . Then  $\tilde{F}'(l) = \hat{f}(l) = \tilde{F}(l)$  for all  $l \in \mathfrak{g}_{gen}^*$  and, for any  $l_0 \in \mathfrak{g}_{gen}^*$  and  $\mathcal{O} = \mathcal{O}(l_0)$ ,

$$\|F(l_0, \cdot, \cdot) - F'(l_0, \cdot, \cdot)\|_2^2 = \int_{\mathcal{O}} |\tilde{F}(l) - \tilde{F}'(l)|^2 d\mu(l) = 0.$$

So  $F'(l_0, \cdot, \cdot) = F(l_0, \cdot, \cdot)$  and  $\pi_{l_0}(f)$  has  $F(l_0, \cdot, \cdot)$  as operator kernel.  $\square$

Finally we get the Paley-Wiener theorem for the case of flat orbits:

**Theorem 9.10.** *Let  $G = (\mathfrak{g}, \cdot_{CBH})$  be a connected, simply connected, nilpotent Lie group with a fixed Jordan-Hölder basis. Assume that there exists a  $G$ -invariant subset  $\mathfrak{g}_{gen}^*$  of  $\mathfrak{g}_{Puk}^*$  (Pukanszky generic elements of  $\mathfrak{g}^*$ ) such that for every  $l \in \mathfrak{g}_{gen}^*$ , the co-adjoint orbit  $\mathcal{O}(l)$  is flat, i. e.  $\mathcal{O}(l) = \mathcal{O}(l)|_{\mathfrak{g}(l)} + \mathfrak{g}(l)^\perp$ . The  $G$ -invariant measures on the quotients  $G/G(l)$  are normalized such that the trace formula holds. Let then  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  be a field of operator kernels. The following are equivalent:*

(i) *There exists  $f \in \mathcal{S}(G)$  such that  $\pi_l(f)$  has  $F(l, \cdot, \cdot)$  as operator kernel for all  $l \in \mathfrak{g}_{gen}^*$ . The function  $f$  may be extended to an entire function on all of  $G_{\mathbb{C}} \equiv \mathfrak{g}_{\mathbb{C}}$  and*

$$|f(z)| \leq \gamma_N (1 + \|z\|)^{-N} e^{2\pi r \|\Im z\|}, \quad \forall z \in G_{\mathbb{C}}, \forall N \in \mathbb{N}.$$

(ii)  $(F(l, \cdot, \cdot))_{l \in \mathfrak{g}_{gen}^*}$  is a  $\mathcal{C}^\infty$ -field with compact support contained in  $rB$ . This means that the function

$$\tilde{F}(l) = \int_{G/G(l)} \left[ \int_{G/P(l)} F(l, v \cdot s, s) ds \right] e^{-2\pi i \langle l, v \rangle} dv$$

may be extended to a  $\mathcal{C}^\infty$ -function to all of  $\mathfrak{g}^*$ , whose support is contained in  $rB$ .

If (ii) holds, then  $f$  is the inverse Fourier transform of the function  $\tilde{F}$ . Its extension to  $G_{\mathbb{C}}$  is obtained by the complex inverse Fourier transform of  $\tilde{F}$ .

If (i) and (ii) hold and if  $\pi_l(f) \neq 0$  for some  $l \in \mathfrak{g}_{gen}^*$ , then  $\mathcal{O}(l) \cap rB \neq \emptyset$ .

**Proof:** This is due to the classical Paley-Wiener theorem, to theorem 9.9 and to the fact that  $\tilde{F}(l) = \hat{f}(l)$  for all  $l \in \mathfrak{g}_{gen}^*$ .  $\square$

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Jean Ludwig, *Département de Mathématiques, Université Paul Verlaine de Metz, Ile de Saulcy, F-57045 Metz cedex 1, France, ludwig@poncelet.sciences.univ-metz.fr*

Carine Molitor-Braun, *Laboratoire de Mathématiques, Université du Luxembourg, 162A, avenue de la Faïencerie, L-1511 Luxembourg, Luxembourg, carine.molitor@uni.lu*