Analysis of Restrictions of Unitary Representations of a Nilpotent Lie Group

by

Ali Baklouti [a,⋆], Hidenori Fujiwara [b] and Jean Ludwig [c]

[a] *Faculté des Sciences de Sfax, Département de Mathématiques, Route de Soukra, 3038 Sfax, Tunisie. e-mail: Ali.Baklouti@fss.rnu.tn(⋆ Corresponding author).*

[b] *Université de Kinki, Faculté de Technologie à Kyushu, 820-8555 Iizuka, Japon. e-mail: fujiwara@fuk.kindai.ac.jp*

[c] *Département de Mathématiques, Faculté des Sciences, Université de Metz, Ile du Saulcy, 57045 Metz CEDEX 01, France. e-mail: ludwig@poncelet.sciences.univ-metz.fr*

Abstract. Let G be a connected simply connected nilpotent Lie group, K an analytic subgroup of G and π an irreducible unitary representation of G. Let $D_\pi(G)^K$ be the algebra of differential operators keeping invariant the space of C^∞ vectors of π and commuting with the action of K on that space. In this paper, we assume that the restriction of π to K has finite multiplicities and we show that $D_\pi(G)^K$ is isomorphic to a subalgebra of the field of rational $K-$invariant functions on the co-adjoint orbit $\Omega(\pi)$ associated to π, and for some particular cases, that $D_\pi(G)^K$ is even isomorphic to the algebra of polynomial $K-$invariant functions on $\Omega(\pi)$. We prove also the Frobenius reciprocity for some restricted classes of nilpotent Lie groups, especially in the cases where K is normal or abelian.

Résumé. Soit G un groupe de Lie nilpotent connexe et simplement connexe, K un sous-groupe analytique de G et π une représentation unitaire et irréductible G. Soit $D_\pi(G)^K$ l'algèbre des opérateurs différentiels qui laissent invariant l'espace des vecteurs C^∞ de π et qui commuttent avec l'action de K sur cet espace. Nous prouvons dans ce papier que sous l'hypothèse que la restriction de π à K est à multiplicités finies, l'algèbre $D_\pi(G)^K$ est isomorphe à une sous-algèbre du corps des fonctions rationnelles $K-$invariantes sur l'orbite co-adjointe $\Omega(\pi)$ assoçiée à π, et dans certains cas particuliers que $D_\pi(G)^K$ est même isomorphe à l'algèbre des fonctions polynomiales $K-$invariantes sur $\Omega(\pi)$. Nous prouvons aussi la réciprocité de Frobenius pour quelques classes de groupes de Lie nilpotents, particulièrement les cas où K est normal ou abélien.

1. Introduction and notations

1.1 It is well known that there exists a strong parallelism between inductions and restrictions of representations of locally compact groups. Monomial representations of nilpotent Lie groups have been analyzed in detail: the canonical central disintegration in [4],[9],[16],[27], Plancherel formula in [5],[6],[14],[17],[31], the associated algebra of invariant differential operators in [2],[11],[19],[20],[21],[22] and the Frobenius reciprocity in [6],[23],[31].

Concerning the restriction, similar investigations have begun, but much less has been done so far: the canonical central disintegration has been studied in [10],[18] and the associated algebra of invariant differential operators in [2],[3]. In this paper we continue the analysis of the restriction by looking at Frobenius vectors and the Frobenius reciprocity.

1.2 Let $G = \exp(\mathfrak{g})$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We denote by $\widehat{G}$ the unitary dual of G, i.e. the set of all equivalence classes of irreducible unitary representations of G. We shall sometimes identify the equivalence class $[\pi]$ with its representative π and we denote the equivalence relation between two representations π_1 and π_2 by $\pi_1 \simeq \pi_2$ or even by $\pi_1 = \pi_2$.

1.3 Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. By Kirillov's orbit theory, $\widehat{G}$ can be realized as the space of coadjoint orbits $\mathfrak{g}^*/G$ by means of Kirillov's mapping $\Theta = \Theta_G : \mathfrak{g}^*/G \to \widehat{G}$ (cf. [26]). We designate by the same notation Θ its pull-back $\mathfrak{g}^* \to \widehat{G}$ too. Let us write $\Omega(\pi) = \Omega_G(\pi)$ for the Kirillov-orbit $\Theta^{-1}(\pi)$ of π and also π_l, $l \in \mathfrak{g}^*$, for the irreducible representation $\Theta_G(G \cdot l)$.

1.4 Let π be an irreducible unitary representation of G. We restrict π to an analytic subgroup $K = \exp(\mathfrak{k})$ of G and we denote by $\pi_{|K}$ this restriction.

1.5 Let us recall the canonical central disintegration of $\pi_{|K}$ (cf. [10],[18]). Let μ_π be a finite measure on $\mathfrak{g}^*$, which is equivalent to the G-invariant measure on the coadjoint orbit $\Omega(\pi)$. We consider the image $\mu = (\Theta_K \circ p)_*(\mu_\pi)$ of μ_π under the mapping $\Theta_K \circ p : \mathfrak{g}^* \to \widehat{K}$, where $p : \mathfrak{g}^* \to \mathfrak{k}^*$ stands for the canonical projection. For $\sigma \in \widehat{K}$, let $n_\pi(\sigma)$ be the number of $K-$ orbits contained in $\Gamma(\pi, \sigma) = \Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$. Then

$$\pi_{|K} \simeq \int_{\widehat{K}}^{\oplus} n_\pi(\sigma)\sigma d\mu(\sigma). \tag{1.5.1}$$

On the other hand, if we disintegrate the representation $\mathrm{ind}_K^G \sigma$ for $\sigma \in \widehat{K}$, it follows from [16] that

$$\mathrm{ind}_K^G \sigma \simeq \int_{\widehat{G}}^{\oplus} n_\pi(\sigma)\pi d\nu(\pi) \tag{1.5.2}$$

for a certain measure ν on $\widehat{G}$. Hence, the Frobenius reciprocity holds in this situation. In these two cases of restriction and induction it happens that the multiplicities appearing in the disintegration are either uniformly bounded or infinite (cf. [9],[10],[27]). According to these situations we say that $\pi_{|K}$ or $\mathrm{ind}_K^G\sigma$ is of finite multiplicities (resp. of infinite multiplicities).

1.6 Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}$ and let $\ker(\pi)$ be the primitif ideal in $\mathcal{U}(\mathfrak{g})$ associated to π. The algebra

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \{A \in \mathcal{U}(\mathfrak{g}); [A, \mathfrak{k}] \subset \ker(\pi)\}$$

and its image $D_\pi(G)^K$ under the homomorphism π have been studied in two preceding papers (cf. [2],[3]).

1.7 Let us introduce other ingredients of the theory. We denote by $\mathcal{H}_\pi, \mathcal{H}_\pi^\infty, \mathcal{H}_\pi^{-\infty}$ the Hilbert space of π, (resp. the subspace of the C^∞-vectors of $\mathcal{H}_\pi$, resp. the anti-dual space of $\mathcal{H}_\pi^\infty$) (cf. [8],[32]). For $a \in \mathcal{H}_\pi^{\pm\infty}$ and $b \in \mathcal{H}_\pi^{\mp\infty}$ we write $\langle a, b\rangle$ for the image of b by a, which gives us the relation $\langle a, b\rangle = \overline{\langle b, a\rangle}$. For an element $W \in \mathcal{U}(\mathfrak{g})$, we then have

$$\langle \pi(W)a, b\rangle = \langle a, \pi(W^*)b\rangle,$$

where $W \mapsto W^*$ denotes the natural involution of $\mathcal{U}(\mathfrak{g})$.

1.8 For a subgroup H of G and a unitary character χ of H, let

$$(\mathcal{H}_\pi^{-\infty})^{H,\chi} = \{a \in \mathcal{H}_\pi^{-\infty};\ \pi(h)a = \chi(h)a, \forall h \in H\}.$$

1.9 Let us consider a unipotent representation of G on a real finite dimensional vector space V. Let $v \in V$ be a G-invariant vector. For a fixed vector $x \in V$, let $L_x = x + \mathbb{R}v$, the line of direction v passing through x. Then we have two possibilities: either $L_x \cap G\cdot x = L_x$ or $L_x \cap G \cdot x = \{x\}$ (cf. [33]). In the first case we say that the $G-$orbit $G \cdot x$ is saturated in the direction $\mathbb{R}v$, in the second case that it is not saturated in the direction $\mathbb{R}v$.

1.10 We shall apply in the following this fact to the coadjoint representation of G (or of a subgroup of G) [33]. Here, the invariant vector is a linear form which is zero on an ideal $\mathfrak{g}'$ of codimension 1 in $\mathfrak{g}$. In this situation we say that the orbit $\Omega = \Omega(\pi)$ in question is either saturated or not saturated with respect to $\mathfrak{g}'$. If the orbit $\Omega(\pi)$ is saturated with respect to $\mathfrak{g}'$, then the projection $\gamma(\Omega(\pi))$ of Ω to $\mathfrak{g}'$ is the union of a one parameter family ω_t ($t \in \mathbb{R}$) of G'-orbits ($G' = \exp(\mathfrak{g}')$) and there exists an element $Y_l \in \mathfrak{g}'$ which depends smoothly on l, such that $\mathrm{Ad}^*(\exp(\mathbb{R}Y_l))l = l + \mathfrak{g}'^{\perp}$ for all $l \in \Omega(\pi)$, where $\mathfrak{g}'^{\perp} = \{\phi \in \mathfrak{g}^*; \phi_{|\mathfrak{g}'} = 0\}$. Fix a vector $X \in \mathfrak{g} \setminus \mathfrak{g}'$ and define the mapping $\iota : \mathfrak{g}'^* \to \mathfrak{g}^*$ by

$$\langle \iota(l'), X\rangle = 0, \iota(l')_{|\mathfrak{g}'} = l',\ l' \in \mathfrak{g}'^*.$$

The mapping

$$\mathbb{R} \times \mathbb{R} \times \omega_0 \to \Omega :\ (s, t, l') \to\ \mathrm{Ad}^*(\exp(tX)) \circ\ \mathrm{Ad}^*(\exp(sY_{\iota(l')}))\iota(l') = \psi(s, t, l')$$

is then a diffeomorphism and the invariant measure $d\mu_\pi$ on Ω can be decomposed as

$$\int_\Omega \varphi(l)d\mu_\pi(l) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\omega_0} \varphi \circ \psi(s, t, l')d\mu_{\pi'}(l')dtds, \varphi \in C_c(\Omega). \tag{1.10.1}$$

Here $d\mu_{\pi'}$ denotes the invariant measure on $\omega_0 = G' \cdot l'$ where $\pi' = \Theta_{G'}(\omega_0)$. According to this decomposition of $\Omega(\pi)$, the representation $\pi_{|G'}$ disintegrates into an integral

$$\pi_{|G'} \simeq \int_{\mathbb{R}}^{\oplus} \pi'_t dt$$

of irreducible representations π'_t (where $\Omega_{G'}(\pi'_t) = \omega_t = \exp(tX) \cdot \omega_0,\ t \in \mathbb{R}$) of G'.

1.11 For a linear form $l \in \Omega(\pi)$, let $\mathfrak{b}[l]$ denote a polarization at l. By $\mathfrak{g}(l)$, we denote the radical of the skew-symmetric bilinear form $B_l : B_l(x,y) = \langle l, [x,y]\rangle,\ x,y \in \mathfrak{g}$. Hence $\mathfrak{g}(l) = \{x \in \mathfrak{g};\ \langle l, [x,y]\rangle = 0,\ \forall y \in \mathfrak{g}\}$.

1.12 We recall here e-central elements of Corwin-Greenleaf [11]. Let $\{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}, \{X_1^*, \ldots, X_n^*\}$ the dual basis of $\mathfrak{g}^*$ and $(l_1, \ldots, l_n)$ the dual coordinates $l_j = l(X_j)$ of an element $l \in \mathfrak{g}^*$. Then, for $1 \leq j \leq n$, $\mathfrak{g}_j = \langle X_1, \ldots, X_j\rangle_{\mathbb{R}} = \sum_{i=1}^{j} \mathbb{R}X_i$ is an ideal of $\mathfrak{g}$, ${\mathfrak{g}_j}^{\perp} = \langle X_{j+1}^*, \ldots, X_n^*\rangle_{\mathbb{R}} \subset \mathfrak{g}^*$ and ${\mathfrak{g}_j}^* \cong \mathfrak{g}^*/{\mathfrak{g}_j}^{\perp}$. Let $p_j : \mathfrak{g}^* \to {\mathfrak{g}_j}^*$ be the restriction mapping, which intertwines the coadjoint actions of G on $\mathfrak{g}^*, {\mathfrak{g}_j}^*$. For $l \in \mathfrak{g}^*$, write $e_j(l) = \dim(G{\cdot}p_j(l)), e(l) = (e_1(l), \ldots, e_n(l))$, and define the set of dimension indices $\mathcal{E} = \{e(l),\ l \in \mathfrak{g}^*\}$. For $e \in \mathcal{E}$, define the G-invariant e-layer $U_e = \{l \in \mathfrak{g}^*; e(l) = e\}$ and, setting $e_0 = 0$, define

$$S(e) = \{1 \leq j \leq n; e_j = e_{j-1} + 1\}, T(e) = \{1 \leq j \leq n; e_j = e_{j-1}\}.$$

Let $e \in \mathcal{E}$. We say that $A \in \mathcal{U}(\mathfrak{g})$ is e-central if $\pi_l(A)$ is scalar for all $l \in U_e$. Then, there is a Zariski open set $\mathcal{L} \subset \mathfrak{g}^*$ such that $\mathcal{L} \cap U_e$ is non-empty and G-invariant, and there exists $A_j \in \mathcal{U}(\mathfrak{g}_j)$ for each $j \in T(e)$, with the following properties:

1) Each A_j is e-central on $\mathcal{L} \cap U_e$, i.e. $\pi_l(A_j)$ is scalar for $l \in \mathcal{L} \cap U_e$, and $A_j = P_j X_j + Q_j$ such that

- i. P_j is a polynomial in the A_k such that $k \in T(e)$ and $k < j$; in particular $P_j \in \mathcal{U}(\mathfrak{g}_{j-1})$,
- ii. P_j is e-central on $\mathcal{L} \cap U_e$,
- iii. $Q_j \in \mathcal{U}(\mathfrak{g}_{j-1})$, in particular $P_1, Q_1 \in \mathbb{C}\mathbb{I}$.

2) $\pi_l(P_j) \neq 0$ for all $l \in \mathcal{L} \cap U_e$.

3) $\pi_l(A_j) = \phi_j(l)\mathbb{I}$ for $l \in \mathcal{L} \cap U_e$, where $\phi_j(l) = \tilde{p}_j(l')l_j + \tilde{q}_j(l'); \tilde{p}_j, \tilde{q}_j$ are rational non-singular functions on $\mathcal{L} \cap U_e$ and depend only on $l' = (l_1, \ldots, l_{j-1})$.

4) $\tilde{p}_j(l)$ is G-invariant and is never zero on $\mathcal{L} \cap U_e$.

Having bitten the Zariski open set $\mathcal{L} \cap U_e$ out of U_e, we may repeat the whole process to get the same result on the remaining part. Thus we can refine the layering, keeping the same orbit parametrization within each sublayer of U_e, and treat each piece as a layer in its own right on which the above result is valid with $\mathcal{L} \cap U_e = U_e$ (cf. also [20]).

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2. Frobenius Vectors

We keep our notations, i.e. $G = \exp(\mathfrak{g})$ is a connected simply connected nilpotent Lie group, $K = \exp(\mathfrak{k})$ an analytic subgroup of G and $\pi \in \widehat{G}$. We begin with the proof of

2.1 Lemma: *The representation $\pi_{|K}$ has finite multiplicities if and only if for μ_π-almost every $l \in \Omega(\pi)$ the subspace $\mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l)$ is lagrangian for the skew-symmetric bilinear form B_l, where $\mathfrak{b}[l_{|\mathfrak{k}}]$ denotes any polarization at $l_{|\mathfrak{k}}$.*

Proof: Let us proceed by induction on $\dim(\mathfrak{g})$. Let $\mathfrak{g}'$ be a subalgebra of codimension 1 containing $\mathfrak{k}$, let $G' = \exp(\mathfrak{g}')$ and $\gamma : \mathfrak{g}^* \to \mathfrak{g}'^*$ be the canonical projection. Finally let $l' = \gamma(l) \in \mathfrak{g}'^*$. If the orbit $\Omega(\pi)$ is not saturated with respect to $\mathfrak{g}'$, then $\gamma(\Omega(\pi)) = G' \cdot l'$ and $\pi' = \pi_{|G'}$ is irreducible. Since $\dim(\mathfrak{g}(l)) = \dim(\mathfrak{g}'(l')) + 1$ and since a subspace $\mathfrak{p}'$ of $\mathfrak{g}'$ is lagrangian for $B_{l'}$ if and only if $\mathfrak{p}' + \mathfrak{g}(l)$ is lagrangian for B_l, we see that the induction hypothesis applied to $\mathfrak{g}'$ and l' gives us the desired result.

If the orbit $\Omega(\pi)$ is saturated with respect to $\mathfrak{g}'$, then $\gamma(\Omega(\pi))$ is the union of a one parameter family $\omega_t (t \in \mathbb{R})$ of G'-orbits. It follows from [3] that $\pi_{|K}$ is of finite multiplicities, if and only if $\pi'_{t|K}$ is of finite multiplicities for almost all $t \in \mathbb{R}$ and the orbit $K \cdot l$ is saturated with respect to $\mathfrak{g}'$ for μ_π-almost all $l \in \Omega(\pi)$. By the induction hypothesis we know that for every $t \in \mathbb{R}$ such that $\pi'_{t|K}$ is of finite multiplicities, the subspaces $\mathfrak{b}[l'_{t|\mathfrak{k}}] + \mathfrak{g}'(l'_t)$ are lagrangian for $B_{l'_t}$ at $\mu_{\pi'_t}$-almost all $l'_t \in \omega_t$. We conclude from this and from the description 1.10 of $\Omega(\pi)$ that if $\pi_{|K}$ is of finite multiplicities, then μ_π-almost everywhere $\mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}'(l')$ is lagrangian with respect to $B_{l'}$. Furthermore, since $\Omega(\pi)$ is saturated with respect to $\mathfrak{g}'$, for every $l \in \Omega(\pi)$ and $l' = \gamma(l) \in \mathfrak{g}'^*$, we have that $\mathfrak{g}'(l') = \mathfrak{g}(l) + \mathbb{R}Y_l$, where Y_l is as in 1.10. Hence for every l such that $K \cdot l \supset l + \mathfrak{g}'^{\perp}$, we can find an Z_l in $\mathfrak{k}(l_{|\mathfrak{k}}) \subset \mathfrak{b}[l_{|\mathfrak{k}}]$ for which $0 \neq Z_l \cdot l := ad^*(Z_l)(l) \in \mathfrak{g}'^{\perp}$. It follows from these considerations that the condition $\pi_{|K}$ is of finite multiplicities implies that

$$\mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l) = \mathfrak{b}[l'_{|\mathfrak{k}}] + \mathfrak{g}'(l')$$

is lagrangian for B_l μ_π- almost everywhere on $\Omega(\pi)$.

If on the other hand $\mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l)$ is lagrangian μ_π-almost everywhere, then we can choose a Z_l in $\mathfrak{k}$ such that $\langle l, [Z_l, \mathfrak{g}'] \rangle = (0)$ and $\langle l, [X, Z_l] \rangle = 1$ and so $K \cdot l$ is saturated with respect to $\mathfrak{g}'$ for all these l's. Hence we also have that $\mathfrak{b}[l'_{|\mathfrak{k}}] + \mathfrak{g}'(l) = \mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l)$ is lagrangian for $B_{l'}$ at those l's. The induction hypothesis and the structure of $\Omega(\pi)$ (see 1.10) tell us now that for almost all t in $\mathbb{R}$, $\pi'_{t|K}$ is of finite multiplicity. Whence $\pi_{|K}$ is of finite multiplicity too.

■

2.2 Remark:

By the Frobenius reciprocity in the disintegrations (1.5.1) and (1.5.2) one might think that generically $\pi_{|K}$ is of finite multiplicities if and only if $\mathrm{ind}_K^G\sigma$ is of finite multiplicities for $\sigma \in \widehat{K}$ μ-almost everywhere. This last statement implies again (see [9]) that $\mathfrak{b}[l_{|\mathfrak{k}}]+\mathfrak{g}(l)$ is generically a lagrangian subspace.

2.3 Let

$$\mathcal{S} : \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$$

a flag of ideals of $\mathfrak{g}$ such that $\dim(\mathfrak{g}_k) = k$ for $0 \leq k \leq n$. Choosing for every $j \in \{1, \cdots, n\}$ a vector Z_j in $\mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$, we obtain a Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ of $\mathfrak{g}$. Let $\mathcal{I}^{\mathfrak{k}} = \mathcal{I} = \{i_1 < \cdots < i_d\}$, $(d = \dim(\mathfrak{k}))$ be the set of indices i $(1 \leq i \leq n)$ such that $\mathfrak{k} \cap \mathfrak{g}_i \neq \mathfrak{k} \cap \mathfrak{g}_{i-1}$. Let us put

$$\mathcal{J}^{\mathfrak{g}/\mathfrak{k}} = \mathcal{J} = \{j_1 < \cdots < j_q\} = \{1, ..., n\} \setminus \mathcal{I}$$

with $q = \dim(\mathfrak{g}/\mathfrak{k})$.

We obtain an increasing sequence of subalgebras $\mathfrak{l}_r, r = 0, \cdots, q$ with $\dim(\mathfrak{l}_r/\mathfrak{l}_{r-1}) = 1, \mathfrak{l}_q = \mathfrak{g}$, of $\mathfrak{g}$ by setting

$$\mathfrak{l}_0 = \mathfrak{k},\ \mathfrak{l}_r = \mathfrak{k} + \mathfrak{g}_{j_r},\ r = 1, \cdots, q. \tag{2.3.1}$$

Considering $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{g}_{i_s}$ $(i_s \in \mathcal{I})$, we produce a flag of ideals of $\mathfrak{k}$:

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_d = \mathfrak{k},\ \dim(\mathfrak{k}_s) = s\ (0 \leq s \leq d). \tag{2.3.2}$$

We fix now a vector Y_s of $\mathfrak{k}_s \setminus \mathfrak{k}_{s-1}$ for $1 \leq s \leq d$ and we obtain a Jordan-Hölder basis $\{Y_1, \cdots, Y_d\}$ of $\mathfrak{k}$. In the same way, extracting an element Y_{d+r} of $\mathfrak{l}_r \setminus \mathfrak{l}_{r-1}$ for $1 \leq r \leq q$, we form a Malcev basis $\{Y_{d+1}, \cdots, Y_n\}$ of $\mathfrak{g}$ relative to $\mathfrak{k}$.

2.4 Let $l \in \Omega(\pi)$. Taking a real polarisation $\mathfrak{b}[l]$ at l, we realize the representation π as $\pi = \mathrm{ind}_{B[l]}^G\chi_l$, where $B[l] = \exp(\mathfrak{b}[l])$ and $\chi_l(\exp(X)) = e^{il(X)},\ X \in \mathfrak{b}[l]$. We can use the flag described in (2.3.2) to construct the Vergne polarization $\mathfrak{b}[l_{|\mathfrak{k}}]$ at $l_{|\mathfrak{k}} \in \mathfrak{k}^*$ (see [36]). Let $B[l_{|\mathfrak{k}}] = \exp(\mathfrak{b}[l_{|\mathfrak{k}}])$. It is easy to see that there exists a Malcev basis of $\mathfrak{b}[l_{|\mathfrak{k}}]$ relative to $\mathfrak{b}[l_{|\mathfrak{k}}] \cap \mathfrak{b}[l]$ which is contained in a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{b}[l]$. From this it follows directly that the formula

$$\langle a_l, \varphi\rangle = \int_{B[l_{|\mathfrak{k}}]/B[l_{|\mathfrak{k}}]\cap B[l]} \overline{\varphi(b)\chi_l(b)}d\dot{b}\ (\varphi \in \mathcal{H}_\pi^\infty), \tag{2.4.1}$$

where $d\dot{b}$ denotes an invariant measure on the homogeneous space $B[l_{|\mathfrak{k}}]/B[l_{|\mathfrak{k}}] \cap B[l]$, defines a generalized semi-invariant vector $a_l \in (\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$. Suppose that $\pi_{|K}$ is of finite multiplicities. Then according to Lemma 2.1, we know that $\mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l)$ is μ_π-almost everywhere a lagrangian subspace for the bilinear form B_l. For such an l, we know that the choice of the polarization $\mathfrak{b}[l]$ does not matter for the definition of the distribution a_l (see [14]). In fact, if $\mathfrak{b}'[l]$ is another polarization at l and $B'[l] = \exp(\mathfrak{b}'[l])$, then we can form the distribution a'_l as in (2.4.1) and if T denotes an intertwining operator for these two realizations of π, then

$$ca_l = a'_l \circ T$$

for some complex number c. **2.5 Theorem**: *Suppose that $\pi_{|K}$ is of finite multiplicities. Then μ_π-almost everywhere on $\Omega(\pi)$, the distribution a_l is an eigen-vector for every element in $D_\pi(G)^K$. In other words, for $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, we have that*

$$\pi(W)a_l = P_W(l)a_l$$

for μ_π-almost all $l \in \Omega(\pi)$ with certain complex scalars $P_W(l)$. Furthermore the function $l \mapsto P_W(l)$ is K-invariant.

Proof: If $G = K$, then the algebra $D_\pi(G)^K$ is reduced to $\mathbb{C}\mathbb{I}$ according to Schur's lemma and there is nothing to prove. Let now $q > 0$ in the sequence (2.3.1) and let us proceed by induction on $\dim(G)$. Let $G' = \exp(\mathfrak{l}_{q-1})$ and $l' = l_{|\mathfrak{l}_{q-1}}$. Suppose first that $\Omega(\pi)$ is saturated with respect to $\mathfrak{l}_{q-1}$. According to 2.4, we can choose the polarization $\mathfrak{b}[l]$ as we want, so we can assume that $\mathfrak{b}[l] \subset \mathfrak{l}_{q-1}$. Let $\pi' = \mathrm{ind}_{B[l]}^{G'}\chi_l$ with $B[l] = \exp(\mathfrak{b}[l])$. Then a_l can be identified with $a_{l'}$. We know from [3], that $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \subset \mathcal{U}(\mathfrak{l}_{q-1}) + \ker(\pi)$. Hence we can apply the induction hypothesis to π' and $a_{l'}$.

Suppose now that $\Omega(\pi)$ is not saturated with respect to $\mathfrak{l}_{q-1}$. Then $\pi' = \pi_{|G'} = \mathrm{ind}_{B[l']}^{G'}\chi_{l'} \in \widehat{G'}$ (where $B[l'] = \exp(\mathfrak{b}[l] \cap \mathfrak{l}_{q-1})$) and a result of Pedersen (see [30]) says that there exists an element $A \in \ker(\pi)$ of the form $A = Y_n + V$ with $V \in \mathcal{U}(\mathfrak{l}_{q-1})$. Let $W = \sum_{j=0}^{L} Y_n^j V_j$ (where $V_j \in \mathcal{U}(\mathfrak{l}_{q-1})$, $0 \leq j \leq L$) be an element in $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Replacing in the expression of W the vector Y_n by the element A, we see that $W \in \mathcal{U}_{\pi'}(\mathfrak{l}_{q-1})^{\mathfrak{k}} + \ker(\pi)$. On the other hand, since $\mathfrak{k} \subset \mathfrak{l}_{q-1}$, we can identify a_l with $a_{l'} \in \mathcal{H}_{\pi'}^{-\infty}$. These considerations allow us to descend to G' and π', where the induction hypothesis applies. The function $P_W(l)$ is easily checked to be K-invariant.

■

3. The function P_W on $\Omega(\pi)$

3.1 We suppose again that $\pi_{|K}$ has finite multiplicities. Putting $\mathfrak{k}_{d+j} = \mathfrak{l}_j$ for $1 \leq j \leq q$, we have a sequence of subalgebras:

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_d = \mathfrak{k} \subset \mathfrak{k}_{d+1} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g},\ \dim(\mathfrak{k}_r/\mathfrak{k}_{r-1}) = 1 \tag{3.1.1}$$

and a Malcev basis $\{Y_1, \cdots, Y_n\}$ of $\mathfrak{g}$. Let $K_j = \exp(\mathfrak{k}_j)$ for $0 \leq j \leq n$. Let $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ and let $P_W : l \mapsto P_W(l)$ be the function defined μ_π-almost everywhere on $\Omega(\pi)$ by Theorem 2.5. We shall show that this function is rational on $\Omega(\pi)$. First we need

3.2 Lemma: *Let $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Then $P_W \equiv 0$, if and only if $W \in$ ker(π).*

Proof: Suppose first that $P_W \equiv 0$. Let us proceed by induction on $\dim(G)$. Put $\mathfrak{g}' = \mathfrak{k}_{n-1}$, $G' = \exp(\mathfrak{g}')$ and denote by $\gamma : \mathfrak{g}^* \to \mathfrak{g}'^*$ the canonical projection. The proof of Theorem 2.5 shows that W reduces modulo $\ker(\pi)$ to an element $W' \in \mathcal{U}(\mathfrak{g}')$ which is contained in $\mathcal{U}_{\pi'}(\mathfrak{g}')^{\mathfrak{k}}$ for $(\Theta_{G'} \circ \gamma)_*(\mu_\pi)$-almost all $\pi' \in \widehat{G'}$ and such that $P_{W'}(l') = 0$ for $\gamma_*(\mu_\pi)$-almost all $l' \in \gamma(\Omega(\pi))$. The induction hypothesis tells us that $\pi'(W') = 0$ for $(\Theta_{G'} \circ \gamma)_*(\mu_\pi)$-almost all $\pi' \in \widehat{G'}$ which implies that $\pi(W') = \pi(W) = 0$.

Assume now that $W \in \ker(\pi)$. Then of course W^* is in $\ker(\pi)$ too and for every $\xi \in \mathcal{H}_\pi^\infty$ we have by (2.4.1) that

$$\langle \pi(W)a_l, \xi\rangle = \langle a_l, \pi(W^*)\xi\rangle = 0 = P_W(l)\langle a_l, \xi\rangle.$$

Hence $P_W(l) \equiv 0$.

$\blacksquare$

3.3 Let us define two sets of indices S_K and T_K contained in $\{1, \cdots, n\}$:

$$S_K = \{j \in \{1, \cdots, n\}, \text{ there exists a Zariski open subset } \mathcal{S}_j \subset \Omega(\pi), \text{ such that}$$
$$\dim(K \cdot (l_{|\mathfrak{k}_j})) = \dim(K \cdot (l_{|\mathfrak{k}_{j-1}})) + 1 \ \ \forall\, l \in \mathcal{S}_j\}$$

and

$$T_K = \{1, \cdots, n\} \setminus S_K$$
$$= \{j \in \{1, \cdots, n\}, \text{ there exists a Zariski open subset } \mathcal{T}_j \subset \Omega(\pi), \text{ such that}$$
$$\dim(K \cdot (l_{|\mathfrak{k}_j})) = \dim(K \cdot (l_{|\mathfrak{k}_{j-1}})) \ \ \forall\, l \in \mathcal{T}_j\}.$$

Putting $\mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} = \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k} \cap \mathcal{U}(\mathfrak{k}_j)$ ($j \in \{1, \cdots, n\}$), we know from [3] that for $j \in S_K$, we have that

$$\mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} = \mathcal{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k} + \mathcal{U}(\mathfrak{k}_j)(\mathcal{U}(\mathfrak{k}_{j-1}) \cap \ker(\pi))$$

and if $j \in T_K$ then

$$\mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} \neq \mathcal{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k} + \mathcal{U}(\mathfrak{k}_j)(\mathcal{U}(\mathfrak{k}_{j-1}) \cap \ker(\pi))$$

and so for μ_π-almost all $l \in \Omega(\pi)$ the subalgebra $\mathfrak{k}_j(l_{|\mathfrak{k}_j})$ is not contained in $\mathfrak{k}_{j-1}$ under the assumption that $\pi_{|K}$ is of finite multiplicities and so there exists (see 1.12) a Corwin-Greenleaf element $W_j \in \mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k}$ having the form

$$W_j = a_j Y_j + b_j \ (a_j, b_j \in \mathcal{U}(\mathfrak{k}_{j-1}), \ a_j \notin \ker(\pi), \ a_j \in \mathcal{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k}, \ j \in T_K).$$

Furthermore, concerning the Corwin-Greenleaf elements $W_j, j \in T_K$, we have that $P_{W_j}(l) = \varphi_j(l) l_j + \psi_j(l)$, where $l_i = l(Y_i)$, $i \in \{1, \cdots, n\}$, $l \in \Omega(\pi)$ and φ_j, ψ_j are rational functions in $l_1, \cdots, l_{j-1}$.

3.4 Theorem: *Suppose that $\pi_{|K}$ is of finite multiplicities. Let $W \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$. Then the function P_W, which is defined μ_π-almost everywhere on $\Omega(\pi)$ by Theorem 2.5, extends to a rational $K-$invariant function on $\Omega(\pi)$. Furthermore, the homomorphism $\mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k} \ni W \mapsto P_W$ defines an imbedding of $D_\pi(G)^K$ into the field $\mathbb{C}(\Omega(\pi))^K$ of the rational K-invariant functions on $\Omega(\pi)$.*

Proof: The definition of the function P_W implies that $P_{W \cdot W'} = P_W P_{W'}$ for any $W, W' \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$.

Let $1 \leq m \leq n$ be the smallest index in the sequence (3.1.1) such that $W \in \mathcal{U}(\mathfrak{k}_m) + \ker(\pi)$.

Let us proceed by induction on m to show that P_W is rational on $\Omega(\pi)$. We first remark that in order to compute the eigenvalue $P_W(l)$ we can consider that $W \in \mathcal{U}(\mathfrak{k}_m)$ and replace $a_l \in \mathcal{H}_\pi^{-\infty}$ by $a_{l_{|\mathfrak{k}_m}} \in \mathcal{H}_\sigma^{-\infty}$, $\sigma = \Theta_{K_m}(l_{|\mathfrak{k}_m}) \in \widehat{K}_m$, repeating step by step the observation made in the proof of Theorem 2.5 to go down from $\mathfrak{g}$ to $\mathfrak{l}_{q-1}$.

If $m = 1$, then W is in the center of $\mathcal{U}(\mathfrak{k})$ modulo $\ker(\pi)$ and then $\Theta_K(f)(W) = \widehat{W}(f)\mathbb{I}$ for every $f \in \mathfrak{k}^*$, where the function $f \mapsto \widehat{W}(f)$ is a K-invariant polynomial function on $\mathfrak{k}^*$ [12]. Hence P_W is the restriction to $p(\Omega(\pi))$ of the polynomial $\widehat{W}$.

Let us write $W = \sum_{k=0}^{r} Y_m^k w_k$ with $r > 0$ and $w_k \in \mathcal{U}(\mathfrak{k}_{m-1})$ for $0 \leq k \leq r$ satisfying $w_r \notin \ker(\pi)$. We shall use another induction on the degree r of W with respect to Y_m. If the element $w_r \in \mathcal{U}(\mathfrak{k}_{m-1})$, which appears in the expression of W is not contained in $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, then we can find elements $X_1, \cdots, X_a \in \mathfrak{k}$, such that $[X_1, [\cdots, [X_a, w_r] \cdots]] \notin \ker(\pi)$, but $[X, [X_1, [\cdots, [X_a, w_r] \cdots]]] \in \ker(\pi)$ for all $X \in \mathfrak{k}$, i.e. such that $u_r = [X_1, [\cdots, [X_a, w_r] \cdots]]$ is contained in $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \cap \mathcal{U}(\mathfrak{k}_{m-1}) \backslash \ker(\pi)$. The element $V = [X_1, [\cdots, [X_a, W] \cdots]$ of $\mathcal{U}(\mathfrak{g})$ is then contained in $\ker(\pi)$ and has the form $V = Y_m^r u_r +$ an element of $\sum_{k=0}^{r-1} Y_m^k \mathcal{U}(\mathfrak{k}_{m-1})$. We apply the induction hypothesis to $\tilde{W} = u_r W - V w_r \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ which is of degree $< r$ modulo $\ker(\pi)$ in Y_m. Hence, by the induction hypothesis on m and r and the multiplicativity property of P_W, the function $P_{\tilde{W}} = P_{u_r} P_W - P_V P_{w_r} = P_{u_r} P_W$ and then also (since $P_{u_r} \neq 0$ by 3.2)

$$P_W = \frac{P_{\tilde{W}}}{P_{u_r}}$$

admits an extension to a rational function on $\Omega(\pi)$. Hence we can assume now that $w_r \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Since $W \in \mathcal{U}(\mathfrak{k}_m)$, we know from [3] that $m \in T_K$ and so generically $\dim(K \cdot l_{|\mathfrak{k}_m}) = \dim(K \cdot l_{|\mathfrak{k}_{m-1}})$ $(l \in \Omega(\pi))$. The finite multiplicity condition implies now that generically $\dim(K_m \cdot l_{|\mathfrak{k}_m}) = \dim(K_m \cdot l_{|\mathfrak{k}_{m-1}})$ $(l \in \Omega(\pi))$ in the case where $m > d = \dim(\mathfrak{k})$. Anyhow we have now a Corwin-Greenleaf element $W_m = a_m Y_m + b_m$ with $a_m, b_m \in \mathcal{U}(\mathfrak{k}_{m-1})$, $a_m \notin \ker(\pi)$ and $a_m \in \mathcal{U}_\pi(\mathfrak{k}_{m-1})^{\mathfrak{k}}$. Hence the element

$$\tilde{W} = a_m^r W - W_m^r w_r$$

of $\mathcal{U}_\pi(\mathfrak{k}_m)^{\mathfrak{k}}$ is of degree $< r$ in Y_m modulo $\ker(\pi)$ and so we can apply the induction hypothesis to it. Hence $P_{\tilde{W}} = (P_{a_m})^r P_W - (P_{W_m})^r P_{w_r}$, P_{a_m} and P_{w_r} and therefore

$$P_W = \frac{P_{\tilde{W}} + (P_{W_m})^r P_{w_r}}{(P_{a_m})^r}$$

are also rational functions on $\Omega(\pi)$.

■

3.5 We have seen that for the Corwin-Greenleaf elements $W_j, j \in T_K$, the functions P_{W_j} have the form $P_{W_j}(l) = \varphi_j(l) l_j + \psi_j(l)$, where $l_i = l(Y_i)$, $i \in \{1, \cdots, n\}$, $l \in \Omega(\pi)$ and φ_j, ψ_j are rational functions in $l_1, \cdots, l_{j-1}$. Hence we obtain as a corollary of Theorem 3.4 the following result (see Theorem 5.4 in [11] for the case of the monomial representations).

3.6 Proposition: *Suppose that* $\pi_{|K}$ *is of finite multiplicities. Let* $A \in \mathcal{U}_\pi(\mathfrak{k}_m)^{\mathfrak{k}}$*. Then there exist two polynomials* β_A *and* γ_A *in the variables* $W_j, j \in T_K, j \leq m$*, such that* $\beta_A A \equiv \gamma_A$ *modulo* $ker(\pi)$*. In particular the functions* $\{P_{W_j};\ j \in T_K\}$ *are rational generators of the algebra* $\mathbb{C}(\Omega(\pi))^K$*.*

3.7 Remark:

Recalling the polynomial conjecture (see [11]) for monomial representations, it is natural to ask whether the functions P_W are polynomials or not.

Question: Does the mapping $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W$ give us by passing to quotients an algebra isomorphism of $D_\pi(G)^K$ onto the algebra $\mathbb{C}[\Omega(\pi)]^K$ of the K-invariant polynomials on $\Omega(\pi)$?

3.8 Proposition: *Assume that* $dim(\Omega(\pi)) \leq 2$*. If* $\pi_{|K}$ *is of finite multiplicities, the algebra* $D_\pi(G)^K$ *is isomorphic to* $\mathbb{C}[\Omega(\pi)]^K$*.*

Proof: As usual we use the induction on $\dim(G)$. It suffices to examine the case where $\mathfrak{k} \neq \mathfrak{g}$ and $\dim(\Omega(\pi)) = 2$. Let $l \in \Omega = \Omega(\pi)$. If l vanishes on a non-trivial ideal $\mathfrak{a}$ of $\mathfrak{g}$, we can descend to the quotient $\mathfrak{g}/\mathfrak{a}$ and apply the induction hypothesis. Suppose hereafter that l does not vanish on any non-trivial ideal of $\mathfrak{g}$, and take the sequence of subalgebras (3.1.1). As we already argued, if Ω is not saturated with respect to $\mathfrak{k}_{n-1}$, we can immediately descend to the subgroup $K_{n-1} = \exp(\mathfrak{k}_{n-1})$ to which applies the induction hypothesis.

Now assume that Ω is saturated with respect to $\mathfrak{k}_{n-1}$. At every point $l \in \Omega$, $\mathfrak{k}_{n-1}$ is a polarization. This together with our assumption implies that $\mathfrak{k}_{n-1}$ is abelian. The Frobenius vector turns out to be the Dirac measure at the unity of G and $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is modulo $\ker(\pi)$ contained in $\mathcal{U}(\mathfrak{k}_{n-1})$ which is identified with the symmetric algebra $S(\mathfrak{k}_{n-1})$ of $(\mathfrak{k}_{n-1})_{\mathbb{C}}$. We hereby get $P_W(l) = \widehat{W}(l)$ $(\forall l \in \Omega)$, hence the result.

■

3.9 Remark:

Let's keep the second situation of the above proposition, i.e. Ω is saturated with respect to $\mathfrak{k}_{n-1}$, and assume that $\mathfrak{g}$ has the one dimensional center $\mathfrak{z} = \mathbb{R}Z$ on which Ω is not trivial. Take a new Jordan-Hölder sequence $\{\mathfrak{g}'_j\}_{j=1}^n$ such that $\mathfrak{g}'_{n-1} = \mathfrak{k}_{n-1}$. Then we have $\mathfrak{g}'_1 = \mathfrak{z}$ and $\mathfrak{g}'_2 = \mathfrak{z} + \mathbb{R}Y$ with $l([Y_n, Y]) \neq 0$ for any $l \in \Omega$. By the finite multiplicity condition there exists for every $l \in \Omega$ a vector $X(l) \in \mathfrak{g}(l)$ satisfying $Y + X(l) \in \mathfrak{k}$. It follows from this that $D_\pi(G)^K \cong \mathbb{C}[Y] \cong \mathbb{C}[P_Y]$, where P_Y is the polynomial function defined by $P_Y(l) = l(Y)$.
In Proposition 3.8, we are mainly led to the case where $\mathfrak{k}$ is included in a one codimensional polarization at $l \in \Omega$. In fact, we can prove the following more general setting. Assume that π is realized by a normal polarization which contains $\mathfrak{k}$. Then the same result holds.

3.10 Proposition: *Assume that the orbit $\Omega(\pi)$ is flat, i.e. $\Omega(\pi) = l + \mathfrak{a}^{\perp}$ with $\mathfrak{a} = \mathfrak{g}(l)$ for any $l \in \Omega(\pi)$. Suppose also that $\mathfrak{k}$ is abelian and that $\pi_{|K}$ is of finite multiplicities. Then, $D_\pi(G)^K \simeq \mathbb{C}[\Omega(\pi)]^K$.*

Proof: In this situation the stabilizer $G(l)$ is normal subgroup of G for $l \in \Omega(\pi)$, and π is square integrable modulo $G(l)$ (cf. [29]). Thus $\mathfrak{b} = \mathfrak{a} + \mathfrak{k}$ is a polarization at any $l \in \Omega(\pi)$. Take a Malcev basis $\{T_1, \dots, T_r\}$ of $\mathfrak{k}$ relative to $\mathfrak{k} \cap \mathfrak{a}$, which is also a Malcev basis of $\mathfrak{b}$ relative to $\mathfrak{a}$. Let $\mathfrak{p} = \sum_{j=1}^{r} \mathbb{R}T_j$. Then by [3], [30], we can easily see that $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \mathcal{U}(\mathfrak{p}) + \ker(\pi)$ and that $D_\pi(G)^K \cong S(\mathfrak{p}) \cong \mathbb{C}[\Omega(\pi)]^K$, where $S(\mathfrak{p})$ denotes the symmetric algebra of $\mathfrak{p}_{\mathbb{C}}$.

■

3.11 Example: Let $\mathfrak{g} = \langle X_1, X_2, \dots, X_{n-1}, X_n \rangle_{\mathbb{R}}$ with non-zero brackets $[X_n, X_j] = X_{j-1}$ for $2 \leq j \leq n-1$ (threadlike algebra). Let $\pi \in \widehat{G}$. We have $\dim(\Omega(\pi)) \leq 2$. Let $\mathfrak{k}$ be a non-central subalgebra of $\mathfrak{g}$, namely $\mathfrak{k} \not\subset \mathfrak{z} = \mathbb{R}X_1$. If $\pi(X_1) \neq 0$, then $\pi_{|K}$ is of finite multiplicities and $D_\pi(G)^K \cong \mathbb{C}[\Omega(\pi)]^K$. More precisely, if $\mathfrak{k} \subset \mathfrak{g}_{n-1} = \sum_{j=1}^{n-1} \mathbb{R}X_j$, then

$$D_\pi(G)^K \cong \mathbb{C}[\Omega(\pi)]^K \cong \mathbb{C}[P_{X_2}].$$

If $\mathfrak{k} \not\subset \mathfrak{g}_{n-1}$, let's take X_n in $\mathfrak{k}$. Two cases would happen. When all K-orbits in $p(\Omega(\pi)) \subset \mathfrak{k}^*$ are points, $\mathfrak{k}$ must be abelian and

$$D_\pi(G)^K \cong \mathbb{C}[\Omega(\pi)]^K \cong \mathbb{C}[P_{X_n}].$$

When $\dim(K \cdot p(l)) = 2$ for generic $l \in \Omega(\pi)$, we have $\dim(K \cdot l) = \dim(\Omega(\pi)) = 2$ and

$$D_\pi(G)^K \cong \mathbb{C}[\Omega(\pi)]^K \cong \mathbb{C}.$$

In this last eventuality, $\pi_{|K}$ turns out to be irreducible.

We have the answer "yes" in another case where $\mathfrak{k}$ is an ideal in $\mathfrak{g}$.

3.12 Theorem: *Assume that $\pi_{|K}$ has finite multiplicities. If $\mathfrak{k}$ is an ideal in $\mathfrak{g}$, then $D_\pi(G)^K \simeq \mathbb{C}[\Omega(\pi)]^K$.*

Proof: We can assume that our flag of ideals $(\mathfrak{g}_j)_{j=0}^{n}$ passes through $\mathfrak{k}$, i.e. $\mathfrak{g}_d = \mathfrak{k}$. In particular the index set $\mathcal{I}^{\mathfrak{k}}$ is now equal to $\{1, \cdots, d\}$ and $\mathfrak{k}_j = \mathfrak{g}_j$ for $j \in \{1, \cdots, n\}$. Fix a base point $l_0 \in \Omega(\pi)$ and let $\mathfrak{b} = \mathfrak{b}[l_0]$, $B = \exp(\mathfrak{b})$, $\pi = \mathrm{ind}_B^G \chi_{l_0}$. Then for every element $l = g \cdot l_0 \in \Omega(\pi)$, the Vergne polarization $\mathfrak{b}[l]$ contains the Vergne polarization $\mathfrak{b}[l_{|\mathfrak{k}}]$ and so the distribution a_l, transferred to $\mathcal{H}_\pi^\infty$, becomes evaluation at the point g:

$$\langle a_l, \varphi \rangle = \int_{B[l_{|\mathfrak{k}}]/B[l_{|\mathfrak{k}}] \cap B[l]} \overline{\varphi(bg)\chi_l(b)} d\dot{b} = \overline{\varphi(g)} \ (\varphi \in \mathcal{H}_\pi^\infty).$$

Furthermore, by the finite multiplicity condition, we know that $\mathfrak{g}(l) + \mathfrak{b}[l_{|\mathfrak{k}}]$ is lagrangian with respect to B_l for μ_π-almost all $l \in \Omega(\pi)$, and since these spaces are now conjugate, $\mathfrak{k}$ being an ideal, it follows that $\mathfrak{b}[l] = \mathfrak{g}(l) + \mathfrak{b}[l_{|\mathfrak{k}}]$ for all $l \in \Omega(\pi)$. Therefore $B[l] = B[l_{|\mathfrak{k}}]G(l)$ since

$[\mathfrak{b}[l_{|\mathfrak{k}}], \mathfrak{g}(l)] \subset \mathfrak{b}[l_{|\mathfrak{k}}]$. Choosing a Malcev basis $\mathcal{Q} = \{Q_1, \cdots, Q_q\}$ of $\mathfrak{g}$ relative to $\mathfrak{b}$ which is extracted from our Jordan-Hölder basis $\mathcal{Z}$ of $\mathfrak{g}$, we obtain a polynomial diffeomorphism

$$E_{\mathcal{Q}} : \mathbb{R}^q \to G/B;\ E_{\mathcal{Q}}(u_1, \cdots, u_q) = \prod_{i=q}^{1} \exp(u_i Q_i)$$

and an identification of $\mathcal{H}_\pi$ with $L^2(\mathbb{R}^q)$. We know also by Kirillov's theorem that $\mathcal{H}_\pi^\infty$ is isomorphic to the Schwartz space $\mathcal{S}(\mathbb{R}^q)$. In particular for any $W \in \mathcal{U}(\mathfrak{g})$, $\pi(W^*)$ becomes a partial differential operator with polynomial coefficients:

$$(\pi(W^*)\varphi) \circ E_{\mathcal{Q}} = \sum_{\alpha \in \mathbb{N}^q} P_\alpha \partial^\alpha (\varphi \circ E_{\mathcal{Q}}). \tag{3.12.1}$$

Take now $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \setminus \ker(\pi)$. We know from 2.5, that

$$P_W(g \cdot l_0)\overline{\varphi(g)} = \langle \pi(W)a_l, \varphi\rangle = \langle a_l, \pi(W^*)\varphi\rangle = \overline{\pi(W^*)\varphi(g)}.$$

Hence by (3.12.1)

$$P_W(E_{\mathcal{Q}}(u) \cdot l_0)\overline{\varphi(E_{\mathcal{Q}}(u))} = \sum_{\alpha \in \mathbb{N}^q} \overline{P_\alpha(u)} \partial^\alpha \overline{(\varphi \circ E_{\mathcal{Q}})(u)},\ u \in \mathbb{R}^q.$$

Now there exists $\gamma \in \mathbb{N}^q$, such that $P_\gamma \not\equiv 0$ since $W^* \notin \ker(\pi)$. We can assume that the length $|\gamma|$ of γ is maximal. Suppose that $|\gamma| > 0$. Take $u \in \mathbb{R}^q$, such that $P_\gamma(u) \neq 0$. We choose $\psi \in \mathcal{S}(G/B)$, for which $\partial^\gamma(\psi \circ E_{\mathcal{Q}})(u) = 1$, but $\partial^\beta(\psi \circ E_{\mathcal{Q}})(u) = 0$ for every $\beta \in \mathbb{N}^q$, $\beta \neq \gamma$ and $|\beta| \leq |\gamma|$. Then

$$\begin{aligned} 0 = P_W(E_{\mathcal{Q}}(u) \cdot l_0)\overline{\psi(E_{\mathcal{Q}}(u))\varphi(E_{\mathcal{Q}}(u))} &= \sum_{|\alpha| \leq |\gamma|} \overline{P_\alpha(u)} \partial^\alpha \overline{((\psi\varphi) \circ E_{\mathcal{Q}})(u)} \\ &= \overline{P_\gamma(u)\varphi(E_{\mathcal{Q}}(u))},\ \varphi \in \mathcal{H}_\pi^\infty. \end{aligned}$$

This contradiction tells us that $|\gamma| = 0$. Hence $P_W(E_{\mathcal{Q}}(u) \cdot l_0) = \overline{P_0(u)},\ u \in \mathbb{R}^q$, and P_W is thus a polynomial function on $\Omega(\pi)$.

Take now a polynomial function $P : \Omega(\pi) \to \mathbb{C}$ on $\Omega(\pi)$, which is K-invariant, i.e. for which $P(k \cdot l) = P(l),\ k \in K,\ l \in \Omega(\pi)$. Define the polynomial function $\tilde{P}$ on G by the formula

$$\tilde{P}(g) = P(g \cdot l_0),\ g \in G.$$

Then $\tilde{P}$ is K-invariant. Since $B = B[l_{0|\mathfrak{k}}]G(l_0) \subset KG(l_0)$, it follows that $\tilde{P}(gb) = \tilde{P}(g),\ g \in G,\ b \in B$, and $\tilde{P}$ is in fact a polynomial function on G/B. The operator $M : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty$ defined through multiplication by $\tilde{P}$ is therefore by Kirillov's theorem contained in $\pi(\mathcal{U}(\mathfrak{g}))$. Hence there exists $W \in \mathcal{U}(\mathfrak{g})$, such that $\pi(W)\varphi = \tilde{P}\varphi,\ \varphi \in \mathcal{H}_\pi^\infty$. The fact that $\tilde{P}$ is K-invariant tells us that $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. It is obvious now that $P = P_W$. This shows that the mapping

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \to \mathbb{C}[\Omega(\pi)]^K; W \to P_W,$$

is a surjectif homomorphism (whose kernel is equal to $\ker(\pi)$ by 3.2).

$\blacksquare$

3.13 Corollary: *Suppose that G is two-step. Then, $\pi_{|K}$ is of finite multiplicities if and only if $D_\pi(G)^K \cong \mathbb{C}[\Omega(\pi)]^K$.*

Proof: In fact, adding the center of $\mathfrak{g}$ to $\mathfrak{k}$, we find ourselves in the case where $\mathfrak{k}$ is an ideal.

$\blacksquare$

4. Frobenius reciprocity

4.1 We suppose again that $\pi_{|K}$ is of finite multiplicities. For the next lemma, the notation $\mathfrak{b}[l_{|\mathfrak{k}}]$ means a (not necessarily Vergne) polarisation at $l_{|\mathfrak{k}} \in \mathfrak{k}^*$. We shall construct a basis of $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ and we shall show that μ_π-almost everywhere the multiplicities $n_\pi(\Theta_K(l_{|\mathfrak{k}}))$ in (1.5.1) are equal to the dimension of $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$, provided that l or K fulfills special conditions, which we call conditions $\mathcal{N}$.

Let us realize π at a generic point $l \in \Omega(\pi)$ as $\pi = \mathrm{ind}_{B[l]}^G \chi_l$ and let us consider $\sigma = \mathrm{ind}_{B[l_{|\mathfrak{k}}]}^K \chi_l$ and its associated orbit $\omega(\sigma)$ in $p(\Omega(\pi)) \subset \mathfrak{k}^*$. Then the closed subset $\Omega(\pi) \cap p^{-1}(\omega(\sigma))$ of $\mathfrak{g}^*$ is a disjoint union of $n_\pi(\sigma)$ connected components $C_1, \cdots, C_{n_\pi(\sigma)}$, which are in fact K-orbits. Furthermore, each intersection $\tilde{C}_j = C_j \cap p^{-1}(l_{|\mathfrak{k}})$ $(1 \leq j \leq n_\pi(\sigma))$ is a $K(l_{|\mathfrak{k}})$-orbit (see [3]). For each j, let us take a g_j in G, such that $g_j \cdot l \in \tilde{C}_j$ and let us define the distribution $a_j : \mathcal{H}_\pi^\infty \to \mathbb{C}$ by the formula

$$\langle a_j, \phi \rangle = \int_{B[l_{|\mathfrak{k}}]/B[l_{|\mathfrak{k}}] \cap g_j B[l] g_j^{-1}} \overline{\phi(b g_j) \chi_l(b)} d\dot{b} \ (\phi \in \mathcal{H}_\pi^\infty). \tag{4.1.1}$$

4.2 Lemma: *For generic $l \in \Omega(\pi)$ the distributions a_j $(1 \leq j \leq n_\pi(\sigma))$ are linearly independent elements in $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$, whose supports are mutually disjoint.*

Proof: Since $p(g_j \cdot l) = l_{|\mathfrak{k}}$ we can rewrite (4.1.1) as

$$\langle a_j, \phi \rangle = \int_{g_j^{-1} B[l_{|\mathfrak{k}}] g_j / g_j^{-1} B[l_{|\mathfrak{k}}] g_j \cap B[l]} \overline{\phi(g_j b) \chi_l(b)} d\dot{b} \ (\phi \in \mathcal{H}_\pi^\infty). \tag{4.2.1}$$

This shows that a_j belongs to $\mathcal{H}_\pi^{-\infty}$ and (4.1.1) implies that a_j is effectively in $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$. We also recall (see 2.4) that the replacement of the polarization $B[l]$ by another one does only multiply the distributions a_j by some constants. We proceed by induction on the dimension of G and we show that their supports $B[l_{|\mathfrak{k}}] \cdot g_j \cdot B[l]$ are mutually disjoint.

In the particular case where $K = G$, evidently $\pi = \sigma$, $n_\pi(\sigma) = 1$. Suppose now that $q \geq 1$. We shall use the notations of 2.1, 2.3 and we set $\mathfrak{g}' = \mathfrak{l}_{q-1}, G' = \exp(\mathfrak{g}'), l' = l_{|\mathfrak{g}'}$. Let us write $g_j = g_j' \exp(x_j Y_n)$ with $g_j' \in G'$ and $x_j \in \mathbb{R}$ $(1 \leq j \leq n_\pi(\sigma))$.

We suppose first that the orbit $\Omega(\pi)$ is not saturated with respect to $\mathfrak{g}'$. In that case we may assume that $Y_n \in \mathfrak{g}(l)$ and then that $g_j = g_j' \in G'$. The distributions a_j can now be considered as elements of $(\mathcal{H}_{\pi'}^{-\infty})^{B[l'_{|\mathfrak{k}}],\chi_{l'}}$ and the induction hypothesis applies.

We assume now that $\Omega(\pi)$ is saturated with respect to $\mathfrak{g}'$. We can then suppose that $B[l]$ is contained in G'. Since two double classes D_j and $D_{j'}$ are disjoint if $x_j \neq x_{j'}$, we may reduce the problem to the case where the numbers x_j are all equal to a fixed number x. Hence we can again descend to the subgroup G' and apply the induction hypothesis to the subgroup $\exp(xY_n)K\exp(xY_n)^{-1}$ and (the generic) linear functional $(\exp(xY_n)\cdot l)_{|\mathfrak{g}'}$.

■

Recalling the Frobenius reciprocity [23] for monomial representations, we ask:

Question: Is the equality $n_\pi(\sigma) = \dim((\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l})$, where $\sigma = \mathrm{ind}_{B[l_{|\mathfrak{k}}]}^K \chi_l$, true μ_π-almost everywhere in $\Omega(\pi)$?

At present we need to give a positive answer some condition which we call condition $\mathcal{N}$. Namely, we assume one of the following three conditions: (1) $\mathfrak{k}$ is an ideal of $\mathfrak{g}$; (2) $\mathfrak{b}[l_{|\mathfrak{k}}]$ is common, denoted by $\mathfrak{h}$, for μ_π-almost all $l \in \Omega(\pi)$, for example $\mathfrak{h} = \mathfrak{k}$ if $\mathfrak{k}$ is abelian; (3) there exists a measurable subset $\mathcal{V}$ of $p(\Omega(\pi))$ whose complement is $p_*(\mu_\pi)$-negligible and such that $[\mathfrak{b}[l_{|\mathfrak{k}}], \mathfrak{g}(l)] \subset \mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l)$ for every $l \in \Omega(\pi) \cap p^{-1}(\mathcal{V})$.

4.3 Theorem: *Suppose that $\pi_{|K}$ is of finite multiplicities, and assume the condition $\mathcal{N}$. Then for μ_π-almost every l in $\Omega(\pi)$ and for every polarization $\mathfrak{b}[l_{|\mathfrak{k}}]$ at $l_{|\mathfrak{k}}$ fulfilling the condition $\mathcal{N}$, we have that*

$$(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l} = \sum_{j=1}^{n_\pi(\sigma)} \mathbb{C}a_j \tag{4.3.1}$$

where $\sigma = \Theta_K(l_{|\mathfrak{k}}) = ind_{B[l_{|\mathfrak{k}}]}^K \chi_l$ and $a_j,\ j = 1, ..., n_\pi(\sigma)$, are as in (4.1.1).

Proof: We treat at first the case where the third assumption of the condition $\mathcal{N}$ is satisfied. This means that there exists a measurable subset $\mathcal{V}'$ of $p(\Omega(\pi))$ whose complement is $p_*(\mu_\pi)$-negligible and such that $\mathfrak{b}[l] = \mathfrak{b}[l_{|\mathfrak{k}}] + \mathfrak{g}(l)$ is a polarization at $l \in \Omega(\pi) \cap p^{-1}(\mathcal{V}')$. Take such an l and realize π as $\pi = \mathrm{ind}_{B[l]}^G \chi_l$, where $B[l] = \exp(\mathfrak{b}[l])$. Recalling the flag $\mathcal{S}$ of ideals and the adapted Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ introduced in 2.3, let $e_j = \dim(G \cdot (l_{|\mathfrak{g}_j})), 0 \leq j \leq n, e = (e_0, e_1, \ldots, e_n)$ and $T = \{1 \leq j \leq n; e_j = e_{j-1}\}$. Then, there are (see 1.12) e-central elements of Corwin-Greenleaf, namely for every $j \in T$ there exists $A_j = P_j Z_j + Q_j$ in $\mathcal{U}(\mathfrak{g}_j)$, where $P_j, Q_j \in \mathcal{U}(\mathfrak{g}_{j-1})$, such that $\pi(P_j) = p_j \mathbb{I} \neq 0, \pi(A_j) = \alpha_j \mathbb{I}$ are scalars. We can choose Z_j in $\mathfrak{g}(l)$ for $j \in T$. For $1 \leq j \leq n$, let's put $\mathfrak{h}_j = \mathfrak{b}[l] \cap \mathfrak{g}_j$, let $\{X_1, \ldots, X_{i_j}\}$ be a basis of $\mathfrak{h}_j$ and

$$\overline{\mathfrak{a}_j} = \sum_{k=1}^{i_j} \mathbb{C}\left(X_k - il(X_k)\right).$$

Finally, we denote by $\mathcal{U}(\mathfrak{g}_j)\overline{\mathfrak{a}_j}$ the left ideal of $\mathcal{U}(\mathfrak{g}_j)$ generated by $\overline{\mathfrak{a}_j}$. Then, we know [13] that $P_j = p_j + U_j$,

$$A_j = (p_j + U_j)(Z_j - il(Z_j)) + ip_j l(Z_j) + il(Z_j)U_j + Q_j = \alpha_j + W_j$$

with certain $U_j \in \mathcal{U}(\mathfrak{g}_{j-1})\overline{\mathfrak{a}_{j-1}}, W_j \in \mathcal{U}(\mathfrak{g}_j)\overline{\mathfrak{a}_j}$. Therefore Q_j can be written as $Q_j = q_j + V_j$ with a certain $V_j \in \mathcal{U}(\mathfrak{g}_{j-1})\overline{\mathfrak{a}_{j-1}}$. Hence $\alpha_j = ip_j l(Z_j) + q_j$ and

$$A_j = \alpha_j + p_j(Z_j - il(Z_j)) + W'_j$$

with a certain $W'_j \in \mathcal{U}(\mathfrak{g}_{j-1})\overline{\mathfrak{a}_{j-1}}$. Now, we apply on $a \in (\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ the above e-central element A_{j_0}, j_0 being the first index $j \in T$ such that $\mathfrak{g}(l) \cap \mathfrak{g}_j \not\subset \mathfrak{b}[l_{|\mathfrak{k}}] \cap \mathfrak{g}_j$, to get

$$(Z_{j_0} - il(Z_{j_0}))a = (\pi(Z_{j_0}) - il(Z_{j_0}))a = 0.$$

This process can be repeated until we conclude that

$$a \in \left(\mathcal{H}_\pi^{-\infty}\right)^{B[l],\chi_l},$$

which implies (cf. [1],[15],[25]) that a is a multiple of (the complex conjugate of) the Dirac distribution at the identity element of G. This settles the case of condition (3) in $\mathcal{N}$.

The theorem being trivial when $\dim(G) = 1$, we employ the induction on $\dim(G)$ and assume that the theorem holds for groups of dimension smaller than $n = \dim(G)$. We can as always assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ is one dimensional and contained in $\mathfrak{k}$ and that $\Omega(\pi)$ is not trivial on $\mathfrak{z}$. Choose the element $Z \in \mathfrak{z}$ for which $l(Z) = 1$ for one (hence for all) elements $l \in \Omega(\pi)$. For our flag of ideals

$$\mathcal{S} : \{0\} \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g},$$

we have now that $\mathfrak{z} = \mathfrak{g}_1$. Take $Y \in \mathfrak{g}_2 \setminus \mathfrak{g}_1$, let $\mathfrak{g}_0$ be the centralizer of Y and $G_0 = \exp(\mathfrak{g}_0)$. Finally choose $X \in \mathfrak{g}$ such that $[X, Y] = Z$, whence $\mathfrak{g} = \mathfrak{g}_0 + \mathbb{R}X$. Since $\Omega(\pi)$ is saturated with respect to $\mathfrak{g}_0$, we can take for every $l \in \Omega(\pi)$ a polarization $\mathfrak{b}[l]$ at l which is contained in $\mathfrak{g}_0$. Let $\pi_0 = \mathrm{ind}_{B[l]}^{G_0} \chi_l$. Then $\pi \simeq \mathrm{ind}_{G_0}^{G} \pi_0$. Put also for $x \in \mathbb{R}$, $g_x = \exp(xX)$ and denote by π_x the representation $g_x \cdot \pi_0$ of G_0 i.e. $\pi_x(g_0) = \pi_0(g_{-x} g_0 g_x) (g_0 \in G_0)$. Let $l_0 = l_{|\mathfrak{g}_0} \in \mathfrak{g}_0^*$. We recall (see for instance [33]) that

$$\pi_{|G_0} \simeq \int_{\mathbb{R}}^{\oplus} \pi_x dx$$

and that the invariant measure μ_π on $\Omega(\pi)$ can be written as $dsdx \otimes \mu_{g_x \cdot \pi_0}$ (see (1.10.1)). Note that the third assertion in $\mathcal{N}$ concerns the polarization $\mathfrak{b}[l_{|\mathfrak{k}}]$ itself but the first one admits all polarizations at $l_{|\mathfrak{k}}$.

If the first assumption of the condition $\mathcal{N}$ is satisfied, $\pi_{|K}$ is either of infinite multiplicities or multiplicity free. We take the flag $\mathcal{S}$ so that $\mathfrak{k} = \mathfrak{g}_m$ for some $1 \leq m \leq n$. If $m = 1$, $\pi_{|K}$ is of finite multiplicities if and only if π is a unitary character, and the result is obvious. So, assume that $2 \leq m$. In particular, $Y \in \mathfrak{k}$.

Suppose in a first time that $\mathfrak{b}[l_{|\mathfrak{k}}] \not\subset \mathfrak{g}_0$, then especially $\mathfrak{k} \not\subset \mathfrak{g}_0$. Let $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ and $K_0 = \exp(\mathfrak{k}_0)$. Then we have that

$$\mathfrak{b}[l_{0|\mathfrak{k}_0}] = \mathfrak{b}[l_{|\mathfrak{k}}] \cap \mathfrak{k}_0 + \mathbb{R}Y$$

is a polarization at $l_{0|\mathfrak{k}_0}$. Fix such an l and pick by the way $X \in \mathfrak{b}[l_{|\mathfrak{k}}] \cap \ker(l)$ for our fixed l. Every element $a \in (\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ is now invariant under the action of the group $\exp(\mathbb{R}X)$. This implies that there exists a unique distribution $a_0 \in \mathcal{H}_{\pi_0}^{-\infty}$ such that

$$\langle a, \phi \rangle = \int_{\mathbb{R}} \langle a_0, \phi(x) \rangle dx \ (\phi \in \mathcal{H}_\pi^\infty),$$

where $\phi(x)$ denotes the element of $\mathcal{H}_{\pi_0}^\infty$ defined by $\phi(x)(g_0) = \phi(g_x g_0)$ $(x \in \mathbb{R}, g_0 \in G_0)$. It is easy to see that a_0 is in fact in $\left(\mathcal{H}_{\pi_0}^{-\infty}\right)^{B[l_{0|\mathfrak{k}_0}],\chi_{l_0}}$, $B[l_{0|\mathfrak{k}_0}] = \exp(\mathfrak{b}[l_{0|\mathfrak{k}_0}])$, and so the vector spaces $\left(\mathcal{H}_{\pi_0}^{-\infty}\right)^{B[l_{0|\mathfrak{k}_0}],\chi_{l_0}}$ and $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ are isomorphic. Analyzing the decomposition formula (1.5.1) for $\pi_{|K}$ and $\pi_{0|K_0}$, we can deduce that the number of K-orbits in $G \cdot l \cap (K \cdot l + \mathfrak{k}^\perp)$ is easily seen to be equal to the number of K_0-orbits in $G_0 \cdot l_0 \cap (K_0 \cdot l_0 + {\mathfrak{k}_0}^{\perp,\mathfrak{g}_0})$. Hence the induction hypothesis applied to (G_0, K_0, l_0) gives us the result.

Suppose now that $\mathfrak{b}[l_{|\mathfrak{k}}] \subset \mathfrak{g}_0$. When $K \not\subset G_0$, we easily check [3] that $\pi_{|K_0}$ remains to be of finite multiplicities. In fact, for μ-almost all $\sigma \in \widehat{K}$, we have:

$$\sigma_{|K_0} \simeq \int_{\mathbb{R}}^{\oplus} \sigma'_t dt$$

with mutually inequivalent irreducible representations σ'_t of K_0. This means the two multiplicities $n_\pi(\sigma), n_\pi(\sigma'_t)$ are equal and we can assume that $K \subset G_0$. Then, it follows that for $\phi \in \mathcal{H}_\pi^\infty$

$$(-Y) \cdot \phi(g_x g_0) = \frac{d}{dt}\phi(\exp(tY) g_x g_0)|_{t=0} = \frac{d}{dt}\phi(g_x g_0 \exp(tY) \exp(-txZ))|_{t=0}$$

$$= \frac{d}{dt} e^{it(x - l(Y))} \phi(g_x g_0)|_{t=0} = i(x - l(Y))\phi(g_x g_0) \ (x \in \mathbb{R}, \ g_0 \in G_0).$$

Hence for $a \in (\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$

$$\langle Y \cdot a, \phi \rangle = \langle a, (-Y) \cdot \phi \rangle = \langle a, i(x - l(Y))\phi \rangle = \langle i(l(Y) - x)a, \phi \rangle = \langle i(l(Y))a, \phi \rangle,$$

which shows that $xa = 0$ and then the support of a is contained in G_0. Hence

$$a = \sum_{j=0}^{s} (\frac{\partial}{\partial x})^j|_{(x=0)} a_j$$

for some integer s and some distributions a_j on G_0. Since a is an eigen-distribution for the action of Y, it follows that $a_j = 0$ for $j \neq 0$ and so a is in fact a distribution on G_0, i.e. $a \in (\mathcal{H}_{\pi_0}^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$. Remark that the restriction of π_0 onto K is of finite multiplicities too. In fact, for $l' \in \mathfrak{g}_0^*$, we note $\mathfrak{k}^{l'}$ the orthogonal in $\mathfrak{g}_0$ of $\mathfrak{k}$ with respect to $B_{l'}$. Recall [3] the fact that the restriction of π_x to K has finite multiplicities if and only if $\mathfrak{k}$ is co-isotropic, i.e. $\mathfrak{k}^{l'}$ is isotropic, relative to $B_{l'}$ for $\mu_{g_x\cdot\pi_0}$-almost all $l'\in\omega_x=\Omega_{G'}(\pi_x)=g_x\cdot\omega_0(x\in\mathbb{R})$. This condition is independent of $x\in\mathbb{R}$ since $g_y\cdot\mathfrak{k}^{l'}=\mathfrak{k}^{g_y\cdot l'}(\forall\ y\in\mathbb{R})$ by our assumption that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. Since the formula

$$\pi_{|K}\simeq\left(\pi_{|G_0}\right)_{|K}\simeq\int_{\mathbb{R}}^{\oplus}(\pi_x)_{|K}dx$$

has finite multiplicities, $(\pi_x)_{|K}$ has finite multiplicities for every $x\in\mathbb{R}$. Hence we can apply the induction hypothesis for G_0 and π_0. In what follows we put the second assumption of the condition $\mathcal{N}$. The same observations as above allow us to assume that $\mathfrak{k}=\mathfrak{h}$.

Case 1. We begin with the case where $K\not\subset G_0$. Let as above $K_0=K\cap G_0$ and $\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{g}_0$. Let us check that $\pi_{0|K_0}$ is also of finite multiplicities. Indeed, if $A\in\mathfrak{g}_0$ has the property that $\langle l,[A,\mathfrak{k}_0]\rangle=\{0\}$ for some generic $l\in\Omega(\pi)$, then it follows that $\langle l,[A+\lambda Y,\mathfrak{k}]\rangle=\{0\}$ for $\lambda=\langle l,[A,X]\rangle$. Since $\mathfrak{k}+\mathfrak{g}(l)$ is lagrangian for B_l, it follows that $A+\lambda Y\in\mathfrak{k}+\mathfrak{g}(l)$ and so $A\in\mathfrak{k}_0+\mathfrak{g}_0(l_0)$ with $l_0=l_{|\mathfrak{g}_0}$. Hence $\mathfrak{k}_0+\mathfrak{g}_0(l_0)$ is lagrangian for B_{l_0} too and by 2.1 we have that $\pi_{0|K_0}$ is of finite multiplicities. Take $X\in\mathfrak{k}$. Every element $a\in(\mathcal{H}_\pi^{-\infty})^{K,\chi_l}$ is now semi-invariant under the action of the group $\exp(\mathbb{R}X)$. This implies as above that there exists a unique distribution $a_0\in\left(\mathcal{H}_{\pi_0}^{-\infty}\right)^{K_0,\chi_{l_0}}$ such that

$$\langle a,\phi\rangle=\int_{\mathbb{R}}\langle a_0,\phi(x)\rangle e^{-ixl(X)}dx\ (\phi\in\mathcal{H}_\pi^\infty),$$

where for $x\in\mathbb{R}$, $\phi(x)$ denotes the element of $\mathcal{H}_{\pi_0}^\infty$ defined by $\phi(x)(g_0)=\phi(g_xg_0)$ $(g_0\in G_0)$. So the vector spaces $(\mathcal{H}_{\pi_0}^{-\infty})^{K_0,\chi_{l_0}}$ and $(\mathcal{H}_\pi^{-\infty})^{K,\chi_l}$ are isomorphic. On the other hand the number of K-orbits in $G\cdot l\cap(K\cdot l+\mathfrak{k}^\perp)$ is easily seen to be equal to the number of K_0-orbits in $G_0\cdot l_0\cap(K_0\cdot l_0+\mathfrak{k}_0^\perp)$. Hence the induction hypothesis for G_0 and l_0 gives us the result.

Case 2. We come now to the case where $K\subset G_0$. By Lemma 2.1, there exists a Zariski open subset Ω^1 of $\Omega(\pi)$, such that for every $l\in\Omega^1$ the restriction $\pi_{l_0},l_0=l_{|\mathfrak{g}_0}$ to K is of finite multiplicities too. Hence, using the induction hypothesis for G_0, according to the decomposition (1.10.1) of $\Omega(\pi)$, we have a subset Z of Lebesgue measure zero in $\mathbb{R}$ and for every $x\in\mathbb{R}\setminus Z$ a subset Z_x in $\Omega_{G_0}(g_x\cdot\pi_0)$ of $\mu_{g_x\cdot\pi_0}$-measure zero, such that the relation (4.3.1) holds for all $l_0\in\Omega_{G_0}(g_x\cdot\pi_0)\setminus Z_x$. Hence the subset Ω^2 of $\Omega(\pi)$, consisting of the l's in $\Omega(\pi)$, such that (4.3.1) is valid for l_0 has full μ_π-measure and so it remains the same for the subset

$$\Omega^{\mathrm{gen}}=\{l\in\Omega(\pi);p^{-1}(l_{|\mathfrak{k}})\subset\Omega^2\}$$

of $\Omega(\pi)$. In fact, let $E=\{l'\in\Omega(\pi);l'_{|\mathfrak{k}}=l_{|\mathfrak{k}}\}$ (l generic in $\Omega(\pi)$), then $K(l_{|\mathfrak{k}})$ acts on E and the finite multiplicity condition implies that the number of $K(l_{|\mathfrak{k}})$- orbits in E has an absolute bound. We take from now on $l\in\Omega^{\mathrm{gen}}$.

We can settle the case where $Y\in\mathfrak{k}$ exactly as before. So, let finally $Y\notin\mathfrak{k}$. We shall first show that the support of a is contained in $\bigcup_{j=1}^{n_\pi(\sigma)}g_jG_0$.
Since $[Y,\mathfrak{k}]=\{0\}$ and since $\mathfrak{k}+\mathfrak{g}(l)$ is μ_π-almost everywhere a lagrangian subspace for the bilinear form B_l, it follows that $Y\in\mathfrak{k}+\mathfrak{g}(l)$ for generic $l\in\Omega(\pi)$. Hence there exists a

minimal index $j_0 \in \{1, \cdots, d\}$, such that $Y \in \mathfrak{k}_{j_0} + \mathfrak{g}(l)$ generically. Let $\mathfrak{k}'_j = \mathfrak{k}_j + \mathbb{R}Y$ and $K'_j = \exp(\mathfrak{k}'_j)$, $j = 1, \cdots, d$. Then of course $\mathfrak{g}(l) \cap \mathfrak{k}'_{j_0} \neq \mathfrak{g}(l) \cap \mathfrak{k}'_{j_0-1}$ for generic $l \in \Omega(\pi)$. Using the theorem 1 of [3] for $\mathfrak{k}' = \mathfrak{k} + \mathbb{R}Y$, we see that we have an element

$$W = \sum_{i=0}^{m} P_i Y_{j_0}^i$$

in $\ker(\pi)$ with $P_i \in \mathcal{U}(\mathfrak{k}'_{j_0-1})(0 \leq i \leq m, m > 0)$ and $P_m \notin \ker(\pi)$. If all of the P_i's, $i = 0, \cdots, m$, are in $\mathcal{U}(\mathfrak{k}_{j_0-1})$, then P_m must be in $\ker(\pi)$. Indeed, let us denote by $p_{j_0}, p_{j_0-1}, p'_{j_0}, p'_{j_0-1}$ the canonical projections of $\mathfrak{g}^*$ onto $\mathfrak{k}^*_{j_0}, \mathfrak{k}^*_{j_0-1}, \mathfrak{k}'^*_{j_0}, \mathfrak{k}'^*_{j_0-1}$ respectively. We consider the Zariski open subset $\Omega_{\max}$ of $\Omega(\pi)$, consisting of all the $l's \in \Omega(\pi)$, for which the ranks of these four projections are maximal. Since $\Omega(\pi)$ is algebraic as G is nilpotent, $p_j(\Omega_{\max})$ is semi-algebraic. We have that

$$\dim(p'_{j_0}(\Omega_{\max})) = \dim(\mathfrak{g}/({\mathfrak{k}'}_{j_0}^{\perp_{B_l}})) = \dim(\mathfrak{g}/({\mathfrak{k}'}_{j_0-1}^{\perp_{B_l}})) = \dim(p'_{j_0-1}(\Omega_{\max})),$$

but

$$\dim(p_{j_0}(\Omega_{\max})) = \dim(\mathfrak{g}/(\mathfrak{k}_{j_0}^{\perp_{B_l}})) = \dim(\mathfrak{g}/(\mathfrak{k}_{j_0-1}^{\perp_{B_l}})) + 1 = \dim(p_{j_0-1}(\Omega_{\max})) + 1.$$

Here, the symbol $\perp_{B_l}$ designates the orthogonal with respect to the bilinear form B_l. This shows that for every $l \in \Omega_{\max}$, there exists an interval $I_l \subset \mathbb{R}$, such that

$$p_{j_0}(\Omega_{\max}) \supset l_{|\mathfrak{k}_{j_0}} + I_l Y^*_{j_0}. \tag{4.3.2}$$

Put $l_j = l(Y_j)$ for $1 \leq j \leq j_0$, and assume that all the P_i's, $i = 0, \cdots, m$, are in $\mathcal{U}(\mathfrak{k}_{j_0-1})$. Clearly, $P_W(l) = P_m(il)(il_{j_0})^m + Q(l)$, where $Q(l)$ is a polynomial of degree $\leq m-1$ in l_{j_0} with coefficients which are polynomial functions of $l_1, \ldots, l_{j_0-1}$. So we conclude by (4.3.2) that

$$0 = P_W(l + tY^*_{j_0}) = P_m(il)(l_{j_0} + t)^m + Q(l + tY^*_{j_0})$$

for $\forall t \in I_l$. Hence $P_m(il) \equiv 0$ and then $P_m \in \ker(\pi)$ by Lemma (3.2). This contradiction shows that $W \in \mathcal{U}(\mathfrak{k}'_{j_0}) \setminus \mathcal{U}(\mathfrak{k}_{j_0})$.

Hence if we rewrite W as

$$W = \sum_{k=0}^{m'} Q_k Y^k$$

with $Q_k \in \mathcal{U}(\mathfrak{k}_{j_0})$, $0 \leq k \leq m'$, we see that necessarily one of the Q_i, $i > 0$, (i.e. $Q_{m'}$) cannot be in $\ker(\pi)$. In fact, if $Q_i \in \ker(\pi)$ for all $1 \leq i \leq m'$, then $Q_0 \in \ker(\pi)$ too. Replacing W by Q_i $(0 \leq i \leq m')$ in the above argument, we see that each coefficient of $Y^i_{j_0}, i > 0$ in the expression of Q_i belongs to $\ker(\pi)$. Namely, if we write

$$Q_i = \sum_{r=0}^{m_i} Q_{ir} Y^r_{j_0},\ Q_{ir} \in \mathcal{U}(\mathfrak{k}_{j_0-1})\ (0 \leq r \leq m_i),$$

then $Q_{ir} \in \ker(\pi)$ for $r > 0$. This is contradictory to the assumption $P_m \notin \ker(\pi)$ $(m > 0)$. We choose now W such that $m' > 0$ is minimal.

Let us now exploit the fact that $W \cdot a = 0$ for every $a \in \mathcal{H}_\pi^{-\infty}$, in particular for our $a \in (\mathcal{H}_\pi^{-\infty})^{K,\chi_l}$. So, we apply $\pi(W)$ to the distribution a. Using the fact that $a \in (\mathcal{H}_\pi^{-\infty})^{K,\chi_l}$, we get $\pi(T)a = i\langle l, T\rangle a$ for any $T \in \mathfrak{k}$. Furthermore, since $Q_j \in \mathcal{U}(\mathfrak{k})$, $j = 0, \cdots, m'$, we have that $Q_j \cdot a = Q_j(il)a$ for all j. This implies that

$$0 = \langle \pi(W)a, \phi\rangle = \sum_{j=0}^{m'} \langle \pi(Q_j)a, \pi(-Y)^j \phi\rangle$$

$$= \sum_{j=0}^{m'} Q_j(il)\langle a, \pi(-Y)^j\phi\rangle = \langle a, \left(\sum_{j=0}^{m'} i^j \overline{Q_j(il)}(x - l(Y))^j\right)\phi\rangle,\ \phi \in \mathcal{H}_\pi^\infty.$$

Therefore the distribution a is annihilated by the multiplication with the polynomial function $g_x g_0 \mapsto P_l(x) = R(g_x g_0) := \sum_{j=0}^{m'} (-i)^j Q_j(il)(x - l(Y))^j$. Remark that $Q_{m'}(il) \not\equiv 0$ by $Q_{m'} \notin \ker(\pi)$ so that this polynomial is not trivial. In particular we see that the support of a is contained in the mutually different zeros $g_{x_1} \cdot G_0, \cdots, g_{x_{m''}} \cdot G_0$ $(m'' \le m')$ of the polynomial R.

We take now a closer look at a in a neighborhood of one of the zero-sets $g_{x_r} \cdot G_0$. Then a can be written as

$$a = \sum_{j=0}^{\kappa} \frac{\partial^j}{\partial x^j} \delta_{x_r} \widehat{\otimes} D_j,$$

where δ_{x_r} is the Dirac distribution at the point x_r and $D_j \in \mathcal{H}_{\pi_{x_r}}^{-\infty}$, $0 \le j \le \kappa$ such that $D_\kappa \neq 0$ by identifying $\mathcal{H}_\pi^\infty$ with $\mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{H}_{\pi_0}^\infty$.

Since $[Y, \mathfrak{k}] = \{0\}$, we know that $\pi(Y)^j a$ is in $(\mathcal{H}_\pi^{-\infty})^{K,\chi_l}$ too for every $j \in \mathbb{N}$. Hence, since $(\pi(Y) - il(Y))\xi(g_x g_0) = -ix\xi(g_x g_0)$ for $x \in \mathbb{R}$ and $\xi \in \mathcal{H}_\pi^\infty$, an easy computation shows that

$$(-i)^\kappa x^\kappa a = (\pi(Y) - il(Y))^\kappa a = (-i)^\kappa \kappa! \delta_{x_r} \widehat{\otimes} D_\kappa \in \left(\mathcal{H}_\pi^{-\infty}\right)^{K,\chi_l}.$$

This tells us that the distribution D_κ is an element of $(\mathcal{H}_{\pi_{x_r}}^{-\infty})^{K_r,\chi_{l_r}}$, where $K_r = g_{x_r}^{-1} K g_{x_r}$ and $l_r = g_{x_r}^{-1} \cdot l$. Hence, by the induction hypothesis, the support of D_κ is contained in the subset

$$\{K_r u B[g_{x_r}^{-1} \cdot l];\ u \cdot (g_{x_r}^{-1} \cdot l_0 + \mathfrak{b}[g_{x_r}^{-1} \cdot l_0]^\perp) \cap p_r{}^{-1}(l_{r|\mathfrak{k}_r}) \neq \emptyset\}$$

of G_0, where $\mathfrak{k}_r = Ad(g_{x_r}{}^{-1})\mathfrak{k}$ and $p_r : \mathfrak{g}^* \to \mathfrak{k}_r^*$ the canonical projection. So, the support of the distribution a (which is equal to the support of the distribution $x^\kappa a$) is contained in the subset

$$\{K u B[l];\ u \cdot (l + \mathfrak{b}[l]^\perp) \cap p^{-1}(l_{|\mathfrak{k}}) \neq \emptyset\} \subset \bigcup_{j=1}^{n_\pi(\sigma)} K g_j B[l] \subset \bigcup_{j=1}^{n_\pi(\sigma)} g_j G_0$$

of G.

Since $\mathcal{H}_\pi^\infty$ is Fréchet-isomorphic to $\mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{H}_{\pi_0}^\infty$, we can write now our distribution a as $a = \sum_{j=1}^{m''} a_j$, where the a_j's have their support in $g_{x_j}G_0$ and so

$$a_j = \sum_{k=0}^{\kappa_j} \frac{\partial^k \delta_{x_j}}{\partial x^k} \otimes D_k^j$$

with distributions $D_k^j \in \mathcal{H}_{\pi_{x_j}}^{-\infty}$ such that $D_{\kappa_j}^j \in (\mathcal{H}_{\pi_{x_j}}^{-\infty})^{K_j,\chi_{g_{x_j}^{-1}\cdot l_0}}$.Let us show that $D_{\kappa_j}^j = 0$ if $\kappa_j \neq 0$. The relation $\pi(W)a_l = 0$ for every $l \in \Omega^{\mathrm{gen}}$ tells us that the polynomial

$$P_l(x) := \sum_{j=0}^{m'} (-i)^j Q_j(il)(x - l(Y))^j = \sum_{j=0}^{m'} R_j(l)x^j,\ l \in \Omega^{\mathrm{gen}},$$

must be without constant term, i.e. $R_0(l) \equiv 0$ in l, since the distributions a_l are of order zero with support in G_0. However the first degree term R_1 of P_l cannot be identically zero in l, which means in particular that $m' \geq 2$. Indeed, otherwise,

$$0 \equiv R_1(l) = \sum_{j=1}^{m'} (-ij)Q_j(il)l(-iY)^{j-1},\ \ l \in \Omega^{\mathrm{gen}},$$

i.e.

$$W' = \sum_{j=1}^{m'} (-ij)Q_j Y^{j-1} \in \ker(\pi).$$

This relation contradicts our choice of W. Let now α be a Schwartz function in one variable, with compact support disjoint from the subsets $g_{x_{j'}}\cdot G_0, j' \neq j$, such that $(\frac{d}{dx})^{\kappa_j - 1}\alpha(x_j) = 1$ and $(\frac{d}{dx})^j \alpha(x_j) = 0$ for $j = 0, \cdots, \kappa_j - 2$. For every $\beta \in \mathcal{H}_{\pi_0}^\infty$ we have then that:

$$0 = \langle \pi(W)a, \alpha \otimes \beta \rangle = \sum_{i=0}^{\kappa_j} \frac{\partial^i \delta_{x_j}}{\partial x^i}(P_l\alpha)(x_j)\langle D_i^j, \beta\rangle = \kappa_j R_1(l)\langle D_{\kappa_j}^j, \beta\rangle.$$

This shows that $D_{\kappa_j}^j$ must be zero if $\kappa_j \neq 0$ and $R_1(l) \neq 0$. Hence,

$$a = \sum_{j=1}^{m''} \delta_{x_j} \otimes D_0^j$$

with $D_0^j \in (\mathcal{H}_{\pi_{x_j}}^{-\infty})^{K_j,\chi_{l_j}}$ for all the $l \in \Omega^{\mathrm{gen}}$ with $R_1(l) \neq 0$. Finally, we check as before that the set

$$\{l \in \Omega^{\mathrm{gen}}; R_1(l') \neq 0, \forall l' \in p^{-1}(l_{|\mathfrak{k}})\}$$

has full μ_π-measure by the finite multiplicity condition. It suffices now to apply the induction hypothesis.

■

4.4 Corollary: *(Frobenius Reciprocity) The multiplicities* n_π^σ *of the disintegration (1.1) are* μ_π*-almost everywhere equal to the dimension of* $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$, *where* $\sigma = ind_{B[l_{|\mathfrak{k}}]}^K \chi_l$*. Here we assume condition* $\mathcal{N}$, *if* $\pi_{|K}$ *is of finite multiplicities.*

Proof: Since the multiplicities $n_\pi(\sigma)$ are either uniformly bounded or uniformly infinite, it suffices according to Theorem 4.3 to show in the case of infinite multiplicities that the dimension of the space $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ is also infinite. Let $l \in \Omega(\pi)$ be generic, which means here that the number of K-orbits in $p^{-1}(K \cdot (l_{|\mathfrak{k}})) \cap \Omega(\pi)$ is infinite. Let us realize again the representation π as $\mathrm{ind}_{B[l]}^G \chi_l$. For any $g \in G$, such that $g \cdot l \in p^{-1}(l_{|\mathfrak{k}})$, we can define as before an element $a_g \in (\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ by the formula

$$\langle a_g, \xi \rangle = \int_{B[l_{|\mathfrak{k}}]/B[l_{|\mathfrak{k}}] \cap gB[l]g^{-1}} \overline{\xi(bg)\chi_l(b)} d\dot{b} \ (\xi \in \mathcal{H}_\pi^\infty).$$

Let us show that $\dim((\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l})$ is infinite. Let $\mathfrak{g}'$ be a subalgebra of codimension 1 containing $\mathfrak{k}$, $G' = \exp(\mathfrak{g}')$ and $\gamma : \mathfrak{g}^* \to \mathfrak{g}'^*$ the canonical projection. If $\Omega(\pi)$ is not saturated with respect to $\mathfrak{g}'$, then $\pi' = \pi_{|G'}$ is irreducible and its Kirillov-orbit $\Theta_{G'}^{-1}(\pi')$ is $G' \cdot l'$, where $l' = \gamma(l) = l_{|\mathfrak{g}'}$. Since $\pi'_{|K} = \pi_{|K}$ is of infinite multiplicities and since $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ can be identified with $(\mathcal{H}_{\pi'}^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$, the induction hypothesis gives us the expected result.

Suppose now that $\Omega(\pi)$ is saturated with respect to $\mathfrak{g}'$. We can assume that $B[l] \subset G'$. Take $X \in \mathfrak{g} \setminus \mathfrak{g}'$. Whence $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}'$. For $t \in \mathbb{R}$, let $l'_t = \exp(tX) \cdot l' \in \mathfrak{g}'^*$. Then $\Omega(\pi)_{|\mathfrak{g}'}$ is divided into a one-parameter family of G'-orbits $\omega_t = G' \cdot l'_t$ and according to this decomposition $\pi_{|G'}$ is disintegrated into a one-parameter family of irreducible unitary representations $\pi'_t = \Theta_{G'}(\omega_t)$:

$$\pi_{|G'} = \int_{\mathbb{R}}^{\oplus} \pi'_t dt.$$

Let $\mathcal{O}$ be the subset of the t's in $\mathbb{R}$, for which $\pi'_{t|K}$ is of infinite multiplicities. If $\mathcal{O} \neq \emptyset$, then we can assume that $0 \in \mathcal{O}$. Since the space $(\mathcal{H}_{\pi'_0}^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ can be identified via the mapping

$$a_0 \mapsto \delta_0 \otimes a_0$$

(where δ_0 denotes the Dirac distribution at 0) with a subspace of $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$, the induction hypothesis tells us that $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$ contains an infinite dimensional subspace.

Suppose now that $\mathcal{O}$ is empty. Denote by $p' : \mathfrak{g}'^* \to \mathfrak{k}^*$ the canonical projection. Since $p^{-1}(K \cdot (l_{|\mathfrak{k}})) \cap \Omega(\pi)$ contains an infinite number of K-orbits, the subset

$$\mathcal{M} = \{t \in \mathbb{R};\ \omega_t \cap p'^{-1}(K \cdot (l_{|\mathfrak{k}})) \neq \emptyset\}$$

must be infinite. Since the supports of the distributions $a_{\exp(tX) \cdot l}$, $t \in \mathcal{M}$, are disjoint we obtain an infinite family of linearly independent elements of $(\mathcal{H}_\pi^{-\infty})^{B[l_{|\mathfrak{k}}],\chi_l}$.

■

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