C^∞-vectors of irreducible representations of exponential solvable Lie groups

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Abstract

Let G be an exponential solvable Lie group, and π be an irreducible unitary representation of G. Then by induction from a unitary character of a connected subgroup, π is realized in an L^2-space of functions on a homogeneous space. We are concerned with C^∞vectors of π from a viewpoint of rapidly decreasing properties. We show that the subspace $\mathcal{SE}$ consisting of vectors with a certain property of rapidly decreasing at infinity can be embedded as the space of the C^∞vectors in an extension of π to an exponential group including G. Using the space $\mathcal{SE}$, we also give a description of the space $\mathcal{ASE}$ related to Fourier transforms of L^1-functions on G. We next obtain an explicit description of C^∞vectors for a special case. Furthermore, we consider a space of functions on G with a similar rapidly decreasing property and show that it is the space of the C^∞vectors of an irreducible representation of a certain exponential solvable Lie group acting on $L^2(G)$.

1 Introduction

Let G be an exponential solvable group with Lie algebra $\mathfrak{g}$, and π be an irreducible unitary representation of G. According to the orbit method, there exists a linear form $l \in \mathfrak{g}^*$ and a real polarization $\mathfrak{h}$ at l such that the representation π is realized as the induced representation $\mathrm{ind}_H^G \chi_l$ from χ_l of H, where $H = \exp \mathfrak{h}$ is the connected and simply connected subgroup with Lie algebra $\mathfrak{h}$ and χ_l is the unitary character of H defined by $\chi_l(\exp X) = e^{il(X)}$ for $X \in \mathfrak{h}$.

Suppose that G is nilpotent, and realize π on $L^2(\mathbb{R}^m)$ by taking a supplementary Malcev basis to $\mathfrak{h}$ and identifying G/H with $\mathbb{R}^m$. Then by results of Kirillov [5] and Corwin-Greenleaf-Penney [4], it is well known that the action of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ forms the algebra of differential operators with polynomial coefficients, and the space of the C^∞vectors is precisely the Schwartz space $\mathcal{S}(\mathbb{R}^m)$.

2000 *Mathematics Subject Classification.* Primary 22E27; Secondary 22E25, 43A85

Key Words and Phrases. Exponential solvable Lie group, unitary representation, C^∞-vector,

*Partially supported by Grant-in-Aid for Scientific Research (C), No. 15540171, Japan Society for the Promotion of Science

However, when G is a general exponential solvable Lie group, the space of the C^∞vectors does not have such simple characterizations. For example, the action of $\mathcal{U}(\mathfrak{g})$ may involve multiplications of exponential functions which require C^∞vectors to have a property of rapidly decreasing at infinity in one direction but do not necessarily require such property in another direction.

In this paper, we investigate structures of the C^∞vectors from a viewpoint of some rapidly decreasing properties. In section 2, under a standard realization of π, we are concerned with the subspace $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ consisting of functions with a rapidly decreasing property defined in Definition 2.3. We shall show that it can be embedded as the space of the C^∞vectors in a space of irreducible representation π_{l_0} of an exponential solvable group $F \supset G$ such that the restriction of π_{l_0} to G is equivalent to π. By using this space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$, we also describe the space $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ introduced by Ludwig [7], which is included in the image of Fourier transforms of L^1-functions on G of finite ranks. In section 3, we shall give an explicit characterization of C^∞vectors when G can be described as $G = N^l N$, where N and N^l are the subgroups corresponding to the nilradical of $\mathfrak{g}$ and its stabilizer for l, respectively. In section 4, we are also concerned with the space $\mathcal{SE}(G)$, a space of functions on G with a similar property of rapidly decreasing at infinity, and we shall show that it is the space of C^∞vectors of an irreducible representation of a certain exponential solvable Lie group acting on $L^2(G)$.

2 The space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ (for details on the theory of exponential solvable Lie groups see [6] and [3]). Let $\mathfrak{n}$ be a nilpotent ideal including $[\mathfrak{g}, \mathfrak{g}]$. (For instance we can take the nilradical of $\mathfrak{g}$.) Let $\pi \in \hat{G}$ be an irreducible unitary representation of G, and $l \in \mathfrak{g}^*$ be a real linear form such that the coadjoint orbit $G \cdot l$ corresponds to π. We denote by $\mathfrak{g}^l = \mathfrak{g}(l)$ and $\mathfrak{n}^l$ the stabilizers defined as follows:

$$\mathfrak{g}^l = \mathfrak{g}(l) := \{X \in \mathfrak{g};\ l([X, \mathfrak{g}]) = \{0\}\},$$
$$\mathfrak{n}^l := \{X \in \mathfrak{g};\ l([X, \mathfrak{n}]) = \{0\}\}.$$

Definition 2.1. (see [9]) We say that a polarization $\mathfrak{h}$ at $l \in \mathfrak{g}^*$ is adapted to $\mathfrak{n}$, if

1. $\mathfrak{h} \cap \mathfrak{n}$ is a polarization at $l_{|\mathfrak{n}}$
2. $[\mathfrak{n}^l, \mathfrak{h} \cap \mathfrak{n}] \subset \mathfrak{h} \cap \mathfrak{n}$.

Then $\mathfrak{h}$ is a Pukanszky polarization and there exists a polarization $\mathfrak{h}_0 \subset \mathfrak{n}^l$ at $l_{|\mathfrak{n}^l}$ such that $\mathfrak{h} = \mathfrak{h}_0 + (\mathfrak{h} \cap \mathfrak{n})$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{n}^l$.

Remark 2.2. (1) For any l and $\mathfrak{n}$, there exists a polarization adapted to $\mathfrak{n}$. For example, a Vergne polarization associated with a refinement of the series of ideals $\{0\} \subset \mathfrak{n} \subset \mathfrak{g}$ is adapted to $\mathfrak{n}$.

(2) Let $\mathfrak{h}_{\mathfrak{n}}$ be a polarization at $l_{|\mathfrak{n}}$, such that

$$[\mathfrak{n}^l, \mathfrak{h}_{\mathfrak{n}}] \subset \mathfrak{h}_{\mathfrak{n}}.$$

If $\mathfrak{h}_0 \subset \mathfrak{n}^l$ denotes any polarization at $l_{|\mathfrak{n}^l}$, then

$$\mathfrak{h} := \mathfrak{h}_0 + \mathfrak{h}_{\mathfrak{n}}$$

is a Pukanszky polarization at l. Let $\mathfrak{m} := \mathfrak{n}^l \cap \mathfrak{n} \cap \ker(l)$. Then $\mathfrak{m}$ is an ideal of $\mathfrak{n}^l$ and $\mathfrak{n}^l/\mathfrak{m}$ is either abelian or a direct sum of a central ideal and a Heisenberg algebra. In particular any polarization $\mathfrak{h}_0 \subset \mathfrak{n}^l$ at $l_{|\mathfrak{n}^l}$ is a Pukanszky polarization, since $\mathfrak{n}^l/\mathfrak{m}$ is at most nilpotent of step 2.

Let $\mathfrak{h}$ be a polarization at l adapted to $\mathfrak{n}$, $H = \exp \mathfrak{h}$, χ_l a unitary character of H such that $d\chi_l = il$. Let $\mathcal{D}(G/H)$ be the space of all continuous functions $f : G \to \mathbb{C}$ with compact support modulo H, such that $f(gh) = \frac{\Delta_H(h)}{\Delta_G(h)} f(g)$ for all $h \in H$ and $g \in G$. On this space there exists a unique positive left invariant linear functional

$$f \mapsto \oint_{G/H} f(g) d\mu_{G/H}(g) \tag{2.1}$$

(see [3]). Then we realize π as $\pi = \pi_{l,H} = \mathrm{ind}_H^G \chi_l$ in $\mathcal{H}_\pi$, where $\mathcal{H}_\pi = L^2(G/H, \chi_l)$ is the completion with respect to the norm $\| \|_\pi$ of the space $\mathcal{D}(G/H, \chi_l)$ of the continuous functions ϕ with compact support modulo H on G such that

1. $\phi(gh) = \chi_l(h)^{-1} \Delta_{H,G}^{1/2}(h) \phi(g)$ for all $h \in H$, $g \in G$.
2. $\|\phi\|_\pi^2 := \oint_{G/H} |\phi(g)|^2 d\mu_{G/H}$,

where Δ_G and Δ_H are the modular functions of G and H, respectively, and $\Delta_{H,G}^{1/2} = (\Delta_H/\Delta_G)^{1/2}$.

Taking coexponential bases $\{T_1, \cdots, T_\nu\}$ for $\mathfrak{n}^l + \mathfrak{n}$ in $\mathfrak{g}$, $\{T_{\nu+1}, \cdots, T_m\}$ for $\mathfrak{n} + \mathfrak{h}$ in $\mathfrak{n}^l + \mathfrak{n}$, $\{R_1, \cdots, R_v\}$ for $\mathfrak{h}$ in $\mathfrak{n} + \mathfrak{h}$, we identify G/NH with $\mathbb{R}^m$, NH/H with $\mathbb{R}^v$ by $t = (t_1, \cdots, t_m) \mapsto E(t) := \exp t_1 T_1 \cdots \exp t_m T_m$ modulo HN, $r = (r_1, \cdots, r_v) \mapsto E(r) := \exp r_1 R_1 \cdots \exp r_v R_v$ modulo H, respectively, and G/H with $\mathbb{R}^{m+v}$ by $(t, r) \mapsto E(t, r) := E(t)E(r)$ modulo H.
We can now express the integral (2.1) as an integral on $\mathbb{R}^{m+v}$:

$$\oint_{G/H} f(g) d\mu_{G/H}(g) = \int_{\mathbb{R}^{m+v}} f(E(t, r)) dt dr, \; f \in \mathcal{D}(G/H),$$

(see [6]).

Definition 2.3. Let $\mathfrak{D}_{t,r}$ be the space of all differential operators on $\mathbb{R}^{m+v}$ with polynomial coefficients and let $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of all functions $\phi \in \mathcal{H}_{\pi_{l,H}}$ such that

1. ϕ is smooth,

2.
$$\|\phi\|_{a,D}^2 := \int_{\mathbb{R}^{m+v}} e^{a\|t\|}|D(\phi\circ E)(t,r)|^2 dtdr < \infty, \quad \forall a\in\mathbb{R}_+, \forall D\in\mathfrak{D}_{t,r}.$$

(Here $\|t\|$ denotes a norm on $\mathbb{R}^{m+v}$.)

Remark that this space is independent of the choice of coexponential bases (see [6]).

2.1 $\mathcal{SE}$-space and C^∞ vectors

We shall define an exponential solvable group $F \supset G$ such that its Lie algebra $\mathfrak{f}$ is of the form $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal and $[\mathfrak{n}+\mathfrak{h}, \mathfrak{a}] = \{0\}$. We also show that any linear functional l_0 of $\mathfrak{f}$ whose restriction to $\mathfrak{g}$ equals l satisfies the condition $\dim(\mathfrak{f}(l_0)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$, where $\mathfrak{f}(l_0) = \{X\in\mathfrak{f};\ l_0([X,\mathfrak{f}]) = \{0\}\}$, which implies that $G\cdot l_0 = F\cdot l_0$, and show that $\mathfrak{p} := \mathfrak{h}+\mathfrak{a}$ is a polarization at l_0 with the Pukanszky condition.

For every l_0, we have that the restriction $\pi_{l_0,P|G}$ of $\pi_{l_0,P}$ to G and $\pi_{l,H}$ are equivalent; the G-equivariant unitary mapping $R_{l_0} : \mathcal{H}_{\pi_{l_0,P}} \to \mathcal{H}_{\pi_{l,H}}$

$$R_{l_0}\phi = \phi_{|G}$$

is a unitary intertwining operator and its inverse S_{l_0} is given by

$$S_{l_0} : \mathcal{H}_{\pi_{l,H}} \to \mathcal{H}_{\pi_{l_0,P}}, \quad S_{l_0}\phi(g\exp A) := e^{-il_0(A)}\phi(g),\ g\in G, A\in\mathfrak{a}.$$

We obtain a new set of norms on the space $\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})$ by letting for every element $U\in\mathcal{U}(\mathfrak{f})$,

$$\|\phi\|_{l_0,U} := \|d\pi_{l_0,P}(U)S_{l_0}\phi\|_{\pi_{l_0}}$$

It is easy to see that for every $U\in\mathcal{U}(\mathfrak{f})$, we have $a\in\mathbb{R}_+$ and an element $D\in\mathfrak{D}_{t,r}$ such that

$$\|\phi\|_{l_0,U} \le \|\phi\|_{a,D}, \text{ for all } \phi\in\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}).$$

Indeed, if we use the coordinates (t,r) for G/H, then for any $X\in\mathcal{U}(\mathfrak{g})$ we have that $d\pi_{l,H}(X)$ is a differential operator with coefficients which are bounded by $e^{a\|t\|}(1+\|r\|)^k$ for some $a,k\in\mathbb{R}_+$. This shows that

$$S_{l_0}(\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})) \subset \mathcal{H}^\infty_{\pi_{l_0,P}}. \tag{2.2}$$

Theorem 2.4. *Let $G=\exp\mathfrak{g}$ be an exponential solvable Lie group, $\mathfrak{n}$ be a nilpotent ideal such that $\mathfrak{n}\supset[\mathfrak{g},\mathfrak{g}]$, $l\in\mathfrak{g}^*$, and $\mathfrak{h}$ be a polarization at l adapted to $\mathfrak{n}$. Then there exists an exponential solvable Lie group F with Lie algebra $\mathfrak{f}=\mathfrak{g}\ltimes\mathfrak{a}$ which satisfies the following:*

(1) $\mathfrak{a}$ *is an abelian ideal of dimension* $2m = 2\dim(\mathfrak{g}/(\mathfrak{n}+\mathfrak{h}))$ *and* $[\mathfrak{n}+\mathfrak{h},\mathfrak{a}] = \{0\}$, *and there exists a coexponential basis* $\{X_j\}_{1\le j\le m}$ *for* $\mathfrak{n}+\mathfrak{h}$ *in* $\mathfrak{g}$ *and a basis* $\{A_1,\cdots,A_m,B_1,\cdots,B_m\}$ *of* $\mathfrak{a}$ *such that*

$$[X_j, A_k] = \delta_{j,k}A_k, \quad [X_j, B_k] = -\delta_{j,k}B_k, \quad 1 \le j,k \le m.$$

(2) For all extension $l_1 \in \mathfrak{f}^*$ *of* l, *we have* $\dim(\mathfrak{f}(l_1)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$, *and the subalgebra* $\mathfrak{p} = \mathfrak{h}+\mathfrak{a}$ *is a Pukanszky polarization at* l_1 *adapted to* $\mathfrak{n}+\mathfrak{a}$.

(3) There exists an extension $l_0 \in \mathfrak{f}^*$ *of* l *such that the family of norms* $\{\| \|_{a,D}, a \in \mathbb{R}_+, D \in \mathfrak{D}_{t,r}\}$ *is equivalent to the family of norms* $\{\| \|_{l_0,U},\ U \in \mathcal{U}(\mathfrak{f})\}$ *and we have that*

$$\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}) = R_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}}),$$

where $P = \exp\mathfrak{p}$.

Proof. By (2.2), we have only to show that $\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}) \supset R_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}})$. We make an induction on the dimension of G. If $\mathfrak{g}$ is abelian or $\mathfrak{n} = \mathfrak{g}$, the statement is trivial. Suppose that $l = 0$ on an abelian ideal $\mathfrak{i} \neq \{0\}$. Then $\mathfrak{h} \supset \mathfrak{i}$. Let $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$, $\dot{\mathfrak{n}} = (\mathfrak{n}+\mathfrak{i})/\mathfrak{i}$, $\dot{\mathfrak{h}} = \mathfrak{h}/\mathfrak{i}$, $\dot{G} = \exp\dot{\mathfrak{g}} = G/I$, $I = \exp\mathfrak{i}$. Then, denoting quotient maps by $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, $Q : G \to G/I$, we have $\dot{\pi} \in \widehat{\dot{G}}$ such that $\dot{\pi} \circ Q = \pi$, and we have $\dot{\pi} = \mathrm{ind}_{\dot{H}}^{\dot{G}}\chi_{\dot{l}}$, where $\dot{l} \circ q = l$. By the induction hypothesis for $(\dot{G},\dot{\mathfrak{n}},\dot{l},\dot{\mathfrak{h}})$, there exists an exponential solvable Lie group $\dot{F} = \exp\dot{\mathfrak{f}}$, $\dot{\mathfrak{f}} = \dot{\mathfrak{g}} \ltimes \dot{\mathfrak{a}}$ and an extension $\dot{l}_0 \in \dot{\mathfrak{f}}^*$ of $\dot{l}$ with the required properties.

Let $\mathfrak{f} = \mathfrak{g} \ltimes \dot{\mathfrak{a}}$ defined by $[X,\dot{A}] := [q(X),\dot{A}]$ for $X \in \mathfrak{g}$, $\dot{A} \in \dot{\mathfrak{a}}$, and an extension $l_0 \in \mathfrak{f}^*$ of l be defined by $l_0|_{\dot{\mathfrak{a}}} = \dot{l}_0$. Then we have that $\mathfrak{f}$ and l_0 has the required properties for $(G,\mathfrak{n},l,\mathfrak{h})$.

Suppose $l \neq 0$ on any non-zero abelian ideal. Let $\mathfrak{g}_1$ be a minimal ideal contained in $\mathfrak{n}$. Then there are following possibilities (see [6]):

(1) $\mathfrak{g}_1$ is non-central. Then $\dim(\mathfrak{g}_1) = 1$ or 2:

a) There exist $Y \in \mathfrak{g}_1$, $\lambda \in \mathfrak{g}^*$, and $X \in \mathfrak{g}^*$ such that $\mathfrak{g}_1 = \mathbb{R}Y$, $l(Y) = 1$,

$$[U,Y] = \lambda(U)Y \quad \text{for all} \quad U \in \mathfrak{g},$$

$$\lambda(X) = 1$$

b) There exist $Y_1, Y_2 \in \mathfrak{g}_1$, $\lambda \in \mathfrak{g}^*$, $\omega \in \mathbb{R}\setminus\{0\}$ and $X \in \mathfrak{g}^*$ such that $l(Y_1) \neq 0$, $\mathfrak{g}_1 = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$, and

$$[U,Y_1] = \lambda(U)(Y_1 - \omega Y_2), \quad [U,Y_2] = \lambda(U)(\omega Y_1 + Y_2) \quad \text{for all } U \in \mathfrak{g}^*,$$

$$\lambda(X) = 1.$$

(2) $\mathfrak{g}_1$ is the center of $\mathfrak{g}$. Then $\mathfrak{g}_1$ is one dimensional because of the assumption of l. Let $Z \in \mathfrak{g}_1$ such that $l(Z) = 1$.

Suppose first that $\mathfrak{g}_1$ is properly contained in $\mathfrak{n}$. Let $\mathfrak{g}_2$ be a minimal ideal modulo $\mathfrak{g}_1$ such that $\mathfrak{g}_2 \subset \mathfrak{n}$. Then

a) $\mathfrak{g}_2$ is two-dimensional and there exist $Y \in \ker(l) \cap \mathfrak{g}_2$, $X \in [\mathfrak{g}, \mathfrak{g}] \cap \ker(l), T \in \ker(l)$, $\lambda, \gamma \in \mathfrak{g}^*$ such that

$$[U, Y] = \lambda(U)Y + \gamma(U)Z \quad \text{for all } U \in \mathfrak{g}, \tag{2.3}$$

$$\lambda(T) = 1,\ \lambda(X) = 0,\ \gamma(T) = 0,\ \gamma(X) = 1.$$

Then we have $[T, X] \in -X + (\ker(\lambda) \cap \ker(\gamma))$, and $\ker(\gamma) + \mathfrak{n} = \mathfrak{g}$.

b) $\mathfrak{g}_2$ is two-dimensional and there exist $Y \in \ker(l) \cap \mathfrak{g}_2$, $X \in \ker(l), \gamma \in \mathfrak{g}^*$ such that

$$[U, Y] = \gamma(U)Z \qquad \text{for all } U \in \mathfrak{g}, \tag{2.4}$$

$$\gamma(X) = 1.$$

Then we have two subcases:

b-1) $\mathfrak{n} + \ker(\gamma) = \mathfrak{g}$.

b-2) $\mathfrak{n} + \ker(\gamma) \neq \mathfrak{g}$ (here λ is necessarily 0).

c) $\mathfrak{g}_2$ is 3-dimensional and there exist $Y_1, Y_2 \in \mathfrak{g}_2 \cap \ker(l)$, $X_1, X_2 \in [\mathfrak{g}, \mathfrak{g}] \cap \ker(l), T \in \ker(l)$, $\lambda, \gamma_1, \gamma_2 \in \mathfrak{g}^*, \omega \in \mathbb{R}^*$, such that for all $U \in \mathfrak{g}$

$$\begin{aligned} [U, Y_1] &= \lambda(U)(Y_1 - \omega Y_2) + \gamma_1(U)Z, \\ [U, Y_2] &= \lambda(U)(\omega Y_1 + Y_2) + \gamma_2(U)Z, \end{aligned} \tag{2.5}$$

$$\gamma_j(X_i) = \delta_{i,j},\ i, j = 1, 2, \quad \gamma_1(T) = \gamma_2(T) = 0,$$
$$0 = \lambda(X_1) = \lambda(X_2),\ \lambda(T) = 1.$$

Then we have $\mathfrak{n} + \ker(\gamma_1) \cap \ker(\gamma_2) = \mathfrak{g}$.

Suppose now that $\mathbb{R}Z = \mathfrak{g}_1 = \mathfrak{n}$. Then $\mathfrak{n}^l = \mathfrak{g}$. Since the center is one dimensional, our $\mathfrak{g}$ is the Heisenberg algebra, if $\mathfrak{g}$ is not abelian, which we assume. We can take a bases $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z\}$ such that $[X_i, Y_j] = \delta_{i,j}Z$ $(i, j = 1, \cdots, n)$ and so that $\mathfrak{h}$ is spanned by $\{Y_1, \cdots, Y_n, Z\}$.

Case (1): Let $\mathfrak{k} := \ker(\lambda)$. Then $\mathfrak{k}$ is an ideal, $\mathfrak{k} \supset \mathfrak{n} + \mathfrak{n}^l$, and $\mathfrak{g}_1 \subset \mathfrak{h} \subset \mathfrak{k}$. We have $\pi = \mathrm{ind}_K^G \pi_1$, where $\pi_1 = \mathrm{ind}_H^K \chi_l$, $K = \exp \mathfrak{k}$.

By the induction hypothesis for $(K, \mathfrak{n}, l|_{\mathfrak{k}}, \mathfrak{h})$, there exists $\tilde{F} = \exp \tilde{\mathfrak{f}}$ such that $\mathfrak{k} \ltimes \tilde{\mathfrak{a}} = \tilde{\mathfrak{f}}$, where $\tilde{\mathfrak{a}}$ is an abelian ideal such that $[\mathfrak{n} + \mathfrak{h}, \tilde{\mathfrak{a}}] = \{0\}$, with the required properties: For all extension $l_1 \in \tilde{\mathfrak{f}}^*$ of $l|_{\mathfrak{k}}$, we have $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{k}(l|_{\mathfrak{k}})) + \dim(\tilde{\mathfrak{a}})$, and the subalgebra

$\tilde{\mathfrak{p}} = \mathfrak{h} + \tilde{\mathfrak{a}}$ is a Pukanszky polarization at l_1. And there exists an extension $\tilde{l} \in \tilde{\mathfrak{f}}^*$ of $l|_{\mathfrak{k}}$ such that the corresponding family of norms are equivalent and such that

$$\mathcal{SE}(K, \mathfrak{n}, l|_{\mathfrak{k}}, \mathfrak{h}) = R_{\tilde{l}}(\mathcal{H}^{\infty}_{\pi_{\tilde{l},\tilde{P}}}).$$

We have in case (1) a) that $[X, Y] = Y$, and in case (1) b) that $[X, Y_1] = Y_1 - \omega Y_2$, $[X, Y_2] = \omega Y_1 + Y_2$, and in both cases that $[\mathfrak{g}_1, \mathfrak{k}] = \{0\}$.

Let $\mathfrak{a} = \tilde{\mathfrak{a}} \oplus \mathbb{R}A \oplus \mathbb{R}B$ be an abelian Lie algebra, and $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ defined by

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{a} = \mathbb{R}X \oplus \mathfrak{k} \oplus \tilde{\mathfrak{a}} \oplus \mathbb{R}A \oplus \mathbb{R}B,$$
$$[A, \tilde{\mathfrak{f}}] = [B, \tilde{\mathfrak{f}}] = \{0\},\ [X, A] = A,\ [X, B] = -B,\ [\tilde{\mathfrak{a}}, X] = \{0\}.$$

Let $\mathfrak{e} = \tilde{\mathfrak{f}} \oplus \mathbb{R}A \oplus \mathbb{R}B = \mathfrak{k} \oplus \tilde{\mathfrak{a}} \oplus \mathbb{R}A \oplus \mathbb{R}B$, $E = \exp \mathfrak{e}$, $\mathfrak{f} = \mathfrak{g} \oplus \mathbb{R}A \oplus \mathbb{R}B$, $F = \exp \mathfrak{f}$.

By the assumption of l and $[\mathfrak{g}_1, \mathfrak{k}] = \{0\}$, we have that $\dim(\mathfrak{g}_1/(\mathfrak{g}(l) \cap \mathfrak{g}_1)) = 1$ and $\dim(\mathfrak{g}(l)) + 1 = \dim(\mathfrak{k}(l|_{\mathfrak{k}}))$. Let $l_0 \in \mathfrak{f}^*$ be an extension of l and $l_1 = l_0|_{\tilde{\mathfrak{f}}}$. We also have that $\dim(\mathfrak{g}_1/(\mathfrak{f}(l_0) \cap \mathfrak{g}_1)) = 1$ and $\dim(\mathfrak{f}(l_0)) = \dim(\tilde{\mathfrak{f}}(l_1)) + 1$. In fact, suppose first that $l_0|_{\mathbb{R}A+\mathbb{R}B} \neq 0$. If $l_0(A)l_0(B) \neq 0$ and $C = \alpha A + \beta B \in \ker(l_0) \setminus \{0\}$, where $\alpha, \beta \in \mathbb{R}$, then $C' := \alpha A - \beta B \notin \ker(l_0)$, $[X, C'] = C$, $[X, C] = C'$, and the mapping $\tilde{\mathfrak{f}}(l_1) \oplus \mathbb{R}C' \ni V \mapsto V - \frac{l_0([X,V])}{l_0(C')} C \in \mathfrak{f}(l_0)$ gives a linear isomorphism of $\tilde{\mathfrak{f}}(l_1) \oplus \mathbb{R}C'$ and $\mathfrak{f}(l_0)$. Similarly, if $l_0(A) = 0$ and $l_0(B) \neq 0$, then taking $Y_0 \in \mathfrak{g}_1 \setminus \mathfrak{f}(l_0)$, $\tilde{\mathfrak{f}}(l_1) \oplus \mathbb{R}A \oplus \mathbb{R}B = \mathfrak{f}(l_0) \oplus \mathbb{R}Y_0$. If $l_0|_{\mathbb{R}A+\mathbb{R}B} = 0$, then $\mathfrak{f}(l_0) \supset \mathbb{R}A \oplus \mathbb{R}B$ and taking $Y_0 \in \mathfrak{g}_1 \setminus \mathfrak{f}(l_0)$, we have $\tilde{\mathfrak{f}}(l_1) \oplus \mathbb{R}A \oplus \mathbb{R}B = \mathfrak{f}(l_0) \oplus \mathbb{R}Y_0$. Since $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{k}(l|_k)) + \dim(\tilde{\mathfrak{a}})$, we have

$$\begin{aligned} \dim(\mathfrak{f}(l_0)) &= \dim(\tilde{\mathfrak{f}}(l_1)) + 1 = \dim(\mathfrak{k}(l|_{\mathfrak{k}})) + \dim(\tilde{\mathfrak{a}}) + 1 \\ &= \dim(\mathfrak{k}(l|_{\mathfrak{k}})) + \dim(\mathfrak{a}) - 1 = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a}). \end{aligned}$$

Since $\tilde{\mathfrak{p}} = \mathfrak{h} + \tilde{\mathfrak{a}}$ is a Pukanszky polarization at $l_0|_{\tilde{\mathfrak{f}}}$ adapted to $\mathfrak{n} + \tilde{\mathfrak{a}}$ and $\mathbb{R}A \oplus \mathbb{R}B$ is central in $\mathfrak{e}$, we also have that $\mathfrak{p} = \tilde{\mathfrak{p}} \oplus \mathbb{R}A \oplus \mathbb{R}B$ is a Pukanszky polarization at l_0 adapted to $\mathfrak{n} + \mathfrak{a}$. Letting l_0 be an extension of $\tilde{l}$ such that $l_0(B) \neq 0$, and $\tilde{\pi} = \pi_{\tilde{l},\tilde{P}} = \mathrm{ind}_{\tilde{P}}^{\tilde{F}} \chi_{\tilde{l}}$, we realize $\tau = \tau_{l_0} = \mathrm{ind}_P^E \chi_{l_0}$ in $\mathcal{H}_{\tilde{\pi}}$ by $\tau(\tilde{x}a)v = \chi_{l_0}(a)\tilde{\pi}(\tilde{x})v$ for $v \in \mathcal{H}_{\tilde{\pi}}$, $\tilde{x} \in \tilde{F}$, $a \in \exp(\mathbb{R}A + \mathbb{R}B)$.

Now, we realize $\pi_{l_0,P} = \mathrm{ind}_P^F \chi_{l_0}$ as $\mathrm{ind}_E^F \tau_{l_0}$ on $L^2(\mathbb{R}, \mathcal{H}_{\tilde{\pi}})$. Then for $\phi = \phi(x) \in L^2(\mathbb{R}, \mathcal{H}_{\tilde{\pi}})$, we have in case (1) a)

$$\begin{aligned} &d\pi_{l_0,P}(X)\phi(x) = -\frac{d}{dx}\phi(x), \\ &d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x), \\ (2.6)\qquad &d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^{x}\phi(x), \\ &d\pi_{l_0,P}(Y)\phi(x) = ie^{-x}\phi(x), \\ &d\pi_{l_0,P}(V)\phi(x) = d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V)(\phi(x)), \qquad V \in \tilde{\mathfrak{f}} \end{aligned}$$

and in case (1) b)

$$
(2.7)\qquad \begin{aligned}
d\pi_{l_0,P}(X)\phi(x) &= -\frac{d}{dx}\phi(x),\\
d\pi_{l_0,P}(A)\phi(x) &= il_0(A)e^{-x}\phi(x),\\
d\pi_{l_0,P}(B)\phi(x) &= il_0(B)e^{x}\phi(x),\\
d\pi_{l_0,P}(Y_1)\phi(x) &= ie^{-x}\left(l(Y_1)\cos(\omega x)+l(Y_2)\sin(\omega x)\right)\phi(x),\\
d\pi_{l_0,P}(Y_2)\phi(x) &= ie^{-x}\left(-l(Y_1)\sin(\omega x)+l(Y_2)\cos(\omega x)\right)\phi(x),\\
d\pi_{l_0,P}(V)\phi(x) &= d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V)(\phi(x)), \qquad V\in\tilde{\mathfrak{f}}.
\end{aligned}
$$

Since we can regard

$$
G/H = (G/K)(K/H) = (G/K)(K/NH)(NH/H),
$$

we can use the coordinates t, r for K/H, and for G/K we use the coordinate x. We show that for $a\in\mathbb{R}$ and $D=\frac{\partial^n}{\partial x^n}\otimes D_{t,r}$, $D_{t,r}\in\mathfrak{D}_{t,r}$ there exists a finite family $\{U_1,\cdots,U_N\}$ in $\mathcal{U}(\mathfrak{f})$, such that $\|\ \|_{a,D}\le\sum_{j=1}^N\|\ \|_{l_0,U_j}$. Indeed, by the induction hypothesis, there exists a finite family $\{\tilde{U}_1,\cdots,\tilde{U}_{\tilde{N}}\}$ in $\mathcal{U}(\tilde{\mathfrak{f}})$, such that

$$
\begin{aligned}
&\int_{K/H} e^{a\|t\|}|D_{t,r}\phi(\exp(xX)E(t,r))|^2 dt dr\\
\le\ & \left(\sum_{j=1}^{\tilde{N}}\left(\int_{\tilde{F}/\tilde{P}}|d\tilde{\pi}(\tilde{U}_j)S_{\tilde{l}}\phi(\exp(xX)k)|^2 d\mu_{\tilde{F}/\tilde{P}}(k)\right)^{1/2}\right)^2\\
\le\ & \tilde{N}^2\sup_{j=1,\cdots,\tilde{N}}\int_{\tilde{F}/\tilde{P}}|d\tilde{\pi}(\tilde{U}_j)S_{\tilde{l}}\phi(\exp(xX)k)|^2 d\mu_{\tilde{F}/\tilde{P}}(k)
\end{aligned}
$$

for all $\phi\in\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})$.

Let d_i be the degree of $\tilde{U}_i$ in $\mathcal{U}(\tilde{\mathfrak{f}})$ and let $\{V_1^i,\cdots,V_{M_i}^i\}$ be a basis of $\mathcal{U}(\tilde{\mathfrak{f}})_{d_i}$, the subspace of $\mathcal{U}(\tilde{\mathfrak{f}})$ consisting of the elements of degree $\le d_i$. Then $\mathrm{Ad}(\exp(xX))\tilde{U}_i = \sum_{j=1}^{M_i}\psi_j^i(x)V_j^i,\ x\in\mathbb{R}$, where the functions ψ_j^i are C^∞ and are bounded by exponential functions. Therefore

$$
(2.8)\qquad \tilde{U}_i = \mathrm{Ad}(\exp(-xX))(\mathrm{Ad}(\exp(xX))\tilde{U}_i) = \sum_{j=1}^{M_i}\psi_j^i(x)\mathrm{Ad}(\exp(-xX))V_j^i
$$

and so

$$
\begin{aligned}
&\|\phi\|_{a,c,D}^2\\
&= \int_{\mathbb{R}}\int_{K/H} e^{c|x|}e^{a\|t\|}\left|\frac{\partial^n}{\partial x^n}D_{t,r}\phi(\exp(xX)E(t,r))\right|^2 dx dt dr
\end{aligned}
$$

$$\leq \int_{\mathbb{R}} e^{c|x|}\tilde{N}^2 \sup_{i=1,\cdots,\tilde{N}} \int_{\tilde{F}/\tilde{P}} \left| d\tilde{\pi}(\tilde{U}_i) S_{\tilde{l}} \frac{\partial^n}{\partial x^n}\phi(\exp(xX)k)\right|^2 d\mu_{\tilde{F}/\tilde{P}}(k)dx$$

$$\leq \int_{\mathbb{R}} \tilde{N}^2 e^{c|x|}$$

$$\sum_{i=1}^{\tilde{N}} M_i^2 \int_{\tilde{F}/\tilde{P}} |\psi_j^i(x)|^2 \sum_{j=1}^{M_i} \left| d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V_j^i) S_{\tilde{l}} \frac{\partial^n}{\partial x^n}\phi(\exp(xX)k)\right|^2 d\mu_{\tilde{F}/\tilde{P}}(k)dx$$

$$\leq \tilde{N}^2 \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{M_i} M_i^2$$

$$\int_{\mathbb{R}} \int_{\tilde{F}/\tilde{P}} e^{c|x|} |\psi_j^i(x)|^2 \left| d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V_j^i) S_{\tilde{l}} \frac{\partial^n}{\partial x^n}\phi(\exp(xX)k)\right|^2 d\mu_{\tilde{F}/\tilde{P}}(k)dx$$

$$\leq \tilde{N}^2 \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{M_i} M_i^2 \int_{\mathbb{R}} \int_{\tilde{F}/\tilde{P}} C_{i,j} e^{\alpha_{i,j}|x|} \left| d\pi_{l_0,P}(V_j^i) \frac{\partial^n}{\partial x^n} S_{l_0}\phi(\exp(xX)k)\right|^2 d\mu_{\tilde{F}/\tilde{P}}(k)dx$$

with some constant $C_{i,j}, \alpha_{i,j} \in \mathbb{R}_+$. It follows now from the formulas in (2.6) and (2.7), that there exists a finite family $U_1, \cdots, U_N$ in $\mathcal{U}(\mathfrak{f})$, such that

$$\| \ \|_{a,c,D} \leq \sum_{j=1}^{N} \| \ \|_{l_0,U_j}.$$

and so

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = R_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}}).$$

Case (2) a), b), c): $\mathfrak{g}_1 \neq \mathfrak{n}$, Let $\mathfrak{k} = \ker(\gamma)$ in case a) and b), resp. $\mathfrak{k} = \ker(\gamma_1) \cap \ker(\gamma_2)$ in case c), and $\mathfrak{k}_0 = \{U \in \mathfrak{g}; [U, \mathfrak{g}_2] = \{0\}\}$. We remark that $\mathfrak{g}_2 \cap \mathfrak{g}(l) = \mathfrak{g}_1$ and $\mathfrak{k}(l|_{\mathfrak{k}}) = \mathfrak{g}(l) + \mathfrak{g}_2$ because of our assumption. Thus we have $\dim(\mathfrak{k}(l|_{\mathfrak{k}})) = \dim(\mathfrak{g}(l)) + 1$ in cases a) and b), resp. $\dim(\mathfrak{k}(l|_{\mathfrak{k}})) = \dim(\mathfrak{g}(l)) + 2$ in case c).

We have two possibilities: either i): $\mathfrak{g}_2 \subset \mathfrak{h}$, or ii): $\mathfrak{g}_2 \not\subset \mathfrak{h}$.

Case (2) a), b), c); i): We begin with case i). Then $\mathfrak{h}$ must be contained in $\mathfrak{k}$. If not, there exists $X' \in \mathfrak{h} \setminus \mathfrak{k}$. But then $X' = \alpha T + \beta X + X_0$ where $\beta \in \mathbb{R}$, $\beta \neq 0$, $X_0 \in \mathfrak{k}_0$ (in case a) , $X' = \alpha X + X_0$ with $X_0 \in \mathfrak{k}$ $\alpha \neq 0$ (in case b), $X' = \alpha T + \beta X_1 + \delta X_2 + X_0$ where $\alpha, \beta, \delta \in \mathbb{R}$, $\beta^2 + \delta^2 \neq 0$, $X_0 \in \mathfrak{k}_0$ (in case c) and so by (2.3), (2.4) and (2.5),

$$\{0\} = l([X', \mathfrak{g}_2]) = l([X', \mathbb{R}Y_1 + \mathbb{R}Y_2]) = l((\beta\mathbb{R} + \delta\mathbb{R})Z) \neq \{0\},$$

since $\mathfrak{g}_2 \subset \mathfrak{h}$ in case c) and similarly in the other two cases. This contradiction tells us that $\mathfrak{h} \subset \mathfrak{k}$.

Hence we have $\pi = \mathrm{ind}_K^G(\mathrm{ind}_H^K \chi_l)$, and by the induction hypothesis for $(K, \mathfrak{k} \cap \mathfrak{n}, l|_{\mathfrak{k}}, \mathfrak{h})$, there exists $\tilde{\mathfrak{f}} = \mathfrak{k} \ltimes \tilde{\mathfrak{a}}$ such that $[\tilde{\mathfrak{a}}, \tilde{\mathfrak{a}}] = \{0\}$, $[\tilde{\mathfrak{a}}, (\mathfrak{k} \cap \mathfrak{n}) + \mathfrak{h}] = \{0\}$, and having the required properties: $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\tilde{\mathfrak{a}}) + \dim(\mathfrak{k}(l|_{\mathfrak{k}}))$ holds for any extension $l_1 \in \tilde{\mathfrak{f}}^*$ of $l|_{\mathfrak{k}}$, the subalgebra $\tilde{\mathfrak{p}} = \mathfrak{h} + \tilde{\mathfrak{a}}$ is a polarization at $\tilde{l}$, and there exists an extension $\tilde{l}$ such that

$$\mathcal{SE}(K, \mathfrak{k} \cap \mathfrak{n}, l|_{\mathfrak{k}}, \mathfrak{h}) = R_{\tilde{l}}(\mathcal{H}^{\infty}_{\pi_{\tilde{l},\tilde{P}}}).$$

We first treat case a), b-1), and c). Recalling $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{k}$ in case a) and b-1), resp. $\mathfrak{g} = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathfrak{k}$, in case c), we define $\mathfrak{a} = \tilde{\mathfrak{a}}$, and $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ by $[\mathbb{R}X, \mathfrak{a}] = \{0\}$, resp. $[\mathbb{R}X_1 + \mathbb{R}X_2, \mathfrak{a}] = \{0\}$. Let $\mathfrak{p} = \mathfrak{h} + \mathfrak{a}$ $(= \tilde{\mathfrak{p}})$. Then, for an extension $l_0 \in \mathfrak{f}^*$ of l and $l_1 = l_0|_{\tilde{\mathfrak{f}}}$, we have $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{f}(l_0)) + 1$ in case a) and b-1) and $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{f}(l_0)) + 2$ in case c), and, by the induction hypothesis $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{a}) + \dim(\mathfrak{k}(l|_{\mathfrak{k}}))$, we have

$$\dim(\mathfrak{f}(l_0)) = \dim(\tilde{\mathfrak{f}}(l_1)) - 1 = \dim(\mathfrak{a}) + \dim(\mathfrak{k}(l|_{\mathfrak{k}})) - 1 = \dim(\mathfrak{a}) + \dim(\mathfrak{g}(l))$$

in a) and b),

$$\dim(\mathfrak{f}(l_0)) = \dim(\tilde{\mathfrak{f}}(l_1)) - 2 = \dim(\mathfrak{a}) + \dim(\mathfrak{k}(l|_{\mathfrak{k}})) - 2 = \dim(\mathfrak{a}) + \dim(\mathfrak{g}(l))$$

in case c). Thus $\mathfrak{p}$ is a polarization at l_0, and $\mathfrak{p}$ is adapted to $\mathfrak{n} + \mathfrak{a}$.

Let l_0 be an extension of $\tilde{l}$, and we realize in case a) and b-1), $\pi_{l_0,P} = \mathrm{ind}_P^F \chi_{l_0}$ as $\mathrm{ind}_{\tilde{F}}^F \tilde{\pi}$, where $\tilde{\pi} = \mathrm{ind}_P^{\tilde{F}} \chi_{l_0}$, on $L^2(\mathbb{R}, \mathcal{H}_{\tilde{\pi}})$. For $\phi = \phi(x) \in C^\infty(\mathbb{R}, \mathcal{H}_{\tilde{\pi}})$, we have

$$d\pi_{l_0,P}(X)\phi(x) = -\frac{d}{dx}\phi(x), \tag{2.9}$$

$$d\pi_{l_0,P}(T)\phi(x) = \left(\frac{1}{2} + x\frac{d}{dx} + d\tilde{\pi}(T + P(x))\right)\phi(x), \tag{2.10}$$

$$d\pi_{l_0,P}(V)\phi(x) = d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V)(\phi(x)), \quad V \in \mathfrak{k}_0 + \mathfrak{a}, \tag{2.11}$$

$$d\pi_{l_0,P}(Y)\phi(x) = -ix(\phi(x)), \tag{2.12}$$

where $P(x)$ is a $\mathfrak{k}_0$-valued polynomial in x, in case a). In case b-1) we have (2.9), (2.11) and (2.12). In case c) we have $G = \exp \mathbb{R}X_1 \exp \mathbb{R}X_2 K$ and we realize $\pi_{l_0,P} = \mathrm{ind}_P^F \chi_{l_0}$ as $\mathrm{ind}_{\tilde{F}}^F \tilde{\pi}$, where $\tilde{\pi} = \mathrm{ind}_P^{\tilde{F}} \chi_{l_0}$, on $L^2(\mathbb{R}^2, \mathcal{H}_{\tilde{\pi}})$. For $\phi = \phi(x_1, x_2) \in C^\infty(\mathbb{R}^2, \mathcal{H}_{\tilde{\pi}})$, we have

$$\begin{aligned}
&d\pi_{l_0,P}(X_1)\phi(x_1,x_2) = -\frac{\partial}{\partial x_1}\phi(x_1,x_2),\\
&d\pi_{l_0,P}(X_2)\phi(x_1,x_2) = -\frac{\partial}{\partial x_2}\phi(x_1,x_2) + d\tilde{\pi}(Q(x_1,x_2))\phi(x_1,x_2),\\
&d\pi_{l_0,P}(T)\phi(x_1,x_2) = \left(1 + (x_1 - \omega x_2)\frac{\partial}{\partial x_1} + (x_1\omega + x_2)\frac{\partial}{\partial x_2} + d\tilde{\pi}(T + R(x_1,x_2))\right)\phi(x_1,x_2)\\
&d\pi_{l_0,P}(V)\phi(x_1,x_2) = d\tilde{\pi}(\mathrm{Ad}(\exp(-x_2X_2)\exp(-x_1X_1))V)(\phi(x_1,x_2)), \quad V \in \mathfrak{k}_0 + \mathfrak{a},\\
&d\pi_{l_0,P}(Y_1)\phi(x_1,x_2) = -ix_1(\phi(x_1,x_2)),\\
&d\pi_{l_0,P}(Y_2)\phi(x_1,x_2) = -ix_2(\phi(x_1,x_2)),
\end{aligned}$$

where $Q(x_1,x_2), R(x_1,x_2)$ are $\mathfrak{k}_0$-valued polynomial in x_1, x_2. Remarking that $\mathrm{Ad}(\exp(-x_2X_2)\exp(-x_1X_1))V$ is also a polynomial in x_1, x_2 since $X_1, X_2 \in [\mathfrak{g}, \mathfrak{g}]$, we can show similarly as in case (1), that the family of norms $\{\| \|_{a,D}, a \in \mathbb{R}_+, D \in \mathfrak{D}_{t,r}\}$ is equivalent to the family of norms $\{\| \|_{l_0,U},\ U \in \mathcal{U}(\mathfrak{f})\}$ and so we have that in case a), case b-1) and c)

$$\phi \in R_{l_0}(\mathcal{H}^{\infty}_{\pi_{l_0,P}}) \iff \phi \in \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}),$$

We next treat case b-2). We define $\mathfrak{a} = \tilde{\mathfrak{a}} + \mathbb{R}A + \mathbb{R}B$ and $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ with

$$[X, A] = A, [X, B] = -B, [A, \tilde{\mathfrak{f}}] = [B, \tilde{\mathfrak{f}}] = \{0\},$$

and $\mathfrak{p} = \mathfrak{h} + \mathfrak{a} = \tilde{\mathfrak{p}} + \mathbb{R}A + \mathbb{R}B$. Let $l_0 \in \mathfrak{f}^*$ be an extension of l. Then it can be deduced as in case (1) that $\dim(\mathfrak{f}(l_0)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$ and $\mathfrak{p}$ is a polarization at l_0 adapted to $\mathfrak{n}+\mathfrak{a}$. Let $\mathfrak{e} = \tilde{\mathfrak{f}} \oplus \mathbb{R}A \oplus \mathbb{R}B$, and $E = \exp \mathfrak{e}$. We take an extension l_0 of $\tilde{l}$ such that $l_0(A) \neq 0$ and $l_0(B) \neq 0$, and realize $\tau = \mathrm{ind}_P^E \chi_{l_0}$ in $\mathcal{H}_{\tilde{\pi}}$ by $\tau(ka)\phi = \chi_{l_0}(a)\tilde{\pi}(k)\phi$ for $\phi \in \mathcal{H}_{\tilde{\pi}}$, $k \in K$, $a \in \exp(\mathbb{R}A + \mathbb{R}B)$. As in case (1), we realize $\pi_{l_0,P} = \mathrm{ind}_P^F \chi_{l_0} = \mathrm{ind}_E^F \tau$ in $L^2(\mathbb{R}, \mathcal{H}_{\tilde{\pi}})$, and we have

$$\begin{aligned} &d\pi_{l_0,P}(X)\phi(x) = -\frac{d}{dx}\phi(x),\\ &d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x),\\ &d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^{x}\phi(x),\\ &d\pi_{l_0,P}(V)\phi(x) = d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V)(\phi(x)), \qquad V \in \tilde{\mathfrak{f}}\\ &d\pi_{l_0,P}(Y)\phi(x) = -ix\phi(x). \end{aligned}$$

We can also show similarly as in case (1) that

$$\phi \in R_{l_0}(\mathcal{H}^{\infty}_{\pi_{l_0,P}}) \iff \phi \in \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}.)$$

Case (2) a), b), c); ii): We come now to case ii). Now $\mathfrak{h} \not\subset \mathfrak{k}$. We take $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{k} + \mathfrak{g}_2$. Since $\mathfrak{n}^l \subset \mathfrak{g}_2^l = \mathfrak{k}$, and $\mathfrak{h}$ is adapted to $\mathfrak{n}$, which implies $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{n}) + (\mathfrak{h} \cap \mathfrak{n}^l)$, we may choose subspaces $\mathcal{X} \subset \mathfrak{n} \cap \ker(l)$ and $\mathcal{Y} \subset \mathfrak{g}_2 \cap \ker(l)$ so that $\mathfrak{h} = \mathcal{X} \oplus (\mathfrak{h} \cap \mathfrak{k})$ and $\mathfrak{h}' = \mathcal{Y} \oplus (\mathfrak{h} \cap \mathfrak{k})$. We remark that $\mathfrak{h}'$ is a polarization at l adapted to $\mathfrak{n}$ and $\dim(\mathcal{X}) = \dim(\mathcal{Y}) (\leq 2)$. Applying the result i) above to $(G, l, \mathfrak{n}, \mathfrak{h}')$, we have $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ with an abelian ideal $\mathfrak{a}$ with the required properties; $[\mathfrak{n} + \mathfrak{h}', \mathfrak{a}] = \{0\}$, $\dim(\mathfrak{f}(l_0)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$ for any extension $l_0 \in \mathfrak{f}^*$ of l, and there exists an extension l_0 such that

$$\phi \in R_{l_0}(\mathcal{H}^{\infty}_{\pi_{l_0,P'}}) \iff \phi \in \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}'),$$

where $\mathfrak{p}' = \mathfrak{h}' + \mathfrak{a}$, $P' = \exp \mathfrak{p}'$.

Let $\mathfrak{p} = \mathfrak{h} + \mathfrak{a} = \mathcal{X} \oplus (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{a}$ and $P = \exp \mathfrak{p}$. Since $\mathcal{X} \subset \mathfrak{n}$, we have $[\mathfrak{n} + \mathfrak{h}, \mathfrak{a}] = \{0\}$, and the subalgebra $\mathfrak{p}$ is also a Pukanszky polarization at l_0 adapted to $\mathfrak{n} + \mathfrak{a}$. We take a subspace $\mathcal{M} \subset \mathfrak{n}$ such that $\mathfrak{n} = \mathcal{M} \oplus \mathcal{X} \oplus (\mathfrak{n} \cap \mathfrak{k})$. (We regard $\mathcal{M} = \{0\}$ if $\mathfrak{h} + \mathfrak{k} = \mathfrak{g}$.) Then we have

$$\begin{aligned} NH/H &= (\exp \mathcal{M} \exp \mathcal{X} (N \cap K)H/G_2H)(G_2H/H)\\ &= (\exp \mathcal{M}(N \cap K)/(N \cap K \cap H)G_2)(\exp \mathcal{Y}),\\ NH'/H' &= \exp \mathcal{X} \exp \mathcal{M}(N \cap K)H'/H'\\ &= \exp \mathcal{X}(\exp \mathcal{M}(N \cap K)/(N \cap K \cap H)G_2). \end{aligned}$$

Remarking that $\mathfrak{n}\cap\mathfrak{k}=\mathfrak{n}\cap\mathfrak{k}_0$, we can take coexponential bases $\{Y_i,R_j\}$ for $\mathfrak{n}\cap\mathfrak{h}$ in $\mathfrak{n}$ and $\{X_i,R_j\}$ for $\mathfrak{n}\cap\mathfrak{h}'$ in $\mathfrak{n}$ so that $\{X_i\}_{i=1,\dim(\mathcal{X})}$ is a basis of $\mathcal{X}$, $\{Y_i\}_{i=1,\dim(\mathcal{Y})}$ is a basis of $\mathcal{Y}$, and $\{R_j\}_{j=1,\cdots,w}$ is a coexponential basis for $\mathfrak{n}\cap\mathfrak{h}'$ $(=(\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{h})+\mathfrak{g}_2=(\mathfrak{n}\cap\mathfrak{k}_0\cap\mathfrak{h})+\mathfrak{g}_2)$ in $\mathcal{M}\oplus(\mathfrak{n}\cap\mathfrak{k})$ $(=\mathcal{M}\oplus(\mathfrak{n}\cap\mathfrak{k}_0))$, where $w:=\dim((\mathcal{M}\oplus(\mathfrak{n}\cap\mathfrak{k}))/(\mathfrak{h}'\cap\mathfrak{n}))$. We identify $NH/H=\mathbb{R}^w\oplus\mathcal{Y}$ by $(r,y)\mapsto E(r)E(y)$, and $NH'/H'=\mathcal{X}\oplus\mathbb{R}^w$ by $(x,r)\mapsto E(x)E(r)$, where $E(r):=\exp r_1R_1\cdots\exp r_wR_w$, $E(x):=\exp(x_1X_1)$, $E(y):=\exp(y_1Y_1)$ (for the case of $\dim(\mathcal{X})=1$), $E(x):=\exp(x_1X_1)\exp(x_2X_2)$, $E(y):=\exp(y_1Y_1)\exp(y_2Y_2)$ (for the case of $\dim(\mathcal{X})=2$.)

The intertwining operator u between the space of $\mathrm{ind}_H^G\chi_l$ and $\mathrm{ind}_{H'}^G\chi_l$ is given by

$$u\phi(g)=\oint_{H'/H'\cap H}\phi(gy)\chi_l(y)\Delta_{G,H'}^{-1/2}(y)dy,\quad \phi\in\mathcal{H}_{\pi_{l,H}}$$

(see [2]), which is in our coordinates (x,r) for NH'/H', (r,y) for NH/H, and t for $G/NH'=G/NH$ given by

$$u\phi(t,x,r)=\int_{\mathbb{R}^{\dim(\mathcal{Y})}}\phi(t,r(x),y)e^{-il([x,y]-[x,g(x)]+h(x))}dy,$$

where $r(x)$ is a $\mathbb{R}^w$-valued polynomial, $h(x)$ is an $\mathfrak{h}\cap\mathfrak{n}\cap\mathfrak{k}_0$-valued polynomial and $g(x)$ is a $\mathcal{Y}$-valued polynomial in x such that $E(x)E(r)E(x)^{-1}=E(r(x))\exp g(x)\exp h(x)$. Hence the operator u maps $\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})$ onto $\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}')$. Therefore,

$$\begin{aligned}\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}) &= u^{-1}(u(\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})))\\ &= u^{-1}(R_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P'}}))\\ &= R_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}}).\end{aligned}$$

Case (2) $\mathfrak{n}=\mathfrak{g}_1$: We define $\mathfrak{f}=\mathfrak{g}\ltimes\mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal spanned by $\{A_1,\cdots,A_n,B_1,\cdots,B_n\}$, by $[X_j,A_k]=\delta_{j,k}A_j$, $[X_j,B_k]=-\delta_{j,k}B_j$, $[Y_j,A_k]=[Y_j,B_k]=0$ $(j,k=1,\cdots,n)$. Let $\mathfrak{p}:=\mathfrak{h}+\mathfrak{a}$. Then for all extension $l_0\in\mathfrak{f}^*$ of l we have that $\mathfrak{f}(l_0)$ is spanned by $\{Z,\ A_j-l_0(A_j)Y_j,\ B_j+l_0(B_j)Y_j,\ 1\le j\le n\}$ and $\dim(\mathfrak{f}(l_0))=\dim(\mathfrak{g}(l))+\dim(\mathfrak{a})$ since $\mathfrak{g}(l)=\mathbb{R}Z$. Thus $\mathfrak{p}$ is a polarization at l_0 adapted to $\mathfrak{n}+\mathfrak{a}$. We choose an extension l_0 such that $l_0(A_j)\neq 0$, $l_0(B_j)\neq 0$ for all $j=1,\cdots,n$, and realize $\pi_{l_0,P}$ as $\mathrm{ind}_P^F\chi_{l_0}$ on $L^2(\mathbb{R}^n)$. Then for smooth functions $\phi=\phi(x_1,\cdots,x_n)\in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}&d\pi_{l_0,P}(X_j)\phi(x_1,\cdots,x_n)=-\frac{\partial}{\partial x_j}\phi(x_1,\cdots,x_n),\\ &d\pi_{l_0,P}(A_j)\phi(x_1,\cdots,x_n)=il_0(A_j)e^{-x_j}\phi(x_1,\cdots,x_n),\\ &d\pi_{l_0,P}(B_j)\phi(x_1,\cdots,x_n)=il_0(B_j)e^{x_j}\phi(x_1,\cdots,x_n),\\ &d\pi_{l_0,P}(Y_j)\phi(x_1,\cdots,x_n)=i(l_0(Y_j)-x_j)\phi(x_1,\cdots,x_n),\qquad j=1,\cdots,n.\end{aligned}$$

Noting that $G/NH=G/H$, we get

$$\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})=R_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}}).$$

□

2.2 $\mathcal{SE}^\infty$-space and $\mathcal{ASE}$-space

Using our $\mathcal{SE}$-space, we shall describe the $\mathcal{ASE}$-space introduced in [7], where it is denoted by $\mathcal{ES}$ (see Remark 2.7). Let $G = \exp \mathfrak{g}$, $\mathfrak{n}$, $l \in \mathfrak{g}^*$ be as above, $\mathfrak{h}$ be a polarization at l adapted to $\mathfrak{n}$, and $\mathfrak{h}_\mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}$. Let $\mathcal{P}(\mathfrak{h})$ be the set of all the polarizations $\check{\mathfrak{h}}$ at l such that $\check{\mathfrak{h}} \cap \mathfrak{n} = \mathfrak{h}_\mathfrak{n}$ and $\check{\mathfrak{h}}$ is adapted to $\mathfrak{n}$. By Remark 2.2, a polarization $\check{\mathfrak{h}}$ belongs to $\mathcal{P}(\mathfrak{h})$ if and only if $\check{\mathfrak{h}} = \mathfrak{h}_0 + \mathfrak{h}_\mathfrak{n}$, where $\mathfrak{h}_0 \subset \mathfrak{n}^l$ is a polarization at $l_{|\mathfrak{n}^l}$. Let $\check{H} = \exp \check{\mathfrak{h}}$, and we denote by $\mathcal{T}_{\mathfrak{h}\check{\mathfrak{h}}} : \mathcal{H}_{\pi_{l,\check{H}}} \to \mathcal{H}_{\pi_{l,H}}$ the intertwining operator of $\mathrm{ind}_{\check{H}}^G \chi_l$ and $\mathrm{ind}_H^G \chi_l$;

$$\mathcal{T}_{\mathfrak{h}\check{\mathfrak{h}}}\phi(g) = \oint_{H/(H\cap\check{H})} \phi(gy)\chi_l(y)\Delta_{G,H}^{-1/2}(y)d\mu_{H/(H\cap\check{H})}(y), \quad \phi \in \mathcal{H}_{\pi_{l,\check{H}}}$$

(see [2]).

Definition 2.5. We define

$$\mathcal{SE}^\infty(G, \mathfrak{n}, l, \mathfrak{h}) := \cap_{\check{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})} \mathcal{T}_{\mathfrak{h}\check{\mathfrak{h}}}(\mathcal{SE}(G, \mathfrak{n}, l, \check{\mathfrak{h}})).$$

We also define the space $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ ([6],[7]). Recall that for defining $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$, we regard $G/H = (G/NH) \times (NH/H) = \mathbb{R}^{m+v}$; now we decompose G/H as

$$G/H = (G/N^lN) \times (N^lN/NH) \times (NH/H) = \mathbb{R}^{\nu+u+v},$$

where $m = \nu + u$, taking coexponential bases $\{T_1, \cdots, T_\nu\}$ for $\mathfrak{n}^l + \mathfrak{n}$ in $\mathfrak{g}$, $\{S_1 := T_{\nu+1}, \cdots, S_u := T_{\nu+u}\}$ for $\mathfrak{n} + \mathfrak{h}$ in $\mathfrak{n}^l + \mathfrak{n}$, $\{R_1, \cdots, R_v\}$ for $\mathfrak{h}$ in $\mathfrak{n} + \mathfrak{h}$, and letting $\mathbb{R}^\nu \ni t = (t_1, \cdots, t_\nu) \mapsto E(t) = \exp t_1T_1 \cdots \exp t_\nu T_\nu$, $\mathbb{R}^u \ni s = (s_1, \cdots, s_u) \mapsto E(s) = \exp s_1S_1 \cdots \exp s_uS_u$, $\mathbb{R}^v \ni r = (r_1, \cdots, r_v) \mapsto E(r) = \exp r_1R_1 \cdots \exp r_vR_v$, and $\mathbb{R}^{\nu+u+v} \ni (t, s, r) \mapsto E(t, s, r) = E(t)E(s)E(r)$.

Definition 2.6. Let $\mathfrak{D}_{t,s,r}$ be the space of all differential operators on $\mathbb{R}^{\nu+u+v}$ with polynomial coefficients and let $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of all functions $\phi \in \mathcal{H}_{\pi_{l,H}}$ such that

1. ϕ is smooth,

2. $$\|\phi\|_{a,b,D}^2 := \int_{\mathbb{R}^{\nu+u+v}} e^{a\|t\|}e^{b\|s\|}|D(\phi \circ E)(t, s, r)|^2 dtdsdr < \infty, \forall\ (a, b) \in \mathbb{R}_+^2, D \in \mathfrak{D}_{t,s,r}.$$

3. The same conditions 1 and 2 hold for the partial Fourier transform $\hat{\phi}_s$ of ϕ in s, where

$$\hat{\phi}_s(t, s, r) = \int_{\mathbb{R}^u} \phi \circ E(t, x, r)e^{i\langle x,s\rangle}dx.$$

Remark 2.7. The space $\mathcal{ASE}$ is also independent of the choice of coexponential bases. We have $\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}) \supset \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$; A function $\phi \in \mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})$ belongs to $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$ if and only if ϕ satisfies the condition 3 above. In the paper [7] this space has been denoted by $\mathcal{ES}(G,\mathfrak{n},l,\mathfrak{h})$. We write here the letter $\mathcal{A}$ in front to indicate that the functions ϕ contained in $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$ are analytic in the direction x. It has been shown in [7] and [1] that for ϕ and ψ in $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$ there exists a function $f \in L^1(G)$ and more precisely in the subalgebra $\mathcal{SE}(G)$ (see section 4), such that

$$\pi_{l,H} f(\xi) = \langle \xi, \psi\rangle \phi, \ \xi \in \mathcal{H}_{\pi_{l,H}}.$$

Theorem 2.8. *Let $G = \exp \mathfrak{g}$, $\mathfrak{n}$, l, $\mathfrak{h}$ be as above. Then we have*

$$\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}) = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h}).$$

Proof. The proof is by induction on $\dim(\mathfrak{g})$. We shall use the framework of induction in the proof of Theorem 2.4.

If $\mathfrak{n} = \{0\}$, the statement is trivial. Suppose that $l = 0$ on an abelian ideal $\mathfrak{i} \neq \{0\}$, and let $\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}}, \dot{\pi}$ be as in the proof of Theorem 2.4. Then we get the conclusion by the induction hypothesis for $(\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}})$ and $\dot{\pi}$ because we can naturally identify $\mathcal{SE}^\infty(\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}})$ with $\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h})$ and $\mathcal{ASE}(\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}})$ with $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$.

Suppose $l \neq 0$ on any non-zero abelian ideal. Taking a minimal ideal $\mathfrak{g}_1$ contained in $\mathfrak{n}$, we use the same notations as those in the proof of Theorem 2.4.

Case (1): Letting $\mathfrak{k} := \ker(\lambda)$, $K = \exp \mathfrak{k}$, we have $\check{\mathfrak{h}} \subset \mathfrak{k}$ for all $\check{\mathfrak{h}} \in \mathcal{P}(\mathfrak{h})$ and

$$\mathcal{SE}^\infty(K,\mathfrak{n},l_{|\mathfrak{k}},\mathfrak{h}) = \mathcal{ASE}(K,\mathfrak{n},l_{|\mathfrak{k}},\mathfrak{h}).$$

by the induction hypothesis. Since $\mathfrak{k}$ is an ideal including $\mathfrak{n}^l + \mathfrak{n}$, we have

$$G/N^l N = (G/K)(K/N^l N),$$

and obtain the conclusion $\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}) = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$.

Case (2) a),b),c), $\mathfrak{g}_1 \neq \mathfrak{n}$: Let $\mathfrak{k} = \ker(\gamma)$ in case a) and b), resp. $\mathfrak{k} = \ker(\gamma_1) \cap \ker(\gamma_2)$ in case c).

Case (2) a),b),c); i): $\mathfrak{g}_2 \subset \mathfrak{h}$. Then any polarization $\check{\mathfrak{h}} \in \mathcal{P}(\mathfrak{h})$ is contained in $\mathfrak{k}$. We have $\mathfrak{n} + \mathfrak{k} = \mathfrak{g}$ in cases a),b-1),c), $\mathfrak{n} \subset \mathfrak{k}$ in case b-2). Since

$$(\mathfrak{k} \cap \mathfrak{n})^{l_{|\mathfrak{k}}} = \mathfrak{n}^l + \mathfrak{g}_2,$$

and

$$G/N^l N = \begin{cases} K/(K\cap N)^{l_{|\mathfrak{k}}}(K\cap N) & \text{cases a),b-1),c)} \\ (G/K)(K/N^l N) = (G/K)(K/(K\cap N)^{l_{|\mathfrak{k}}}(K\cap N)) & \text{case b-2),} \end{cases}$$

$$N^l N/NH = N^l(K\cap N)/(K\cap N)H = (K\cap N)^{l_{|\mathfrak{k}}}(K\cap N)/(K\cap N)H$$

we can deduce the conclusion from the induction hypothesis

$$\mathcal{SE}^\infty(K, \mathfrak{k}\cap\mathfrak{n}, l_{|\mathfrak{k}}, \mathfrak{h}) = \mathcal{ASE}(K, \mathfrak{k}\cap\mathfrak{n}, l_{|\mathfrak{k}}, \mathfrak{h}).$$

Case (2) a),b),c); ii): $\mathfrak{g}_2 \not\subset \mathfrak{h}$. Then $\check{\mathfrak{h}} \not\subset \mathfrak{k}$ for any polarization $\check{\mathfrak{h}} \in \mathcal{P}(\mathfrak{h})$. Let $\mathfrak{h}' = (\mathfrak{h}\cap\mathfrak{k}) + \mathfrak{g}_2$, and according to the notations in case (2) a),b),c) ii) of the proof of Theorem 2.4, we have $\mathfrak{h} = \mathcal{X}\oplus(\mathfrak{h}\cap\mathfrak{k})$, $\mathcal{X}\subset\mathfrak{n}\cap\ker(l)$ and $\mathfrak{h}' = (\mathfrak{h}\cap\mathfrak{k})\oplus\mathcal{Y}$, $\mathcal{Y}\subset\mathfrak{g}_2\cap\ker(l)$, and we identify $NH/H = \mathbb{R}^w\oplus\mathcal{Y}$ by $(r,y)\mapsto E(r)E(y)$, and $NH'/H' = \mathcal{X}\oplus\mathbb{R}^w$ by $(x,r)\mapsto E(x)E(r)$. Then the intertwining operator $\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}$ is given by

$$\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}\phi(t,s,x,r) = \int_{\mathbb{R}^{\dim(\mathcal{Y})}} \phi(t,s,r(x),y)e^{-il([x,y]-[x,g(x)]+h(x))}dy, \tag{2.13}$$

where $r(x)$, $h(x)$ and $g(x)$ are polynomials whose values are in $\mathbb{R}^w$, $\mathfrak{h}\cap\mathfrak{n}\cap\mathfrak{k}_0$ and $\mathcal{Y}$, respectively. Thus we have

$$\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}(\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})) = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h}').$$

For any polarization $\check{\mathfrak{h}} = \mathfrak{h}_0+\mathfrak{h}_{\mathfrak{n}}\in\mathcal{P}(\mathfrak{h})$, letting $\check{\mathfrak{h}}' = (\check{\mathfrak{h}}\cap\mathfrak{k})+\mathfrak{g}_2$, we also get the expression of $\mathcal{T}_{\check{\mathfrak{h}}'\check{\mathfrak{h}}}$ as (2.13), and we have

$$\mathcal{T}_{\check{\mathfrak{h}}'\check{\mathfrak{h}}}(\mathcal{SE}(G,\mathfrak{n},l,\check{\mathfrak{h}})) = \mathcal{SE}(G,\mathfrak{n},l,\check{\mathfrak{h}}').$$

Applying the result i) above, we have

$$\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}') = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h}').$$

The set $\mathcal{P}(\mathfrak{h}')$ consists of polarizations $\mathfrak{h}_1 = \mathfrak{h}_0 + (\mathfrak{h}'\cap\mathfrak{n}) = \mathfrak{h}_0 + (\mathfrak{h}\cap\mathfrak{k}\cap\mathfrak{n}) + \mathfrak{g}_2$, with some polarization $\mathfrak{h}_0\subset\mathfrak{n}^l\subset\mathfrak{k}$ at $l_{|\mathfrak{n}^l}$. Thus we have $\mathcal{P}(\mathfrak{h}') = \{(\check{\mathfrak{h}}\cap\mathfrak{k})+\mathfrak{g}_2; \check{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})\}$. Writing $\check{\mathfrak{h}}' = (\check{\mathfrak{h}}\cap\mathfrak{k})+\mathfrak{g}_2$ for each $\check{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})$, we have

$$\begin{aligned}
&\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}(\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h})) = \cap_{\check{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})}\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}\circ\mathcal{T}_{\mathfrak{h}\check{\mathfrak{h}}}(\mathcal{SE}(G,\mathfrak{n},l,\check{\mathfrak{h}}))\\
&= \cap_{\check{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})}\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}\circ\mathcal{T}_{\mathfrak{h}\check{\mathfrak{h}}'}\circ\mathcal{T}_{\check{\mathfrak{h}}'\check{\mathfrak{h}}}(\mathcal{SE}(G,\mathfrak{n},l,\check{\mathfrak{h}})) = \cap_{\check{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})}\mathcal{T}_{\mathfrak{h}'\mathfrak{h}}\circ\mathcal{T}_{\mathfrak{h}\check{\mathfrak{h}}'}(\mathcal{SE}(G,\mathfrak{n},l,\check{\mathfrak{h}}'))\\
&= \cap_{\check{\mathfrak{h}}'\in\mathcal{P}(\mathfrak{h}')}\mathcal{T}_{\mathfrak{h}'\check{\mathfrak{h}}'}(\mathcal{SE}(G,\mathfrak{n},l,\check{\mathfrak{h}}')) = \mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}')\\
&= \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h}') = \mathcal{T}_{\mathfrak{h}'\mathfrak{h}}(\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})).
\end{aligned}$$

Thus we have $\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}) = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$.

Case (2) $\mathfrak{n} = \mathfrak{g}_1$ whence $\mathfrak{n}^l = \mathfrak{g}$. We can take a bases $\{X_1,\cdots,X_n,Y_1,\cdots,Y_n,Z\}$ of $\mathfrak{g}$ such that $[X_i,Y_j] = \delta_{i,j}Z$, $l(Z)=1$, $l(X_i)=l(Y_i)=0$ $(i,j=1,\cdots,n)$, and $\mathfrak{h}$ is spanned by $\{Y_1,\cdots,Y_n,Z\}$. Let $\mathfrak{h}_2$ be a subalgebra spanned by $\{X_1,\cdots,X_n,Z\}$. Then $\mathfrak{h}_2\in\mathcal{P}(\mathfrak{h})$. We identify G/H with $\mathcal{X} := \mathbb{R}X_1\oplus\cdots\oplus\mathbb{R}X_n$ and G/H_2 with $\mathcal{Y} := \mathbb{R}Y_1\oplus\cdots\oplus\mathbb{R}Y_n$, and realize $\mathrm{ind}_H^G\chi_l$ and $\mathrm{ind}_{H_2}^G\chi_l$, respectively. Then the intertwining operator $\mathcal{T}_{\mathfrak{h}\mathfrak{h}_2}$ is described by

$$\mathcal{T}_{\mathfrak{h}\mathfrak{h}_2}\phi(x_1,\cdots,x_n) = \int_{\mathcal{Y}}\phi(y_1,\cdots,y_n)e^{-i(x_1y_1+\cdots+x_ny_n)}dy_1\cdots dy_n, \quad \phi\in\mathcal{H}_{\pi_{l,H_2}}.$$

Let $\phi_0 \in \mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}) \cap \mathcal{T}_{\mathfrak{h}\mathfrak{h}_2}(\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}_2))$. Then $\phi_0 = \mathcal{T}_{\mathfrak{h}\mathfrak{h}_2}\phi$ with $\phi \in \mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}_2)$, and we have that ϕ is obtained by Fourier transform of ϕ_0, $\phi = c\hat{\phi_0}$ with some constant c, and ϕ satisfies the conditions 1 and 2 of Definition 2.6, which implies that $\phi_0 \in \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$. Conversely, let $\mathfrak{h}'$ be a polarization at l. Taking subspaces $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{V}$ such that $\mathfrak{h} = \mathcal{Y} \oplus (\mathfrak{h} \cap \mathfrak{h}')$, $\mathfrak{h}' = \mathcal{W} \oplus (\mathfrak{h} \cap \mathfrak{h}')$, $\mathcal{W} \subset \ker(l)$, and $\mathfrak{g} = \mathcal{V} \oplus \mathcal{W} \oplus \mathcal{Y} \oplus (\mathfrak{h} \cap \mathfrak{h}')$, we identify G/H with $\mathcal{V} \oplus \mathcal{W}$ by $(V,W) \mapsto \exp V \exp W$, G/H' with $\mathcal{V} \oplus \mathcal{Y}$ by $(V,Y) \mapsto \exp V \exp Y$. Then we have

$$\mathcal{T}_{\mathfrak{h}\mathfrak{h}'}\phi(V,W) = \int_{\mathcal{Y}} \phi(V,Y) e^{-il([W,Y])} dY, \quad \phi \in \mathcal{H}_{\pi_{l,H'}}.$$

If $\phi_0 \in \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$, then the function $\psi := \phi_0 \circ E$ has the property that all its partial Fourier transforms are exponentially decreasing. Hence $\phi_0 = \mathcal{T}_{\mathfrak{h},\mathfrak{h}'}\mathcal{T}^{-1}_{\mathfrak{h},\mathfrak{h}'}\phi_0$ with $\phi := \mathcal{T}^{-1}_{\mathfrak{h},\mathfrak{h}'}\phi_0 \in \mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}')$. Thus we have $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h}) \subset \mathcal{T}_{\mathfrak{h}\mathfrak{h}'}(\mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h}'))$, and therefore, we have the conclusion $\mathcal{SE}^{\infty}(G,\mathfrak{n},l,\mathfrak{h}) = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$.

□

3 The case $G = N^l N$

Let $\mathfrak{n}$ again be the nilradical of $\mathfrak{g}$ and suppose that $\mathfrak{g} = \mathfrak{n}^l + \mathfrak{n}$. Let $\mathfrak{h}_{\mathfrak{n}}$ be a polarization at $l_{|\mathfrak{n}}$, such that $[\mathfrak{n}^l, \mathfrak{h}_{\mathfrak{n}}] \subset \mathfrak{h}_{\mathfrak{n}}$. and let $\mathfrak{h}_0 \subset \mathfrak{n}^l$ be any polarization at $l_{|\mathfrak{n}^l}$. As we have seen in section 2, the subalgebra $\mathfrak{h} := \mathfrak{h}_0 + \mathfrak{h}_{\mathfrak{n}}$ is a Pukanszky-polarisation at l. Taking a subspace $\mathfrak{x} \subset \mathfrak{n}^l$ such that $\mathfrak{g} = \mathfrak{x} \oplus (\mathfrak{n} + \mathfrak{h})$, and a coexponential basis $\{T_1, \cdots, T_m\}$ for $\mathfrak{h}_{\mathfrak{n}}$ in $\mathfrak{n}$, we identify G/H with $\mathbb{R}^m \times \mathfrak{x}$ through the mapping

$$(t_1, \cdots, t_m, X) \mapsto E(t,X) = \exp t_1 T_1 \cdots \exp t_m T_m \exp X.$$

Let $H = \exp \mathfrak{h}, H_{\mathfrak{n}} := \exp(\mathfrak{h}_n), H_0 = \exp \mathfrak{h}_0$. The invariant linear functional $\oint_{G/H} d\mu_{G/H}$ is given in these coordinates by

$$\oint_{G/H} f(g) d\mu_{G/H}(g) = \int_{\mathbb{R}^m \times \mathfrak{x}} f(E(t,X)) \Delta_{G/H}(\exp X) dt dX, \tag{3.14}$$

where $\Delta_{G/H}(\exp X) := e^{-\mathrm{tr}_{\mathfrak{n}/\mathfrak{h}_{\mathfrak{n}}}(\mathrm{ad}X)}, X \in \mathfrak{x}$.

In order to see this, let us denote by $\nu(f)$ the concrete expression on the right of equation (3.14). Since $\mathfrak{h}_{\mathfrak{n}}$ is $\mathfrak{n}^l$-invariant, we see that the positive functional ν is N-invariant. If we take $Y \in \mathfrak{n}^l$, denoting by $\lambda(g)$, $g \in G$, left translation by g, then

$$\begin{aligned}
\nu(\lambda(\exp Y)f) &= \int_{\mathbb{R}^m \times \mathfrak{x}} f(\exp(-Y)E(t)\exp(X)) \Delta_{G/H}(\exp X) dt dX \\
&= \int_{\mathbb{R}^m \times \mathfrak{x}} \Delta_{G/H}(\exp(-Y)) f(E(t)\exp(-Y)\exp(X)) \Delta_{G/H}(\exp X) dt dX \\
&= \int_{\mathbb{R}^m \times \mathfrak{x}} f(E(t)\exp(-Y)\exp(X)) \Delta_{G/H}(\exp(-Y+X)) dt dX \\
&= \nu(f).
\end{aligned}$$

The uniqueness of $\oint_{G/H} d\mu(g)$ tells us that equation (3.14) is valid. In particular, for every ξ in the Hilbert space $\mathcal{H}_{\pi_{l,H}} = L^2(G/H, \chi_l)$, the L^2-norm of ξ is given by

$$\|\xi\|_2^2 = \oint_{\mathbb{R}^m \times \mathfrak{r}} |\xi(E(t, X))|^2 \Delta_{G/H}(\exp X) dt dX.$$

Let χ_l be the unitary character of H whose differential is the linear functional $il_{|\mathfrak{h}}$ and let $\pi = \pi_{l,H} = \mathrm{ind}_H^G \chi_l$.

Definition 3.1. Let $\mathfrak{D}_{t,\mathfrak{r}}$ be the space of all differential operators on $\mathbb{R}^m \times \mathfrak{r}$ with polynomial coefficients and let $\mathcal{S}_{t,\mathfrak{r}} = \mathcal{S}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of all functions $\phi \in \mathcal{H}_{\pi_{l,H}}$ such that

1. ϕ is smooth,
2. $$\|\phi\|_D^2 := \int_{\mathbb{R}^m \times \mathfrak{r}} |D(\phi \circ E)(t, X)|^2 \Delta_{G/H}(\exp X) dt dX < \infty, \quad \forall D \in \mathfrak{D}_{t,\mathfrak{r}}.$$

Denote by $\mathcal{S}(V)$ the Schwartz space of rapidly decreasing smooth functions on the real finite dimensional vector space V. With this notation, we see that

$$\mathcal{S}_{t,\mathfrak{r}} = \{\phi : G \to \mathbb{C},\ (\Delta_{G/H} \cdot \phi) \circ E \in \mathcal{S}(\mathbb{R}^m \times \mathfrak{r})\},$$

since the mapping $D \mapsto M_\Delta \circ D \circ M_\Delta^{-1}$, $D \in \mathfrak{D}_{t,\mathfrak{r}}$, where M_Δ denotes multiplication with the function $\Delta_{G/H}$, is a bijection of $\mathfrak{D}_{t,\mathfrak{r}}$.

Theorem 3.2. *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group, let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and let $l \in \mathfrak{g}^*$. Suppose that $\mathfrak{g} = \mathfrak{n}^l + \mathfrak{n}$. Choose a polarization $\mathfrak{h}_\mathfrak{n}$ at $l_{|\mathfrak{n}}$, such that $[\mathfrak{n}^l, \mathfrak{h}_\mathfrak{n}] \subset \mathfrak{h}_\mathfrak{n}$. Then the space $\mathcal{H}^\infty := \mathcal{H}^\infty_{\pi_{l,H}}$ of the C^∞ vectors of the representation $\pi := \pi_{l,H}$ and the Fréchet space $\mathcal{S}_{t,\mathfrak{r}}$ coincide.*

Proof. By a theorem of [8], the C^∞ vectors of the representation π are smooth functions. For fixed $X \in \mathfrak{r}$ and $\xi \in \mathcal{H}^\infty$, we see that the function

$$N \ni n \mapsto \xi_X(n) = \xi(n \exp(X))$$

satisfies the covariance condition

$$\xi_X(nh) = \chi_l(h^{-1})\xi_X(n),\ h \in H_\mathfrak{n}, n \in N,$$

since $\chi_l(\exp(X)h\exp(-X)) = \chi_l(h)$ for all $X \in \mathfrak{n}^l$ and all $h \in H_n$. Therefore, multiplying ξ with a smooth function $\varphi \in C_c^\infty(G/H)$, we obtain an element $\eta := (\varphi\xi)_X \in \mathcal{H}^\infty_{\pi_\mathfrak{n}}$ (where $\pi_\mathfrak{n} := \mathrm{ind}_{H_\mathfrak{n}}^N \chi_{l_{|\mathfrak{n}}}$) and hence by Kirillov's theorem, for any element D in $\mathfrak{D}_t$, there exists a u_D in the enveloping algebra $\mathcal{U}(\mathfrak{n})$ such that $D = d\pi_\mathfrak{n}(u_D)$ on $\mathcal{H}^\infty_{\pi_\mathfrak{n}}$. Now, if we let run φ through an approximate unit, we see that

$$D(\xi) = d\pi(u_\mathcal{D})\xi,\ \xi \in \mathcal{H}^\infty. \tag{3.15}$$

Hence $\mathfrak{D}_t \subset d\pi(\mathcal{U}(\mathfrak{g}))$. For $Y \in \mathfrak{n}^l$, we have that

$$\pi(\exp(Y))\xi(E(t, X)) = \xi(\exp(t_1 Ad(\exp(-Y)T_1)) \cdots \exp(t_m Ad(\exp(-Y)T_m)) \exp(-Y) \exp(X)),$$

for $t \in \mathbb{R}^m, X, Y \in \mathfrak{n}^l$. This shows that

$$(d\pi(Y)(\varphi\xi)) \circ E(t, X) = D_Y(\varphi\xi \circ E)(t, X) + d\pi_0(Y)((\varphi\xi)_t)(\exp(X)),$$

where D_Y is some element in $\mathfrak{D}_t$ (acting only on the variable t), where $(\varphi\xi)_t$ is the function $(\varphi\xi)_t(\exp X) := \varphi\xi(E(t)\exp(X))$, which is contained in the Hilbert space $\mathcal{H}_0$ of the representation $\pi_0 := \mathrm{ind}_{H_0}^{N^l} \chi_{l_{|\mathfrak{n}^l}}$. Together with the relation (3.15) and the fact that $\mathfrak{D}_\mathfrak{r} = d\pi_0(\mathcal{U}(\mathfrak{n}^l))$ this shows that $\mathfrak{D}_\mathfrak{r}$ is contained in $d\pi(\mathcal{U}(\mathfrak{g}))$ and finally that

$$\mathfrak{D}_{t,\mathfrak{r}} = d\pi(\mathcal{U}(\mathfrak{g})). \tag{3.16}$$

In particular, the function $(\Delta_{G/H}\xi) \circ E$ is contained in $\mathcal{S}(\mathbb{R}^m \times \mathfrak{r})$. Conversely, because of (3.16) every smooth function η defined on G satisfying the covariance condition for H and χ_l, such that $\Delta_{G/H}\eta \circ E$ is in $\mathcal{S}_{t,\mathfrak{r}}$ is also contained in $\mathcal{H}^\infty$.

□

4 The space $\mathcal{SE}(G)$

Let again $G = \exp \mathfrak{g}$ be an exponential solvable Lie group. We shall introduce special coordinates on G, which will allow us write the product in G in a particularly simple way. Let $\mathfrak{n}$ again be the nilradical of $\mathfrak{g}$. Take an element T of $\mathfrak{g}$ which is in general position with respect to the roots of $\mathfrak{g}$. This means that for any two distinct roots λ, λ' of $\mathfrak{g}$ we have that $\lambda(T) - \lambda'(T) \neq 0$. This means that the mapping $\lambda \to \lambda(T)$ is an injection. For a root λ let

$$\mathfrak{g}_{\lambda,\mathbb{C}} = \{X \in \mathfrak{g}_\mathbb{C};\ (\lambda(T)\mathbb{I}_{\mathfrak{g}_\mathbb{C}} - \mathrm{ad}(T))^d(X) = 0, \text{ for some } d \in \mathbb{N}^*\}$$

By the usual rules we have that

$$[\mathfrak{g}_{\lambda,\mathbb{C}}, \mathfrak{g}_{\lambda',\mathbb{C}}] \subset \mathfrak{g}_{\lambda+\lambda',\mathbb{C}}$$

for two roots λ, λ'. Then $\mathfrak{g}_0 := \mathfrak{g}_{0,\mathbb{C}} \cap \mathfrak{g}$ is thus a nilpotent Lie subalgebra of $\mathfrak{g}$. Since by Jordan's theorem

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g}_{0,\mathbb{C}} + \sum_{\lambda \neq 0} \mathfrak{g}_{\lambda,\mathbb{C}}$$

and since $\mathfrak{g}_{\lambda,\mathbb{C}},\ \lambda \neq 0$, is contained in $[\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}] \subset \mathfrak{n}_\mathbb{C}$, we see that

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{n}.$$

Let us choose a subspace $\mathfrak{t} \subset \mathfrak{g}_0$, such that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}.$$

We can now define a Lie group structure on the Lie algebra $\mathfrak{k} := \mathfrak{t} \oplus \mathfrak{n}_{\mathbb{C}}$. We use on the complexification $\mathfrak{n}_{\mathbb{C}}$ of $\mathfrak{n}$ the Campbell-Baker-Hausdorff multiplication $\cdot_C$ and we can write for $S, S' \in \mathfrak{t}$

$$S \cdot_C S' = S + S' + \frac{1}{2}[S, S'] + \cdots = (S + S') \cdot_C m(S, S'),$$

where $m : \mathfrak{t} \times \mathfrak{t} \to \mathfrak{n} \cap \mathfrak{g}_0$ is a polynomial mapping.

We define now on $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{n}_{\mathbb{C}}$ a multiplication $\cdot$ in the following way:

$$(4.17)\quad (S + U) \cdot (S' + U') := S + S' + m(S, S') \cdot_C (e^{\mathrm{ad}(-S')}U) \cdot_C U',\ U, U' \in \mathfrak{n}_{\mathbb{C}}, S, S' \in \mathfrak{t}.$$

In particular we have the relations

$$S \cdot U = S + U,\ U \cdot S = S + e^{-ad(S)}U,\ S \in \mathfrak{t},\ U \in \mathfrak{n}_{\mathbb{C}}.$$

It is easy to check that we obtain in this fashion a simply connected exponential solvable Lie group $K = (\mathfrak{k}, \cdot)$ and that this new Lie group contains a closed subgroup $(\mathfrak{g}, \cdot)$, which is isomorphic to G, since $\mathfrak{g} \subset \mathfrak{k}$. Denote also by $N_{\mathbb{C}}$ the subgroup $(\mathfrak{n}_{\mathbb{C}}, \cdot_C)$ of the Lie group $(\mathfrak{k}, \cdot)$.

The Haar measure on the group $(\mathfrak{g}, \cdot)$ is given by Lebesgue measure dx on the vector space $\mathfrak{g}$. Indeed, for a continuous function δ with compact support on $\mathfrak{g}$, we have that

$$\int_{\mathfrak{g}} \delta(x)dx = \int_{\mathfrak{t}\times\mathfrak{n}} \delta(T \cdot U)dUdT$$

and the left-invariance of this measure follows from the multiplication rule (4.17).

We define now a space of smooth functions on G, which will replace the well known Schwartz space of nilpotent Lie groups.

Definition 4.1. Let $\mathfrak{D}_{\mathfrak{t},\mathfrak{n}}$ be the space of all differential operators on $\mathfrak{t}+\mathfrak{n}$ with polynomial coefficients and let $\mathcal{SE}(G)$ be the space of all functions $\phi : G \to \mathbb{C}$ such that

1. ϕ is smooth,
2. $$\|\phi\|^2_{a,D} := \int_{\mathfrak{t}+\mathfrak{n}} e^{a\|t\|}|D(\phi)(t + U)|^2 dtdU < \infty,\ \forall\ a \in \mathbb{R}_+, D \in \mathfrak{D}_{\mathfrak{t},\mathfrak{n}}.$$

The space $\mathcal{SE}(G)$ is in fact independent of the choice of the subspace $\mathfrak{t}$. Indeed, for any subspace $\mathfrak{s}$ of $\mathfrak{g}_0$ such that $\mathfrak{s} \oplus \mathfrak{n} = \mathfrak{g}$, the mapping $E : \mathfrak{s} \times \mathfrak{n} \to \mathfrak{g}, E(S, U) := S \cdot U$ is a diffeomorphism, whose coordinate functions are polynomials in $U \in \mathfrak{n}$ and all the partial derivatives of them are exponentially bounded in S. This allows us to write

$$\begin{aligned}\mathcal{SE}(G) \quad = \quad \Big\{&\phi : G \to \mathbb{C}; \phi \text{ smooth}, \\ &\int_{\mathfrak{s}\times\mathfrak{n}} e^{a\|S\|}|D(\phi)(S \cdot U)|^2 dSdU < \infty, \\ &\forall\ a \in \mathbb{R}_+, D \in \mathfrak{D}_{\mathfrak{s},\mathfrak{n}}\Big\}.\end{aligned}$$

We shall show in this section that the space $\mathcal{SE}(G)$ is the space of the C^∞ vectors of an irreducible representation of a certain exponential solvable Lie group $\mathcal{G}$ acting on $L^2(G)$.

Let $\mathcal{T} := \{T_1, \cdots, T_m\}$ be a basis of $\mathfrak{t}$. Choose a Jordan-Hölder basis $\mathcal{B} = \{T_1, \cdots, T_m, U_1, \cdots, U_p\} =: \{X_1, \cdots, X_n\}$ of $\mathfrak{k}$ and for every $i = 1, \cdots, n$, we choose a Jordan-Hölder basis $\mathcal{B}_i = \{U_1^i, \cdots, U_p^i\}$ for the endomorphism $\mathrm{ad}X_i$ of $\mathfrak{n}_\mathbb{C}$. Then the coefficients $a_{k,l}^i(t_i), t_i \in \mathbb{R}$, of the matrix of the endomorphism $\mathrm{Ad}(\exp(t_i X_i))$ with respect to the basis $\mathcal{B}_i$ are polynomials in t_i for $i > m$, are 0 for $k > l$ and for $k = l$ they are exponential functions of the form $e^{t_i\alpha(T_i)}, i \leq m$, where α denotes a root of $\mathfrak{g}$. Hence, by replacing the basis $\mathcal{B}_i$ by the basis $\mathcal{B}$, for $T = \sum_{i=1}^m t_i T_i$ in $\mathfrak{t}$ and $U \in \mathfrak{n}$, the coefficients $a_{k,l}(T \cdot U)$ of the matrix of $\mathrm{Ad}(T \cdot U)$ with respect to the basis $\mathcal{B}$ are polynomials in $(t_1, \cdots, t_d, U)$ multiplied by exponential functions χ_α of the form $\chi_\alpha(T \cdot U) = e^{a_1 t_1 + \cdots + a_m t_m}$. We denote by $\mathcal{R}'$ the family of all these complex valued linear functionals α which appear in this way. Let also $\mathcal{R}''$ be the family of complex linear functionals of $\mathfrak{k}$ obtained as sums of j elements of $\mathcal{R}'$, with $j \leq 2p$ and let

$$\mathcal{R} = \{\pm\beta, \beta \in \mathcal{R}''\}$$

and let $E_\mathcal{R}$ be the (finite) family of exponential functions of the form $e^\alpha, \alpha \in \mathcal{R}$.

Definition 4.2. For a function f defined on a group K, let the left and right translates of f be defined by

$$\lambda(t)f(g) := f(t^{-1}g), \ \rho(t)f(g) := f(gt), \ g, t \in K.$$

Let now W be the span of all the left and right translates by elements of $N_\mathbb{C}$ of the complex polynomial functions of degree 1 defined on $\mathfrak{n}_\mathbb{C}$. Then every element of W is of total degree $\leq 2\mathrm{dim}(\mathfrak{n}) = 2p$ and so W is finite dimensional and left and right $N_\mathbb{C}$-invariant. Let $(P_j)_{j=1}^d$ be a basis of W. For $g \in N$ we have that the matrix coefficients $a_{i,j}, b_{i,j}$ defined by

$$\lambda(g)P_j = \sum_{i=1}^d a_{i,j}(g)P_i, \ \rho(g)P_j = \sum_{i=1}^d b_{i,j}(g)P_i.$$

are also elements of W, hence they are polynomial functions of total degree $\leq 2p$. It follows that for every $P \in W$, there exists two finite families of elements of W, $(P_i)_i, (Q_i)_i$, such that

$$P(U \cdot U') = \sum_i P_i(U)Q_i(U'), U, U' \in \mathfrak{n}_\mathbb{C}. \tag{4.18}$$

We consider now the linear span V of the left translates of all linear functionals $l : \mathfrak{k} \to \mathbb{C}$. Since for every couple $(T, U), (T', U')$ the multiplication of these 2 elements is given by

$$(T' + U') \cdot (T + U) = T + T' + m(T, T') \cdot_C (e^{-\mathrm{ad}(T)}(U')) \cdot_C U,$$

it follows from 4.18 that the the translate of $l \in \mathfrak{k}_\mathbb{C}^*$ is given by

$$\begin{aligned} \lambda((T' + U')^{-1})l(T + U) &= l(T) + l(T') + \\ &+ \sum_{i,j} P_i(m(T, T'))Q_{i,j}((e^{-\mathrm{ad}(T)}(U'))R_{i,j,}(U), \end{aligned}$$

where the different polynomial functions $P_i, Q_{i,j}$ and $R_{i,j}$ are contained in W. Hence $\lambda((T'+U')^{-1})l$ is a finite linear combination of polynomial functions of degree $\leq 2p$ in U, of degree $\leq 4p^2$ in T multiplied by exponential functions in T, which are all contained $E_{\mathcal{R}}$. Hence V is a finite dimensional left invariant space of functions on $\mathfrak{k}$ and so is the vector space $\mathcal{V}$ of real valued functions on $\mathfrak{g}$, which is generated as a vector space by the restrictions to $\mathfrak{g}$ of the real parts of the elements of V and by the exponential functions $e^{\pm \mathrm{Re}\, \alpha}$, $\alpha \in \mathcal{R}$.

We obtain the group $\mathbb{G}$ as the semi-direct product of G with $\mathcal{V}$, i.e $\mathbb{G} = G \times \mathcal{V}$ with the multiplication defined by

$$(g', \varphi') \cdot (g, \varphi) := (g'g, \lambda(g^{-1})\varphi' + \varphi).$$

This group acts on $L^2(G)$ by left translations with the elements of G and by multiplication with the functions $\chi_\varphi = e^{-i\varphi}$, i.e. for $(g, \varphi) \in \mathbb{G}$, $f \in L^2(G), s \in G$ we let

$$\Pi(g, \varphi)f(s) := e^{-i\varphi(g^{-1}s)} f(g^{-1}s).$$

It is easy to check that $(\Pi, L^2(G))$ is a unitary representation of $\mathbb{G}$ in the Hilbert space $L^2(G)$.

Theorem 4.3. *The representation* $(\Pi, L^2(G))$ *of* $\mathbb{G}$ *is irreducible and the* C^∞ *vectors of* Π *are the elements of* $\mathcal{SE}(G)$.

Proof. Since every real valued linear functional l is contained in $\mathcal{V}$, it follows that for $\phi = l$,

$$\Pi(\phi)\xi = e^{-il}\xi,\ d\Pi(\phi)\xi = -il\xi,\ \xi \in L^2(G)^\infty.$$

Furthermore, for any $\alpha \in \mathcal{R}$ and $\phi = e^{\pm \mathrm{Re}\, \alpha} \in \mathcal{V}$, we have that

$$\Pi(\phi)\xi = e^{-ie^{\pm \mathrm{Re}\alpha}}\xi,\ d\Pi(\phi)\xi = -ie^{\pm \mathrm{Re}\alpha}\xi,\ \xi \in L^2(G)^\infty.$$

This shows that any C^∞-vector of Π is contained in our space $\mathcal{SE}(G)$. Conversely, every function $f \in \mathcal{SE}(G)$ will be mapped by $\mathfrak{g}$ into $\mathcal{SE}(G) \subset L^2(G)$ and therefore $\mathcal{SE}(G) \subset L^2(G)^\infty$.

In order to prove that Π is irreducible, let $(0) \neq \mathcal{H}_0$ be a closed Π-invariant subspace of $L^2(G)$ and let $\xi' \in \mathcal{H}_0^\perp$ and $0 \neq \eta' \in \mathcal{H}_0$. We replace ξ' and η' by $\xi = \Pi(\delta)\xi'$ resp. by $\eta = \Pi(\delta)\eta'$, where δ is a continuous function on G with a small compact support. Then ξ and η are themselves continuous functions and we have that

$$\langle \Pi(\varphi)\Pi(g)\eta, \Pi(g')\xi \rangle_2 = 0, \quad \text{for all } g, g' \in G, \varphi \in \mathcal{V}.$$

In particular for $\varphi = l \in \mathfrak{g}^*$ we get

$$\int_{\mathfrak{g}} e^{-il(x)} \lambda(g)\eta(x)\overline{\lambda(g')\xi(x)}dx = 0.$$

Hence for every $g, g' \in G$, we have that

$$\lambda(g)\eta(x)\overline{\lambda(g')xi(x)} = 0 \text{ for all } x \in G.$$

This shows that $\xi = 0$ whenever $\eta \neq 0$. Finally $\xi' = 0$ and Π is irreducible.

□

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