Generation of L^2 of a nilpotent Lie group by eigenvectors of invariant differential operators.

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Abstract

Let $G = \exp(\mathfrak{g})$ be a non-abelian, connected, simply connected, nilpotent Lie group. We generate $L^2(G)$ by the eigenvectors of a finite number of families of left invariant differential operators and their conjugates. The restriction of the left regular representation to each one of these (left invariant) eigenspaces disintegrates into irreducible unitary representations with multiplicities 0 and 1 only.[1]

Introduction

A fundamental problem raised in a lot of areas of mathematics is the question of how to generate a given mathematical object by its "elementary constituencies". In representation theory, this is the question of the decomposition of a given representation space into subspaces with special properties. It is a problem of this type that we takle in this paper.

Let $G = \exp(\mathfrak{g})$ be a non-abelian, connected, simply connected, nilpotent Lie group. By a fixed algorithm we construct a finite number of families of left invariant differential operators. These families of operators have nontrivial kernels in $\mathcal{S}(G)$ and $L^2(G)$. Moreover, there are plenty of solutions to the corresponding left invariant differential equations. In fact, the kernels of these families of left invariant operators, together with their conjugates, generate the whole L^2-space. On the other hand, these kernels are "small enough" in the following sense: They are of course invariant subspaces for the left regular representation and the restriction of the left regular representation to each one of them disintegrates into irreducible representations with multiplicities 0 and 1 only. This shows again that there are "sufficiently many" such families of left invariant differential operators. Let us insist on the fact that such a result is not possible in the abelian case, since nontrivial eigenfunctions of constant coefficient operators are never in L^2. Here are some details of this construction:

First we decompose a dense subset of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Let d be half the dimension of a coadjoint orbit of maximal dimension. We then construct a Zariski open dense subset $\mathcal{W}$ of $\mathfrak{g}^*$ and a collection of open subsets $\mathcal{A}_\varepsilon$, $\varepsilon \in \{-1,1\}^d$, of $\mathcal{W}$ such that

$$\mathcal{W} = \dot{\bigcup}_{\varepsilon\in\{-1,1\}^d} \mathcal{A}_\varepsilon$$

(disjoint union).

To each one of these finitely many $\mathcal{A}_\varepsilon$'s, such that $\mathcal{A}_\varepsilon \neq \emptyset$, we associate a closed, left invariant subspace $L^2_\varepsilon(G)$ of $L^2(G)$ in the following way: By a precise algorithm we construct differential

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operators $U_{1,\varepsilon},\dots,U_{d,\varepsilon}\in\mathfrak{U}(\mathfrak{g})$ (the enveloping algebra of $\mathfrak{g}$), which are independent if considered as operators acting on the Schwartz algebra $\mathcal{S}(G)$ from the right. We then take

$$\mathcal{S}_\varepsilon(G)=\{f\in\mathcal{S}(G)\mid f*U_{j,\varepsilon}=0,1\le j\le d\}$$

and

$$L^2_\varepsilon(G)=\overline{\mathcal{S}_\varepsilon(G)}^{L^2(G)}.$$

These subspaces are non-trivial and it may be shown that $L^2_\varepsilon(G)$ coincides with the set of weak solutions (in $L^2(G)$) of the system of differential equations $f*U_{j,\varepsilon}=0$, $1\le j\le d$. Finally, for any $u\in G$, let's note $f^u(g)=f(ugu^{-1})$ and $U^u_{j,\varepsilon}$ for the element of $\mathfrak{U}(\mathfrak{g})$ obtained by the action of G on $\mathfrak{U}(\mathfrak{g})$. We put

$$\mathcal{S}_{u,\varepsilon}(G)=\{f\in\mathcal{S}(G)\mid f*U^u_{j,\varepsilon}=0,1\le j\le d\}$$

and

$$L^2_{u,\varepsilon}(G)=\overline{\mathcal{S}_{u,\varepsilon}(G)}^{L^2(G)}.$$

Then $\mathcal{S}_{u,\varepsilon}(G)=(\mathcal{S}_\varepsilon(G))^u$, $L^2_{u,\varepsilon}(G)=(L^2_\varepsilon(G))^u$ and the space $L^2(G)$ is given by

$$L^2(G)=\overline{\overset{\perp}{\bigoplus_{\varepsilon\in\{-1,1\}^d,\mathcal{A}_\varepsilon\neq\emptyset}}\Big(\sum_{u\in G}L^2_{u,\varepsilon}(G)\Big)}^{L^2(G)}.$$

We have thus written the space $L^2(G)$ as a closure of a sum of left invariant subspaces. The subspaces $L^2_{u,\varepsilon}(G)$ of this sum coincide with the set of weak solutions of a system of left invariant partial differential equations. Different parameters ε give orthogonal subspaces. The construction of the corresponding differential operators is given by a fixed algorithm.

Let us now consider the left regular representation ρ, which is the representation of $L^1(G)$ on $L^2(G)$ defined by

$$\rho(f)(g)=f*g,\quad\forall f\in L^1(G),\forall g\in L^2(G).$$

It is known as a consequence of the Plancherel theorem that in the disintegration of ρ into irreducible unitary representations, every class of irreducible representations appears with an infinite multiplicity. So one may ask the question how these infinitely many copies of a same irreducible representation may be realized within the left regular representation. One such realization has been described in ([Ba-Lu]). This paper gives a new approach to the analysis of the left regular representation. In fact, it is easily seen that every subspace $L^2_{u,\varepsilon}(G)$ is invariant under the left regular representation. Moreover, the restriction of this left regular representation ρ to $L^2_{u,\varepsilon}(G)$ disintegrates into irreducible representations with multiplicities 0 and 1 only. This shows of course that there are "sufficiently many" subspaces of the form $L^2_{u,\varepsilon}(G)$.

In the case of the Heisenberg group H_n, the sum in the decomposition of $L^2(H_n)$ may be replaced by a direct sum if we restrict to carefully chosen conjugates of the $L^2_\varepsilon(H_n)$'s and these subspaces may be characterized very precisely. This has already been done in ([Lu-Mo]).

A main point in the argument is the construction of the $U_{1,\varepsilon},\dots,U_{d,\varepsilon}\in\mathfrak{U}(\mathfrak{g})$ and the characterization of the solutions of $f*U_{j,\varepsilon}=0$, $1\le j\le d$. This is done on the Lie algebra level. In fact, if we note the irreducible unitary representations by π_l, $l\in\mathfrak{g}^*$ (given by the Kirillov orbit method), then

$$\begin{aligned}
&f*U_{j,\varepsilon}=0 && 1\le j\le d\\
&\Leftrightarrow d\pi_l(U^*_{j,\varepsilon})\pi_l(f^*)=0 && 1\le j\le d,\ \text{for almost all } l\in\mathfrak{g}^*\\
&\Leftrightarrow \mathfrak{Im}\pi_l(f^*)\subset\cap_{j=1}^d\ \mathrm{Ker}d\pi_l(U^*_{j,\varepsilon}) && \text{for almost all } l\in\mathfrak{g}^*.
\end{aligned}$$

But it may be shown that $\cap_{j=1}^{d}\mathrm{Ker} d\pi_l(U_{j,\varepsilon}^*)$ is one-dimensional if $l \in \mathcal{A}_\varepsilon$ and $\{0\}$ otherwise. This implies that $\pi_l(f^*) = 0$ if $l \notin \mathcal{A}_\varepsilon$ and that $\pi_l(f^*)$ is a rank one operator or 0 if $l \in \mathcal{A}_\varepsilon$. Finally the solutions f are obtained by a Fourier inversion type result applied to the solutions $\xi = (\xi_l)_l$ of $d\pi_l(U_{j,\varepsilon}^*)\xi_l = 0$ for almost all l.

1 Parametrization of the coadjoint orbits

1.1

The following material is standard and is treated for instance in ([Co-Gr]). Let $\mathfrak{g}$ be a nilpotent Lie algebra and $G = \exp \mathfrak{g}$ the corresponding connected, simply connected nilpotent Lie group. Let

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

be a fixed Jordan-Hölder sequence. In particular, $[\mathfrak{g}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}$, for all k. We get an associated strong Malcev basis

$$\{X_1, X_2, \ldots, X_n\}$$

by choosing $X_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$. In $\mathfrak{g}^*$ we consider the dual basis $\{X_1^*, X_2^*, \cdots, X_n^*\}$.
For $l \in \mathfrak{g}^*$, let's denote by $\mathcal{O}_l = \mathrm{Ad}^*(G)(l) = G \cdot l$ the orbit of l under the coadjoint action and let

$$\mathfrak{g}(l) = \{X \in \mathfrak{g} \mid \ < l, [X, \mathfrak{g}] >= \{0\}\}$$

be the Lie algebra of the stabilizer of l.
An index $j \in \{1, 2, \cdots, n\}$ is called a *jump index* for l if

$$\mathfrak{g}(l) + \mathfrak{g}_j \supsetneq \mathfrak{g}(l) + \mathfrak{g}_{j-1}.$$

Let's denote

$$e(l) = \{j \mid j \text{ is a jump index for } l\}$$

the set of jump indices for l. It may be shown that there exist two disjoint sets $\mathcal{S}, \mathcal{T} \subset \{1, 2, \ldots, n\}$ such that $\mathcal{S} \cup \mathcal{T} = \{1, 2, \ldots, n\}$ and such that the following is true: The function P defined on $\mathfrak{g}^*$ by

$$P(l) = \det\big(< l, [X_i, X_j] >\big)_{i,j \in \mathcal{S}}$$

is a G-invariant polynomial on $\mathfrak{g}^*$ and the subset $\mathcal{U} = \{l \in \mathfrak{g}^* \mid e(l) = S\}$ of $\mathfrak{g}^*$ is obtained by

$$\mathcal{U} = \{l \in \mathfrak{g}^* \mid P(l) \neq 0\}.$$

Hence $\mathcal{U}$ is a Zariski-open, G-invariant, dense subset of $\mathfrak{g}^*$ and its elements are said to be in *general position* or *generic in the sense of Pukanszky* for the given Jordan-Hölder sequence. Let's write $\mathfrak{g}^*_{Puk} = \mathcal{U}$. In fact, the sets $\mathcal{U}$ and $\mathcal{S}$ are defined by the following condition: Let's note $V_j = \mathfrak{g}_j^\perp = \mathbb{R} - \mathrm{span}\{X_{j+1}^*, \ldots, X_n^*\}$. Then $l \in \mathcal{U}$ if and only if the dimension of the orbit of l (mod V_j) in $\mathfrak{g}^*/V_j$ is maximal for each j. One may show that the elements of $\mathcal{U}$ all have the same set of jump indices, called $\mathcal{S}$ and conversely that every l having $\mathcal{S}$ as a jump index set is in $\mathcal{U}$ (see [Co-Gr]). The sets $\mathfrak{g}^*_{Puk} = \mathcal{U}, \mathcal{S}, \mathcal{T}$ depend of course on the choice of the Jordan-Hölder sequence.

1.2

There exists a common parametrization of the orbits (which are always of even dimension) of all the elements of $\mathfrak{g}^*_{Puk}$: Let $\mathcal{S} = \{j_1 < j_2 < \cdots < j_{2d}\}$, let's write $l = \sum_{i=1}^n l_i X_i^*$ and identify l with an element of $\mathbb{R}^n$. There are functions $Q_1, Q_2, \dots, Q_n$ satisfying:
(i) The functions $Q_i(l,t)$ are rational nonsingular on $\mathfrak{g}^*_{Puk} \times \mathbb{R}^{2d}$. For fixed l in $\mathfrak{g}^*_{Puk}$, they are polynomial in $t \in \mathbb{R}^{2d}$.
(ii) For each $l = \sum_{i=1}^n l_i X_i^*$ fixed in $\mathfrak{g}^*_{Puk}$, the function $Q(l,t) = \sum_{i=1}^n Q_i(l,t) X_i^*$ maps $\mathbb{R}^{2d}$ diffeomorphically onto the orbit $G \cdot l$, which is a closed sub-manifold of $\mathfrak{g}^*$.
(iii) For fixed l, the function $Q_j(l,t)$ depends only on those t_i such that $j_i \leq j$.
(iv) If $j \notin S$, then $Q_j(l,t) = l_j + R_j(l_1, \dots, l_{j-1}, t_1, \dots, t_i)$ where i is the largest index such that $j_i < j$ and R_j is rational. Moreover, $Q_1(l,t) = l_1$.
(v) $Q_{j_i}(l,t) = t_i$.
(vi) $\mathfrak{g}^*_{Puk}$ is G-invariant and if $t \in \mathbb{R}^{2d}$ is fixed, each $Q_j(l,t)$ is a rational non-singular function on $\mathfrak{g}^*_{Puk}$, constant on G-orbits.
(vii) For sufficiently large $N \in \mathbb{N}$, the functions $P(l)^N Q_j(l,t)$ and $P(l)^N R_j(l,t)$, $j \in \{1, 2, \dots, n\}$, are polynomials. (see [Co-Gr])

1.3

In order to make all the induction steps properly in the proofs, we have to restrict the choice of the elements $l \in \mathfrak{g}^*$ in the following way: Let $P_1, \dots, P_n$ be the different polynomials such that the set $(\mathfrak{g}_k)^*_{Puk}$ of generic elements (in the sense of Pukanszky) f of $\mathfrak{g}_k^*$ are given by $P_k(f) \neq 0$ for every k. We then put $l_k = l|_{\mathfrak{g}_k}$ for all k and

$$\begin{aligned} \mathcal{V} &= \{l \in \mathfrak{g}^* \mid l|_{\mathfrak{g}_k} \in (\mathfrak{g}_k)^*_{Puk}, \forall k\} \\ &= \{l \in \mathfrak{g}^* \mid P_k(l_k) \neq 0, \forall k\} \\ &= \{l \in \mathfrak{g}^* \mid \prod_{k=1}^n P_k(l_k) \neq 0\} \\ &\subset \mathfrak{g}^*_{Puk} \end{aligned}$$

Hence $\mathcal{V}$ is a Zariski open, dense subset of $\mathfrak{g}^*$. Because the ideals $\mathfrak{g}_k$ form a Jordan-Hölder sequence, the classical proof shows that each one of the polynomials P_k is G-invariant, as well as the Zariski open set $\mathcal{V}$.

2 Different situations in an induction proof

2.1

Let $G = \exp(\mathfrak{g})$ be a connected, simply connected nilpotent Lie group. Let $\tilde{\mathfrak{g}}$ be a codimension one ideal in $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \tilde{\mathfrak{g}}$ and let $X \in \mathfrak{g}$ be such that $\mathfrak{g} = \mathbb{R}X \oplus \tilde{\mathfrak{g}}$. For every $l \in \mathfrak{g}^*$, let's denote $\tilde{l} = p(l) = l|_{\tilde{\mathfrak{g}}}$ the restriction of l to $\tilde{\mathfrak{g}}$. Let's assume that the Jordan-Hölder basis is chosen such that $\tilde{\mathfrak{g}} = < X_1, \dots, X_{n-1} >$.
Let's recall that the generic elements of $\mathfrak{g}^*$ (with respect to the given basis) are given by the condition

$$P(l) = \det\bigl(< l, [X_i, X_j] >\bigr)_{i,j \in \mathcal{S}} \neq 0.$$

As $[\mathfrak{g},\mathfrak{g}] \subset \tilde{\mathfrak{g}}$, the polynomial P does not depend on the coordinate of l in the direction of X^*. It may hence be regarded as a polynomial on $\tilde{\mathfrak{g}}^*$, i. e. $P(l) = P(l|_{\tilde{\mathfrak{g}}}) = P(\tilde{l})$. Hence, the projection of $\mathfrak{g}^*_{Puk}$ on $\tilde{\mathfrak{g}}^*$

$$p(\mathfrak{g}^*_{Puk}) = \{l|_{\tilde{\mathfrak{g}}} \mid l \in \mathfrak{g}^* \text{ and } P(l) \neq 0\} = \{\tilde{l} \in \tilde{\mathfrak{g}}^* \mid P(\tilde{l}) \neq 0\}$$

is a non-void Zariski open set of $\tilde{\mathfrak{g}}^*$. Similarly, the set $\tilde{\mathfrak{g}}^*_{Puk}$ of generic elements of $\tilde{\mathfrak{g}}^*$ is given by

$$\tilde{P}(\tilde{l}) = \det\big(< \tilde{l}, [X_i, X_j] >\big)_{i,j \in \tilde{\mathcal{S}}} \neq 0$$

where $\tilde{\mathcal{S}}$ is the set of indices of the generic elements of $\tilde{\mathfrak{g}}$. Hence $p(\mathfrak{g}^*_{Puk}) \cap \tilde{\mathfrak{g}}^*_{Puk}$ is a Zariski open dense subset of $\tilde{\mathfrak{g}}^*$.

2.2

One has to distinguish the following two cases:

Case I: For every $l \in \mathfrak{g}^*_{Puk}$,

$$\tilde{\mathfrak{g}} + \mathfrak{g}(l) \subsetneq \mathfrak{g}.$$

In that case,

$$\mathfrak{g}(l) \subset \tilde{\mathfrak{g}}(\tilde{l}) \subset \tilde{\mathfrak{g}}$$

and, for $\tilde{l} = l|_{\tilde{\mathfrak{g}}}$, the projected image of the orbit $p(\mathcal{O}_l)$ is a disjoint union $\bigcup_{t\in\mathbb{R}} (\mathrm{Ad}^* \exp(tX))\mathcal{O}_{\tilde{l}}$ of distinct $\mathrm{Ad}^*\tilde{G}$-orbits in $\tilde{\mathfrak{g}}^*$. The orbit $\mathcal{O}_l$ in $\mathfrak{g}^*$ is p-saturated, i. e. $\mathcal{O}_l = p^{-1}(p(\mathcal{O}_l)) = p^{-1}(\mathcal{O}_{\tilde{l}})$.

Case II: For all $l \in \mathfrak{g}^*_{Puk}$,

$$\tilde{\mathfrak{g}} + \mathfrak{g}(l) = \mathfrak{g}.$$

In that case,

$$\mathfrak{g}(l) \not\subset \tilde{\mathfrak{g}},$$

the projected image of the orbit $p(\mathcal{O}_l)$ is the single $\mathrm{Ad}^*\tilde{G}$-orbit $\mathcal{O}_{\tilde{l}}$ and $p : \mathcal{O}_l \to \mathcal{O}_{\tilde{l}}$ is a bijection. (see [Co-Gr])

If we start with a fixed Jordan-Hölder sequence as in section (1) and if $\tilde{\mathfrak{g}} = \mathfrak{g}_{n-1}$ and $X = X_n$, then case I means that n is a jump index for every $l \in \mathfrak{g}^*_{Puk}$ and case II means that no $l \in \mathfrak{g}^*_{Puk}$ has n as a jump index (as all the elements of $\mathfrak{g}^*_{Puk}$ have the same jump indices).

2.3

In **case I**, every polarization $\tilde{\mathfrak{p}}(\tilde{l})$ for $\tilde{l}$ in $\tilde{\mathfrak{g}}$ is also a polarization for l in $\mathfrak{g}$, written as $\mathfrak{p}(l) = \tilde{\mathfrak{p}}(\tilde{l})$. If we write $P(l) = \exp(\mathfrak{p}(l)) \subset G$ and $\tilde{P}(\tilde{l}) = \exp(\tilde{\mathfrak{p}}(\tilde{l})) \subset \tilde{G}$, $P(l) = \tilde{P}(\tilde{l}) \subset \tilde{G} \subset G$. Then χ_l and $\chi_{\tilde{l}}$ defined by $\chi_l(x) = \chi_{\tilde{l}}(x) = e^{-i<l,\log x>}$ for every $x \in P(l) = \tilde{P}(\tilde{l})$ are unitary characters on $P(l) \subset G$, resp. $\tilde{P}(\tilde{l}) \subset \tilde{G}$. If we write $\pi_l = \mathrm{ind}_{P(l)}^G \chi_l$ and $\pi_{\tilde{l}} = \mathrm{ind}_{\tilde{P}(\tilde{l})}^{\tilde{G}} \chi_{\tilde{l}}$ for the corresponding unitary representations, then π_l is unitary equivalent to $\tilde{\pi}_l$ defined by

$$\tilde{\pi}_l = \mathrm{ind}_{\tilde{G}}^G \big(\mathrm{ind}_{\tilde{P}(\tilde{l})}^{\tilde{G}} \chi_{\tilde{l}}\big) = \mathrm{ind}_{\tilde{G}}^G \pi_{\tilde{l}}.$$

This unitary equivalence is obtained in the following way: Let $\mathfrak{H}_{\pi_l}$ and $\mathfrak{H}_{\pi_{\tilde{l}}}$ denote the representation spaces of π_l and $\pi_{\tilde{l}}$ respectively. The representation space of $\tilde{\pi}_l$ may then be identified with $L^2(\mathbb{R}, \mathfrak{H}_{\pi_{\tilde{l}}})$ (endowed with the appropriate covariance condition) and the unitary equivalence of π_l and $\tilde{\pi}_l$ is obtained by

$$\begin{array}{rcl} U : \mathfrak{H}_{\pi_l} & \to & \mathfrak{H}_{\tilde{\pi}_l} \\ \xi & \mapsto & \tilde{\xi} \end{array}$$

defined by

$$\tilde{\xi}(x)(\tilde{g}) = \tilde{\xi}(x,\tilde{g}) := \xi\Big(\exp(xX)\cdot\tilde{g}\Big), \quad \forall x\in\mathbb{R}, \forall\tilde{g}\in\tilde{G},$$

or, more generally,

$$\tilde{\xi}\Big(\exp(xX)\cdot\tilde{g}_1\Big)(\tilde{g}) = \xi\Big(\exp(xX)\cdot\tilde{g}_1\tilde{g}\Big), \quad \forall x\in\mathbb{R}, \forall\tilde{g}_1,\tilde{g}\in\tilde{G}.$$

Then $\tilde{\xi}$ is well defined and satisfies the correct covariance condition. We shall very often identify π_l and $\tilde{\pi}_l$. The following computation gives the expression of the representation $d\tilde{\pi}_l$ restricted to the Lie algebra $\tilde{\mathfrak{g}}$ and to the enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$:

$$\begin{aligned} d\tilde{\pi}_l(U)\tilde{\xi}(t,\tilde{g}) &= d\pi_l(U)\xi(\exp(tX)\cdot\tilde{g}) \\ &= \frac{d}{ds}\xi(\exp(-sU)\exp(tX)\cdot\tilde{g})|_{s=0} \\ &= \frac{d}{ds}\tilde{\xi}\big(t,\exp(-s\mathrm{Ad}(\exp(-tX))U)\cdot\tilde{g}\big)|_{s=0} \\ &= d\pi_{\tilde{l}}(\mathrm{Ad}(\exp(-tX))U)\tilde{\xi}(t,\cdot)(\tilde{g}) \end{aligned}$$

for every $U\in\tilde{\mathfrak{g}}$. The formula

$$d\tilde{\pi}_l(U)\tilde{\xi}(t,\tilde{g}) = d\pi_{\tilde{l}}(\mathrm{Ad}(\exp(-tX))U)\tilde{\xi}(t,\cdot)(\tilde{g})$$

then remains true for every $U\in\mathfrak{U}(\tilde{\mathfrak{g}})$.

2.4

In **case II** we have, by ([Co-Gr]), for $l\in\mathfrak{g}^*_{Puk}$:

$$\tilde{\mathfrak{g}}+\mathfrak{g}(l)=\mathfrak{g}$$
$$\tilde{\mathfrak{g}}(\tilde{l})\subsetneq\mathfrak{g}(l)$$

By ([Co-Gr]) we also know that in this case, for every polarization $\mathfrak{p}=\mathfrak{p}(l)$ of an element $l\in\mathfrak{g}^*_{Puk}$, $\tilde{\mathfrak{p}}=\mathfrak{p}\cap\tilde{\mathfrak{g}}$ is a polarization for $\tilde{l}$ in $\tilde{\mathfrak{g}}$, $\mathfrak{p}=\tilde{\mathfrak{p}}+\mathfrak{g}(l)$ and $\tilde{\mathfrak{p}}=\tilde{\mathfrak{p}}(\tilde{l})$ is of codimension one in $\mathfrak{p}(l)$. Finally

$$\pi_{\tilde{l}}\simeq\pi_l|_{\tilde{G}}$$

where $\pi_{\tilde{l}}=\mathrm{ind}_{\tilde{P}}^{\tilde{G}}\chi_{\tilde{l}}$ and $\pi_l=\mathrm{ind}_P^G\chi_l$. In fact, one may in that case choose the same complementary basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$ and to $\tilde{\mathfrak{p}}(\tilde{l})$ in $\tilde{\mathfrak{g}}$. Hence the representation spaces $\mathfrak{H}_{\pi_l}$ and $\mathfrak{H}_{\pi_{\tilde{l}}}$ may both be identified with an $L^2(\mathbb{R}^d)$ (with the appropriate covariance conditions respectively) with respect to the same basis and, with this identification, the intertwining operator A is the identity. Otherwise, i. e. if A is considered as an operator from $\mathfrak{H}_{\pi_l}$ to $\mathfrak{H}_{\pi_{\tilde{l}}}$, $A\xi=\xi|_{\tilde{G}}$.

3 Generalized Kirillov lemma

3.1

Let $\mathcal{V}\subset\mathfrak{g}^*$ be as in (1.3). Let's assume that n is a jump index (in the sense of Pukanszky), i. e. that

$$\mathfrak{g}_{n-1}+\mathfrak{g}(l)\subsetneq\mathfrak{g}_n+\mathfrak{g}(l)=\mathfrak{g}+\mathfrak{g}(l)=\mathfrak{g}, \quad \forall l\in\mathcal{V}.$$

Then obviously we are in **case I** for the step that goes from $\mathfrak{g}_{n-1}$ to $\mathfrak{g}_n = \mathfrak{g}$. Let's use the notations of section 2, i. e. let's put $\tilde{\mathfrak{g}} = \mathfrak{g}_{n-1}$ and $\tilde{l} = l_{n-1} = l|_{\tilde{\mathfrak{g}}}$. Let's also write $\tilde{\mathcal{S}}$ for the set of jump indices of $\tilde{\mathfrak{g}}$ and $\tilde{\mathcal{V}} = p(\mathcal{V}) = \{ \tilde{l} \in \tilde{\mathfrak{g}}^* \mid \exists l \in \mathcal{V} \text{ such that } l|_{\tilde{\mathfrak{g}}} = \tilde{l} \}$.
As the dimension of every orbit is even, the number of jump indices must also be even. As, moreover, $\mathfrak{g}(l)$ is of codimension one in $\tilde{\mathfrak{g}}(\tilde{l})$, there must be exactly one jump index j_r for generic elements in $\mathfrak{g}^*$ distinct from n, which is not a jump index for generic elements in $\tilde{\mathfrak{g}}$. This index must be the same for all $l \in \mathcal{V}$. Hence $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{j_r, n\}$ and

$$\tilde{P}(\tilde{l}) = \det\big(< \tilde{l}, [X_i, X_j] >\big)_{i,j \in \mathcal{S}\setminus\{j_r,n\}}.$$

Let's recall that $\tilde{P}(\tilde{l})$, $\tilde{\mathcal{V}} = p(\mathcal{V})$ and $\tilde{\mathfrak{g}}^*$ are Ad^*G-invariant. In particular, if $\tilde{l}$ is a generic element of $\tilde{\mathfrak{g}}^*$ (resp. if $\tilde{l} \in \tilde{\mathcal{V}}$), this is also the case for $\mathrm{Ad}^*(\exp(tX))(\tilde{l})$, for every $t \in \mathbb{R}$.
Details about the representations associated to the elements of $\mathcal{V}$, resp. $\tilde{\mathcal{V}}$ have been given in (2.3).

3.2 About invariant polynomials

We shall first prove that there exists on $\tilde{\mathfrak{g}}^*$ a polynomial function $\tilde{R}$ that is $\mathrm{Ad}^*\tilde{G}$-invariant, but not Ad^*G-invariant. The orbits of the elements of $\tilde{\mathcal{V}}$ in $\tilde{\mathfrak{g}}^*$ are described as in (1.2) by

$$\tilde{Q}(\tilde{l}, t) = \sum_{i=1}^{n-1} \tilde{Q}_i(\tilde{l}, t) X_i^*.$$

As j_r is not a jump index for $(\tilde{l}, \tilde{\mathfrak{g}}^*)$,

$$\tilde{Q}_{j_r}(\tilde{l}, t) = \tilde{l}_{j_r} + \tilde{R}_{j_r}(\tilde{l}_1, \ldots, \tilde{l}_{j_r-1}, t_1, \ldots, t_{r-1})$$

where $\tilde{R}_{j_r}$ is rational. Moreover, there exists N sufficiently large such that the function

$$\tilde{P}^N(\tilde{l})\tilde{R}_{j_r}(\tilde{l}, t)$$

is polynomial in $\tilde{l}$ and t. Here $\tilde{P}$ is Ad^*G-invariant and $\tilde{R}_{j_r}$ is $\mathrm{Ad}^*\tilde{G}$-invariant. Hence $\tilde{P}^N(\tilde{l})\tilde{R}_{j_r}(\tilde{l}, t)$ is $\mathrm{Ad}^*\tilde{G}$-invariant. Let's assume that for every $t \in \mathbb{R}^{2d-2}$, the function $\tilde{P}^N(\tilde{l})\tilde{R}_{j_r}(\tilde{l}, t)$ is also Ad^*G- invariant, in particular that

$$\tilde{R}_{j_r}(\mathrm{Ad}^*(\exp sX)\tilde{l}, t) = \tilde{R}_{j_r}(\tilde{l}, t), \quad \forall s \in \mathbb{R}, \forall t \in \mathbb{R}^{2d-2}.$$

Hence, for any $l \in \mathcal{V}$ such that $p(l) = \tilde{l}$, $p(\mathcal{O}_l) = \cup_{s\in\mathbb{R}} \mathrm{Ad}^*(\exp(sX))\tilde{l}$ cannot be saturated in the direction of $X_{j_r}^*$. On the other hand, as j_r is a jump index for $(l, \mathfrak{g})$, $\mathcal{O}_l$ and $p(\mathcal{O}_l)$ have to be saturated in the direction of $X_{j_r}^*$. This contradiction shows that there exists $t^0 \in \mathbb{R}^{2d-2}$ such that

$$\tilde{R}(\tilde{l}) = P^N(\tilde{l})\tilde{R}_{j_r}(\tilde{l}, t^0)$$

is an $\mathrm{Ad}^*\tilde{G}$-invariant polynomial, that is not Ad^*G-invariant. By construction, this polynomial is defined on the dense open subset $\tilde{\mathcal{V}}$ of $\tilde{\mathfrak{g}}^*$. But it may of course be extended to the whole of $\tilde{\mathfrak{g}}^*$ and remains then $\mathrm{Ad}^*\tilde{G}$-invariant, by continuity.

3.3 Center of the enveloping algebra

Let's denote by $\mathfrak{Z}\mathfrak{U}(\mathfrak{g})$ the center of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$, i. e.

$$\mathfrak{Z}\mathfrak{U}(\mathfrak{g}) = \{A \in \mathfrak{U}(\mathfrak{g}) \mid (\mathrm{ad}X)A = XA - AX = 0, \forall X \in \mathfrak{g}\}.$$

Let $\mathbb{C}[\mathfrak{g}^*]^G$ be the set of G-invariant polynomials on $\mathfrak{g}^*$ i. e.

$$\mathbb{C}[\mathfrak{g}^*]^G = \{f \in \mathbb{C}[\mathfrak{g}^*] \mid (\mathrm{Ad}g)f = f, \forall g \in G\},$$

where the action of G on $\mathbb{C}[\mathfrak{g}^*]$ is defined by $(\mathrm{Ad}g)f(l) = f(\mathrm{Ad}^*(g^{-1})l)$, for all $l \in \mathfrak{g}^*$ and $f \in \mathbb{C}[\mathfrak{g}^*]$. It is shown in ([Co-Gr]) that there exists an isomorphism

$$\begin{array}{rcl} \mathfrak{Z}\mathfrak{U}(\mathfrak{g}) & \rightarrow & \mathbb{C}[\mathfrak{g}^*]^G \\ W & \mapsto & P_W \end{array}$$

such that $(\mathrm{Ad}g)P_W = P_{(\mathrm{Ad}g)W}$. Moreover, for every $W \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$ and every $\pi_l \in \hat{G}$, $d\pi_l(W)$ is scalar on $\mathfrak{H}_{\pi_l}^{\infty}$, as W is central, and

$$d\pi_l(W) = P_W(il)\mathbf{1}_{\mathfrak{H}_{\pi_l}}.$$

3.4

As $\tilde{\mathfrak{g}}$ is an ideal of $\mathfrak{g}$, we may consider $\mathfrak{U}(\tilde{\mathfrak{g}})$ as a G-invariant subset of $\mathfrak{U}(\mathfrak{g})$. Similarly for $\mathfrak{S}(\tilde{\mathfrak{g}})$ and $\mathfrak{S}(\mathfrak{g})$. Moreover, polynomials on $\tilde{\mathfrak{g}}^*$ may be identified with polynomials on $\mathfrak{g}^*$ (that don't depend on the component $l_n =< l, X_n >$). In that sense, $\mathbb{C}[\tilde{\mathfrak{g}}^*] \subset \mathbb{C}[\mathfrak{g}^*]$. In particular, for $W \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}) \cap \mathfrak{U}(\tilde{\mathfrak{g}}) \subset \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$ we write P_W as well for the polynomial associated to W in $\mathbb{C}[\tilde{\mathfrak{g}}^*]$ and in $\mathbb{C}[\mathfrak{g}^*]$. This is justified by the following argument: Let P_W be the G-invariant polynomial such that $d\pi_l(W) = P_W(il)\mathbf{1}_{\mathfrak{H}_{\pi_l}}$ and $\tilde{P}_W$ the $\tilde{G}$-invariant polynomial such that $d\pi_{\tilde{l}}(W) = \tilde{P}_W(i\tilde{l})\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}}}}$. As π_l and $\tilde{\pi}_l$ are unitary equivalent, we also have $d\tilde{\pi}_l(W) = P_W(il)\mathbf{1}_{\mathfrak{H}_{\tilde{\pi}_l}}$. Hence

$$\begin{aligned} [d\tilde{\pi}_l(W)\tilde{\xi}(t,\cdot)](\tilde{g}) &= [P_W(il)\tilde{\xi}(t)](\tilde{g}) \\ &= P_W(il)[\tilde{\xi}(t,\cdot)(\tilde{g})] \end{aligned}$$

and, on the other hand,

$$\begin{aligned} [d\tilde{\pi}_l(W)\tilde{\xi}(t,\cdot)](\tilde{g}) &= d\tilde{\pi}_l(W)\tilde{\xi}(t,\tilde{g}) \\ &= d\pi_{\tilde{l}}\big(\mathrm{Ad}(\exp(-tX))W\big)\tilde{\xi}(t,\cdot)(\tilde{g}) \\ &= d\pi_{\tilde{l}}(W)\tilde{\xi}(t,\cdot)(\tilde{g}) \\ &= \tilde{P}_W(i\tilde{l})\tilde{\xi}(t,\cdot)(\tilde{g}) \\ &= \tilde{P}_W(i\tilde{l})[\tilde{\xi}(t)(\tilde{g})] \end{aligned}$$

as W is central. This proves that $P_W(il) = \tilde{P}_W(i\tilde{l})$ for all l and that $P_W(il)$ depends only on $\tilde{l}$. As $\tilde{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$, the group G acts on $\mathbb{C}[\tilde{\mathfrak{g}}^*]$ as it does on $\mathbb{C}[\mathfrak{g}^*]$.

3.5

Let's recall that there exists $\tilde{R} \in \mathbb{C}[\tilde{\mathfrak{g}}^*] \subset \mathbb{C}[\mathfrak{g}^*]$ that is $\mathrm{Ad}^*\tilde{G}$-invariant, but not Ad^*G-invariant. Hence there exists $W \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$ such that $\tilde{R} = P_W$ and $W \notin \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$. This implies in particular that $[X, W] \neq 0$. It is easy to see that $[X, W] \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$. If $[X, W] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$, we put $Y = W$ and $0 \neq Z = [X, W] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$. Otherwise we replace W by $[X, W]$ and repeat the previous argument. Because of nilpotency, we find after a finite number of steps $0 \neq \tilde{W} = [X, \ldots, [X, W] \ldots] \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$ such that $\tilde{W} \notin \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$ and $0 \neq [X, \tilde{W}] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$. We then put $Y = \tilde{W} \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$, $0 \neq Z = [X, \tilde{W}] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}) \cap \mathfrak{U}(\tilde{\mathfrak{g}})$ and we have

$$[X, Y] = Z.$$

Because of this relation we speak about the *generalized Kirillov lemma*. As $[X, Y] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$,

$$[X, \mathrm{Ad}(\exp(sX))Y] = [X, Y + sZ] = [X, Y] = Z, \quad \forall s \in \mathbb{R},$$

$\mathrm{Ad}(\exp(sX))Y = Y + sZ \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$ and we may replace Y by $\mathrm{Ad}(\exp(sX))Y = Y + sZ$ with an appropriate s in the following arguments. Moreover, $d\pi_{\tilde{l}}(Y) = \tilde{P}_Y(i\tilde{l})\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}}}}$ and $d\pi_{\tilde{l}}(Z) = P_Z(i\tilde{l})\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}}}}$, where $\tilde{P}_Y \in \mathbb{C}[\tilde{\mathfrak{g}}^*]^{\tilde{G}}$ and $P_Z \in \mathbb{C}[\tilde{\mathfrak{g}}^*]^G \subset \mathbb{C}[\mathfrak{g}^*]^G$ by (3.4), as $Z \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}) \cap \mathfrak{U}(\tilde{\mathfrak{g}})$. Hence, according to (2.3), we have for $l \in \mathfrak{g}^*$ and $\tilde{l} = l|_{\tilde{\mathfrak{g}}}$,

$$\begin{aligned} d\tilde{\pi}_l(Y)\tilde{\xi}(t, \tilde{g}) &= d\pi_{\tilde{l}}\big(\mathrm{Ad}(\exp(-tX))Y\big)\tilde{\xi}(t, \cdot)(\tilde{g}) \\ &= d\pi_{\tilde{l}}(Y - tZ)\tilde{\xi}(t, \cdot)(\tilde{g}) \\ &= [\tilde{P}_Y(i\tilde{l}) + (-t)P_Z(i\tilde{l})]\tilde{\xi}(t, \cdot)(\tilde{g}) \end{aligned}$$

for all $t \in \mathbb{R}$, all $\tilde{g} \in \tilde{G}$.
We may consider the polynomials $\tilde{P}_Y$ and P_Z as being defined on $\mathfrak{g}^*$ (and depending only on $\tilde{\mathfrak{g}}^*$). Let's put $Q_0(l) = \tilde{P}_Y(il)$, $Q_1(l) = -P_Z(il)$ and $Q(t, l) = Q_0(l) + tQ_1(l)$. Then Q_0 is a $\tilde{G}$-invariant polynomial and Q_1 is a G-invariant polynomial (for the action of G on $\tilde{\mathfrak{g}}^*$). They both depend only on $\tilde{l} = l|_{\tilde{\mathfrak{g}}^*}$.
The polynomial Q_0 cannot be G-invariant, because otherwise $\tilde{P}_Y$ would also be G-invariant, which is not the case.
The polynomial $Q_1(\tilde{l})$ is not identically zero. Otherwise

$$Q(t, l) = Q(t, \tilde{l}) = \tilde{P}_{Y-tZ}(i\tilde{l}) = \tilde{P}_{\mathrm{Ad}(\exp(-tX))Y)}(i\tilde{l}) = \tilde{P}_Y(i\mathrm{Ad}^*(\exp(tX))(\tilde{l}))$$

would be independent of t and $\tilde{P}_Y$ would be G-invariant, which is not the case. Let's notice that we shall have to restrict ourselves to the Zariski open set of all for $l \in \mathcal{V}$ such that $Q_1(\tilde{l}) = Q_1(l) \neq 0$. This will automatically be the case if l belongs to the Zariski open set $\mathcal{W}$ to be defined in (5.3).

3.6

The elements Y and Z of $\mathfrak{U}(\mathfrak{g})$ may be chosen to be anti-hermitian, such that the polynomial $Q_1(l) = Q_1(\tilde{l})$ is non-zero and purely imaginary. In fact, let's recall that

$$\begin{aligned} &X^* = -X \quad (\text{as } X \in \mathfrak{g}) \\ &([U, V])^* = -[U^*, V^*], \quad \forall\, U, V \in \mathfrak{U}(\mathfrak{g}) \end{aligned}$$

and hence

$$\begin{aligned}&[X, Y^*] = Z^*\\ &[X, \frac{1}{2}(Y+Y^*] = \frac{1}{2}(Z+Z^*)\\ &[X, \frac{1}{2i}(Y-Y^*)] = \frac{1}{2i}(Z-Z^*).\end{aligned}$$

Then $\frac{1}{2}(Z+Z^*)$ and $\frac{1}{2i}(Z-Z^*)$ are self-adjoint elements of $\mathfrak{Z}\mathfrak{U}(\mathfrak{g})$. One of them at least is non-zero, as $Z \neq 0$. If $\frac{1}{2}(Z+Z^*) \neq 0$, then $\frac{1}{2}(Y+Y^*) \neq 0$ and $\tilde{P}_{\frac{1}{2}(Y+Y^*)} \not\equiv 0$ as, for every $\tilde{l} \in p(\mathcal{V})$, dense subset of $\tilde{\mathfrak{g}}^*$,

$$d\pi_{\tilde{l}}\big(\frac{1}{2}(Y+Y^*)\big) = \tilde{P}_{\frac{1}{2}(Y+Y^*)}(i\tilde{l}) \cdot \mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}}}}.$$

As, in general, $\tilde{P}_{U^*}(i\tilde{l}) = \overline{\tilde{P}_U(i\tilde{l})}$ for all $U \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$, for all $\tilde{l} \in \tilde{\mathfrak{g}}^*$, and as $\frac{1}{2}(Y+Y^*)$ is a self-adjoint element of $\mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$, $\tilde{P}_{\frac{1}{2}(Y+Y^*)}(i\tilde{l})$ has to be real. In this case, replace Y by $\frac{i}{2}(Y+Y^*)$ and Z by $\frac{i}{2}(Z+Z^*)$ and call them again Y and Z. This proves that in this case we may assume Y, Z anti-hermitian and $\tilde{P}_Y(il) = \tilde{P}_Y(i\tilde{l})$ purely imaginary for all $\tilde{l}$, as $\tilde{P}_{iW} = i\tilde{P}_W$ for all $W \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$. If $\frac{1}{2i}(Z-Z^*) \neq 0$, we proceed similarly.

So, in the future, let's assume that Y has been chosen anti-hermitian so that $\tilde{P}_Y(i\tilde{l})$ is purely imaginary. Hence

$$Q(t,l) = Q(t,\tilde{l}) = \tilde{P}_Y(i\mathrm{Ad}^*(\exp tX)(\tilde{l})) = Q_0(\tilde{l}) + Q_1(\tilde{l})t = Q_0(l) + Q_1(l)t$$

is purely imaginary for all t and all l. So $Q_0(l)$ (if non-zero) and $Q_1(l)$ are purely imaginary. Let's just mention that if $Y, Z \in \mathfrak{g}$ (as in the case of the Heisenberg group), then they are anti-hermitian and $\tilde{P}_Y(il)$, $Q_0(l)$ (if non zero), $Q_1(l)$ are in effect purely imaginary. Let's put $R = iQ_1$. Then R is a non-zero, G-invariant, real valued polynomial that depends only on $\tilde{l} = l|_{\tilde{\mathfrak{g}}}$.

4 Polarizations and bases

4.1

Let's come back to the situation described in section 1. Let's recall that

$$\mathfrak{g}_k =< X_1, \ldots, X_k >$$

is an ideal in $\mathfrak{g}$ for all $k \in \{1, \ldots, n\}$. We write $l_k = l|_{\mathfrak{g}_k}$ and

$$\mathfrak{g}_k(l_k) = \{U \in \mathfrak{g}_k \mid \ < l, [U, \mathfrak{g}_k] > \equiv 0\}.$$

Then

$$\mathfrak{p}(l) = \sum_{j=1}^{n} \mathfrak{g}_j(l_j)$$

is the Vergne polarization for l in $\mathfrak{g}$ and

$$\mathfrak{p}_k(l_k) = \sum_{j=1}^{k} \mathfrak{g}_j(l_j)$$

is the Vergne polarization for $l_k = l|_{\mathfrak{g}_k}$ in $\mathfrak{g}_k$, given the fixed Jordan-Hölder sequence.

4.2

Assume that k is not a jump index, i. e. that

$$\mathfrak{g}_{k-1} + \mathfrak{g}(l) = \mathfrak{g}_k + \mathfrak{g}(l).$$

It is then easy to check that also

$$\mathfrak{g}_{k-1} + \mathfrak{g}_k(l_k) = \mathfrak{g}_k + \mathfrak{g}_k(l_k) = \mathfrak{g}_k.$$

To go from $\mathfrak{g}_{k-1}$ to $\mathfrak{g}_k$ we are in case II. The corresponding Vergne polarizations satisfy $\mathfrak{p}_k(l_k) = \mathfrak{p}_{k-1}(l_{k-1}) + \mathfrak{g}_k(k_k)$, their dimension increases by one and we may take the same complementary basis to $\mathfrak{p}_{k-1}(l_{k-1})$ in $\mathfrak{g}_{k-1}$ and to $\mathfrak{p}_k(l_k)$ in $\mathfrak{g}_k$.

4.3

If k is a jump index, i. e. if

$$\mathfrak{g}_{k-1} + \mathfrak{g}(l) \subsetneq \mathfrak{g}_k + \mathfrak{g}(l),$$

two situations are possible:
(i) Either we have

$$\mathfrak{g}_{k-1} + \mathfrak{g}_k(l_k) = \mathfrak{g}_k + \mathfrak{g}_k(l_k) = \mathfrak{g}_k.$$

We may then draw exactly the same conclusions as in the previous section.
(ii) The other possibility is that

$$\mathfrak{g}_{k-1} + \mathfrak{g}_k(l_k) \subsetneq \mathfrak{g}_k + \mathfrak{g}_k(l_k) = \mathfrak{g}_k.$$

Definition: We say that k is a *local jump index* for l if

$$\mathfrak{g}_{k-1} + \mathfrak{g}_k(l_k) \subsetneq \mathfrak{g}_k + \mathfrak{g}_k(l_k) = \mathfrak{g}_k.$$

This means that k is a Pukanszky jump index for l_k in $\mathfrak{g}_k$. Let's recall that for $l \in \mathcal{V}$, $l_k \in (\mathfrak{g}_k^*)_{Puk}$, by definition of $\mathcal{V}$ (1.3). Moreover, we know that all the elements of $(\mathfrak{g}_k^*)_{Puk}$ have the same Pukanszky jump indices in $\mathfrak{g}_k^*$. This implies that the elements of $\mathcal{V}$ have the same local jump indices. We may hence put:

$$\mathcal{R} := \{k \in \mathbb{N} \mid k \text{ is a local jump index } \forall l \in \mathcal{V}\} = \{i_1, i_2, \ldots, i_d\}.$$

Then $\mathcal{R} \subset \mathcal{S}$ (where $\mathcal{S}$ denotes the set of Pukanszky jump indices).

4.4

Let's assume that k is a local jump index. To go from $\mathfrak{g}_{k-1}$ to $\mathfrak{g}_k$ we are in case I. We have $\mathfrak{g}_k(l_k) \subset \mathfrak{g}_{k-1}(l_{k-1})$ and the Vergne polarizations satisfy $\mathfrak{p}_{k-1}(l_{k-1}) = \mathfrak{p}_k(l_k)$. We may add X_k to the complementary basis to $\mathfrak{p}_{k-1}(l_{k-1})$ in $\mathfrak{g}_{k-1}$ to get a complementary basis to $\mathfrak{p}_k(l_k)$ in $\mathfrak{g}_k$. Let's finally notice that in this case $X_k \notin \mathfrak{p}_r(l_r) = \sum_{j=1}^r \mathfrak{g}_j(l_j)$ for every $r \geq k$ and that hence X_k may be put into the complementary basis to $\mathfrak{p}_r(l_r)$ in $\mathfrak{g}_r$. In particular, X_k may be put into the complementary basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$. In fact, as $\mathfrak{p}_{k-1}(l_{k-1}) = \mathfrak{p}_k(l_k)$, $X_k \notin \mathfrak{p}_k(l_k)$. Then $< l, [X_k, \mathfrak{p}_k(l_k)] > \neq 0$. Otherwise $\mathfrak{p}'_k = \mathfrak{p}_k(l_k) + \mathbb{R}X_k$ would also satisfy $< l, [\mathfrak{p}'_k, \mathfrak{p}'_k] >= 0$ and $\mathfrak{p}_k(l_k)$ would not be a polarization. Hence, there exists $Y_k \in \mathfrak{p}_k(l_k) \subset \mathfrak{p}_r(l_r)$ such that $< l, [X_k, Y_k] > \neq 0$. So $X_k \notin \mathfrak{p}_r(l_r)$, as $\mathfrak{p}_r(l_r)$ is a polarization of l_r in $\mathfrak{p}_r$.

For the elements of the Zariski open set $\mathcal{V}$ (see 1.3), we shall make the following choices: Given the fixed basis $\{X_1, \ldots, X_n\}$ as in section 1, given $l \in \mathcal{V}$, we associate to l its Vergne polarization $\mathfrak{p}(l) = \sum_{j=1}^n \mathfrak{p}_j(l_j)$. We then take

$$\{X_k \mid k \in \mathcal{R}\} = \{X_{i_1}, X_{i_2}, \ldots, X_{i_d}\}$$

as complementary basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$. This basis does not depend on the choice of $l \in \mathcal{V}$. We may hence identify the representation spaces $\mathfrak{H}_{\pi_l} = L^2(G/P(l), \chi_l)$ with $L^2(\mathbb{R}^d)$ and $\mathfrak{H}_{\pi_l}^\infty = \mathcal{S}(G/P(l), \chi_l)$ with $\mathcal{S}(\mathbb{R}^d)$ with respect to the same basis for every $l \in \mathcal{V}$, but of course with a covariance condition depending on l. Let's finally notice that the index d is well chosen, as the dimension of $\mathfrak{g}/\mathfrak{p}(l)$ is half the dimension of the orbit of l.

5 A decomposition of a dense subset of the dual of the Lie algebra

5.1

Let's now consider the generalized Kirillov situation studied in section 3. For any $k \in \mathcal{R} = \{i_1, \ldots, i_d\}$, there exists $Y_k \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$ such that

$$0 \neq Z_k = [X_k, Y_k] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_k).$$

We shall show that we may in fact choose $Y_k \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$ such that

$$0 \neq Z_k = [X_k, Y_k] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}).$$

To do this, let's first point out the following facts:
(i) As $\mathfrak{g}$ is nilpotent and as we start with a Jordan-Hölder sequence, $[\mathfrak{g}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}$.
(ii) If $\tilde{\mathfrak{g}}$ is any ideal of $\mathfrak{g}$, then $(\mathfrak{U}(\tilde{\mathfrak{g}}), [\cdot,\cdot])$ is an ideal in $(\mathfrak{U}(\mathfrak{g}), [\cdot,\cdot])$. In particular, all the $(\mathfrak{U}(\mathfrak{g}_k), [\cdot,\cdot])$'s are ideals in $(\mathfrak{U}(\mathfrak{g}), [\cdot,\cdot])$.
(iii) If $\tilde{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$, then $(\mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}}), [\cdot,\cdot])$ is an ideal in $(\mathfrak{U}(\mathfrak{g}), [\cdot,\cdot])$. In particular, $(\mathfrak{Z}\mathfrak{U}(\mathfrak{g}_k), [\cdot,\cdot])$ is an ideal in $(\mathfrak{U}(\mathfrak{g}), [\cdot,\cdot])$ for all k.
In fact, let $Z \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$, $X \in \mathfrak{U}(\mathfrak{g})$ and $U \in \mathfrak{U}(\tilde{\mathfrak{g}})$. Then

$$\begin{aligned}\big[[X,Z],U\big] &= -\big[[Z,U],X\big] - \big[[U,X],Z\big] \\ &= 0,\end{aligned}$$

as $Z \in \mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$ and as $U, [U,X] \in \mathfrak{U}(\tilde{\mathfrak{g}})$, because $\mathfrak{U}(\tilde{\mathfrak{g}})$ is an ideal in $(\mathfrak{U}(\mathfrak{g}), [\cdot,\cdot])$. This shows that $\mathfrak{Z}\mathfrak{U}(\tilde{\mathfrak{g}})$ is an ideal in $(\mathfrak{U}(\mathfrak{g}), [\cdot,\cdot])$.

Let's now assume that $Y_k \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$ such that $0 \neq [X_k, Y_k] = Z_k \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_k)$ and let's show that we may in fact choose Y_k such that $0 \neq [X_k, Y_k] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$. In fact, if $Z_k = [X_k, Y_k] \notin \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$, then there exists $W \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$ such that

$$0 \neq [W, Z_k] = \big[W, [X_k, Y_k]\big] = -\big[X_k, [Y_k, W]\big] - \big[Y_k, [W, X_k]\big] = \big[X_k, [W, Y_k]\big],$$

as $[W, X_k] \in [\mathfrak{g}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}$ and as $Y_k \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$. Moreover $[W, Y_k]$ and $\big[X_k, [W, Y_k]\big]$ belong to $\mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$ as $\mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$ is an ideal. If $0 \neq \big[X_k, [W, Y_k]\big] \notin \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$, we continue with this procedure. Because of nilpotency, there exist $W_1, W_2, \ldots, W_r \in \mathfrak{g}$ such that $0 \neq [W_1, [W_2, \ldots [W_r, Y_k] \ldots]] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{k-1})$ and

$$0 \neq \big[X_k, [W_1, [W_2, \ldots [W_r, Y_k] \ldots]]\big] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}).$$

We then write Y_k instead of $[W_1, [W_2, \ldots [W_r, Y_k] \ldots]]$ and Z_k instead of $\big[X_k, [W_1, [W_2, \ldots [W_r, Y_k] \ldots]]\big]$ and our claim is proven.

5.2

Hence, by (3.5), for each $k \in \mathcal{R} = \{i_1, \dots, i_d\}$, there exist Q_k, $Q_{0,k}$ and $Q_{1,k}$ such that

$$Q_k(t, l_{k-1}) = P_{Y_k}(i\mathrm{Ad}^*(\exp tX_k)(l_{k-1})) = Q_{0,k}(l_{k-1}) + Q_{1,k}(l_{k-1})t.$$

Here $Q_{0,k}$ and $Q_{1,k}$ are polynomials on $\mathfrak{g}^*_{k-1}$, but they may be considered as polynomials on all of $\mathfrak{g}^*$ that don't depend on the last $n-k+1$ coordinates of l. Moreover, $Q_{1,k}(l)$ is a G-invariant polynomial as

$$Q_{1,k}(l) = Q_{1,k}(l_k) = -P_{[X_k,Y_k]}(il_k)$$

and as $[X_k, Y_k] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$. As previously, they may be assumed to be purely imaginary and Y_k, Z_k may be assumed to be anti-hermitian.

Finally, if we define $R_k = iQ_{1,k}$, then $R_1, R_2, \dots, R_d$ are the G-invariant polynomials. The polynomial R_k depends only on the coordinates of $l_{k-1} = l|_{\mathfrak{g}_{k-1}}$.

5.3

Let's define

$$\mathcal{W} = \{l \in \mathcal{V} \mid R_k(l) \neq 0, \forall k\}$$

and, for every $\varepsilon \in \{-1, 1\}^d$,

$$\mathcal{A}_\varepsilon = \{l \in \mathcal{W} \mid \varepsilon_k R_k(l) > 0, \forall k\}.$$

Then $\mathcal{W}$ is a G-invariant, Zariski open dense subset of $\mathfrak{g}^*$ and $\mathcal{A}_\varepsilon$ is an open subset of $\mathfrak{g}^*$ such that

$$\mathcal{W} = \dot{\cup}_{\varepsilon \in \{-1,1\}^d} \mathcal{A}_\varepsilon.$$

Some of the $\mathcal{A}_\varepsilon$'s may be empty.

6 Function spaces

6.1

Let's assume that the Jordan-Hölder basis has been fixed. Let's take $\mathcal{T}$ as in (1.1) and $V_\mathcal{T} = \sum_{j \in \mathcal{T}} \mathbb{R}X_j^*$. Then $\mathcal{W}_\mathcal{T} := \mathcal{W} \cap V_\mathcal{T}$ and $\mathcal{A}_{\varepsilon,\mathcal{T}} := \mathcal{A}_\varepsilon \cap V_\mathcal{T}$ are orbit sections of the elements of $\mathcal{W}$, resp. of $\mathcal{A}_\varepsilon$. Let $\mathfrak{p}(l)$ be the Vergne polarization for the given basis, $P(l) = \exp(\mathfrak{p}(l))$ and $\{X_j \mid j \in \mathcal{R}\}$ for the complementary basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$.

6.2

Let $\mathcal{F}_\varepsilon$ be the set of all $\mathcal{C}^\infty$-functions

$$\begin{array}{rcl} \xi : \mathcal{A}_{\varepsilon,\mathcal{T}} \times \mathbb{R}^d & \to & \mathbb{C} \\ (l, x) & \mapsto & \xi(l,x) := \xi_l(x) \end{array}$$

such that for every compact set $K \subset \mathcal{A}_{\varepsilon,\mathcal{T}}$, for all $A, B, C \in \mathbb{N}$,

$$\|\xi\|_{K,A,B,C} = \sup_{l \in K; x \in \mathbb{R}^d} \big[\sup_{|\alpha| \leq A; |r| \leq B; |s| \leq C} \big|x^r \frac{\partial^\alpha}{\partial l^\alpha} \frac{\partial^s}{\partial x^s} \xi_l(x)\big|\big] < +\infty.$$

Here we used the notations

$$\begin{aligned}
x^r &= x_1^{r_1}\cdots x_d^{r_d} \quad \text{if } x=(x_1,\dots,x_d)\in\mathbb{R}^d \text{ and } r=(r_1,\dots,r_d)\in\mathbb{N}^d\\
\frac{\partial^\alpha}{\partial l^\alpha} &= \frac{\partial^{\alpha_1}}{\partial l_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial l_n^{\alpha_n}} \quad \text{if } l=l_1X_1^*+\cdots+l_nX_n^*,\ \alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n\\
\frac{\partial^s}{\partial x^s} &= \frac{\partial^{s_1}}{\partial x_1^{s_1}}\cdots\frac{\partial^{s_d}}{\partial x_d^{s_d}} \quad \text{if } x=(x_1,\dots,x_d)\in\mathbb{R}^d,\ s=(s_1,\dots,s_d)\in\mathbb{N}^d\\
|\alpha| &= \alpha_1+\cdots+\alpha_n\\
|r| &= r_1+\cdots+r_d\\
|s| &= s_1+\cdots+s_d
\end{aligned}$$

Thanks to the given basis we then identify ξ_l with a function on G/P_l by

$$\xi_l(x_1,\dots,\xi_d)\equiv\xi_l\big(\exp(x_1X_{i_1})\dots\exp(x_dX_{i_d})\big),$$

where $\mathcal{R}=\{i_1,\dots,i_d\}$, and with a function on G with the covariance relation

$$\xi_l(x\cdot u):=\overline{\chi_l(u)}\xi_l(x),\quad \forall u\in P_l.$$

The function ξ, considered as a function in l, may even be extended to the whole orbit by

$$\xi\big((\mathrm{Ad}^*g)(l),x\big):=\xi(l,x\cdot g).$$

This means that the function ξ_l is transformed into the function $\xi_{\mathrm{Ad}^*(g)(l)}$ by the intertwining operator giving the unitary equivalence of π_l and $\pi_{\mathrm{Ad}^*(G)(l)}$. Hence ξ may be considered as a function from $\mathcal{A}_\varepsilon\times G$ to $\mathbb{C}$. We shall write $\xi=(\xi_l)_l$.
If, for instance, we write $\mathcal{A}_\varepsilon=\cup_{m=1}^{+\infty}K_m$ as a countable union of compact sets such that $K_m\subset\mathrm{int}(K_{m+1})$ and if we consider the topology of $\mathcal{F}_\varepsilon$ generated by the semi-norms $\|\cdot\|_{K_m,A,B,C}$, then $\mathcal{F}_\varepsilon$ becomes a Fréchet space.

7 Main result on the Lie algebra level

7.1

We use the definitions and notations of sections 4 and 5. We then have the following theorem:

Theorem 7.2. *Let $G=\exp(\mathfrak{g})$ be a connected, simply connected, non-abelian, nilpotent Lie group. Let*

$$d=\max\{\ \dim(\mathfrak{g}/\mathfrak{p})\mid \exists l\in\mathfrak{g}^* \text{ s. t. } \mathfrak{p}=\mathfrak{p}(l) \text{ is a polarization for } l \text{ in } \mathfrak{g}\ \}.$$

Let

$$\{0\}=\mathfrak{g}_0\subset\mathfrak{g}_1\subset\cdots\subset\mathfrak{g}_n=\mathfrak{g}$$

be a fixed Jordan-Hölder sequence ($[\mathfrak{g},\mathfrak{g}_i]\subset\mathfrak{g}_{i-1}$, for all i) and $\{X_1,\dots,X_n\}$ a corresponding Jordan-Hölder basis (with $X_i\in\mathfrak{g}_i\setminus\mathfrak{g}_{i-1}$). For every $l\in\mathfrak{g}^*_{Puk}$, $\mathfrak{p}=\mathfrak{p}(l)$ will always denote the corresponding Vergne polarization for l in $\mathfrak{g}$. Let $\mathcal{V}=\{l\in\mathfrak{g}^*\mid l|_{\mathfrak{g}_k}\in(\mathfrak{g}_k)^*_{Puk},\forall k\}$ and $\mathcal{R}=\{i_1,\dots,i_d\}$ the set of local jump indices for the elements of $\mathcal{V}$. Then $\{X_{i_1},\dots,X_{i_d}\}$ is a coexponential basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$ for every $l\in\mathcal{V}$. Thanks to this basis, the spaces $\mathfrak{H}_{\pi_l}$ and $\mathfrak{H}^\infty_{\pi_l}$ are identified with $L^2(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ respectively. We have the following results:

(i) There exists a Zariski open, dense subset $\mathcal{W} \subset \mathcal{V}$ of $\mathfrak{g}^*_{Puk}$ and open subsets $\mathcal{A}_\varepsilon$ of $\mathfrak{g}^*$, for every $\varepsilon \in \{-1,1\}^d$, such that

$$\mathcal{W} = \dot{\cup}_{\varepsilon \in \{-1,1\}^d} \mathcal{A}_\varepsilon$$

(disjoint union). Some of the sets $\mathcal{A}_\varepsilon$ may be empty. Moreover, given $\mathcal{W}$ and the $\mathcal{A}_\varepsilon$'s, the following properties (ii) to (vi) are satisfied.
(ii) Given a fixed ε such that $\mathcal{A}_\varepsilon \neq \emptyset$, there exist $V_{1,\varepsilon}, \ldots, V_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ and there exists $\xi = (\xi_l)_l \in \mathcal{F}_\varepsilon$ satisfying $\xi_l \neq 0$, for all $l \in \mathcal{A}_\varepsilon$ and

$$d\pi_l(V_{j,\varepsilon})\xi_l = 0, \quad \forall j \in \{1, \ldots, d\}, \forall l \in \mathcal{A}_\varepsilon.$$

Moreover, there exists a fixed algorithm to construct $V_{1,\varepsilon}, \ldots, V_{d,\varepsilon}$, as well as the ξ_l's.
(iii) For every $k \in \{1, \ldots, d\}$, there exists $\zeta_k = (\zeta_{k,l})_l \in \mathcal{F}_\varepsilon$ such that

$$\begin{aligned} &\zeta_{k,l} \neq 0, \quad \forall l \in \mathcal{A}_\varepsilon, \forall k \in \{1, \ldots, d\} \\ &d\pi_l(V_{j,\varepsilon})\zeta_{k,l} = 0 \quad if\ j \neq k, \forall l \in \mathcal{A}_\varepsilon \\ &d\pi_l(V_{k,\varepsilon})\zeta_{k,l} \neq 0, \quad for\ at\ least\ one\ l \in \mathcal{A}_\varepsilon. \end{aligned}$$

(iv) The element $V_{d,\varepsilon}$ of the enveloping algebra is of the form

$$V_{d,\varepsilon} = X_{i_d} - i\varepsilon_d Y_{i_d}$$

where X_{i_d} is the last element of the basis of $\mathfrak{g}/\mathfrak{p}(l)$, $Y_{i_d} \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g}_{i_d - 1})$ and $Z_{i_d} = [X_{i_d}, Y_{i_d}] \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$.
(v) For every $l \in \mathcal{A}_\varepsilon$,

$$\bigcap_{j=1}^{d} \mathrm{Ker} d\pi_l(V_{j,\varepsilon}) = \mathbb{C}\xi_l$$

and, for $l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon$,

$$\bigcap_{j=1}^{d} \mathrm{Ker} d\pi_l(V_{j,\varepsilon}) = \{0\}.$$

(vi) The functions ξ_l, considered as elements of $\mathfrak{H}^\infty_{\pi_l}$, satisfy the following covariance and compatibility relations:

$$\xi_l(x \cdot u) = \overline{\chi_l(u)}\xi_l(x), \quad \forall x \in G, \forall u \in P_l$$

and

$$\xi_{Ad^*(u)(l)}(x) = \xi_l(x \cdot u), \quad \forall l \in \mathcal{W}, \forall u \in G.$$

This last relation means that the function ξ_l is transformed into the function $\xi_{Ad^*(u)(l)}$ by the intertwining operator giving the unitary equivalence of π_l and $\pi_{Ad^*(u)(l)}$.

Proof: It is a proof by induction and will be developed in the next sections. For $\varepsilon \in \{-1,1\}^d$ fixed, the elements $V_{1,\varepsilon}, \ldots, V_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ will be constructed by a fixed algorithm.

8 Proof of the Lie algebra result

8.1 The Heisenberg group

The first step in the induction consists in studying the situation of the Heisenberg group H_1. This has been done in details in ([Lu-Mo]). In this text we shall just recall the main results of this study. Let $< X, Y, Z >$ with $[X, Y] = Z$ be the generators of the Lie algebra $\mathfrak{h}_1$ of H_1. For any $\lambda \in \mathbb{R} \setminus \{0\}$, let $l_\lambda = \lambda Z^* \in \mathfrak{h}_1^*$. The orbit of l_λ is the horizontal plane through $(0, 0, \lambda)$

and a polarization for l_λ in $\mathfrak{h}_1^*$ is given by $\mathfrak{p}_\lambda = \mathbb{R}Y + \mathbb{R}Z$. Let $P_\lambda = \exp(\mathfrak{p}_\lambda)$. The orbit of l_λ is associated to the class of the infinite dimensional unitary irreducible representation $\pi_\lambda = \mathrm{ind}_{P_\lambda}^{H_1}\chi_{l_\lambda}$, where $\chi_{l_\lambda}\big(\exp(yY)\exp(zZ)\big) = e^{-i\lambda z}$, by the Kirillov mapping. Then $\mathcal{W}$ coincides with $\mathbb{R}X^* + \mathbb{R}Y^* + \mathbb{R}^*Z^*$, $d = 1$, $\varepsilon \in \{-1, 1\}$ and the polynomial R is given by $R(a, b, \lambda) = R(aX^* + bY^* + \lambda Z^*) = -\lambda$.
For $\varepsilon = 1$, $\mathcal{A}_\varepsilon = \mathcal{A}_1 = \{aX^* + bY^* + \lambda Z^* \mid \lambda < 0\}$ and $V_\varepsilon = V_1 = X - iY$. If $l = \lambda Z^*$, the function $\xi_l = \xi_\lambda \in \mathcal{S}(\mathbb{R}) \equiv \mathfrak{H}_\lambda^\infty$ is given by

$$\xi_\lambda(s) = e^{\frac{\lambda}{2}s^2}.$$

It is then defined on the rest of the orbit of l_λ by the formula

$$\xi_{Ad^*(u)(l_\lambda)}(s) = \xi_\lambda(\exp(sX) \cdot u), \quad \forall s \in \mathbb{R}, \forall u \in H_1,$$

using the covariance relation.
Similarly for $\varepsilon = -1$, where we have to take $\mathcal{A}_\varepsilon = \mathcal{A}_{-1} = \{aX^* + bY^* + \lambda Z^* \mid \lambda > 0\}$, $V_\varepsilon = V_{-1} = X + iY$ and

$$\xi_\lambda(s) = e^{-\frac{\lambda}{2}s^2}.$$

See [Lu-Mo]) for more details.

8.2 Induction in the case I situation

Let's first assume that n is a local jump index, i. e. that $n = i_d \in \mathcal{R}$. We are in the generalized Kirillov situation. Let's put $X = X_n = X_{i_d}$, $Y = Y_n = Y_{i_d}$, $Z = Z_n = Z_{i_d}$, $Q_0 = Q_{0,n} = Q_{0,i_d}$, $Q_1 = Q_{1,n} = Q_{1,i_d}$. The argument is divided into several steps:

(i) *Construction of particular elements in the enveloping algebra*
Let's use the notations of section (2), where $\tilde{\mathfrak{g}} = \mathfrak{g}_{n-1}$. Let now be $\varepsilon \in \{-1, 1\}^d$ such that $\mathcal{A}_\varepsilon \neq \emptyset$ and let's write

$$\tilde{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{d-1}) \text{ if } \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{d-1}, \varepsilon_d) \text{ for some } \varepsilon_d.$$

Then $\tilde{\varepsilon}$ corresponds to the restriction of $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$. By construction,

$$\emptyset \neq \tilde{\mathcal{A}}_{\tilde{\varepsilon}} = p(\mathcal{A}_\varepsilon) \subset \{\tilde{l} \in \tilde{\mathcal{W}} \mid \varepsilon_j R_j(\tilde{l}) > 0, 1 \leq j \leq d-1\ \} \neq \emptyset.$$

Let's assume that the results of theorem 7.2 are true for $\tilde{\mathfrak{g}}$. Let $\tilde{V}_{i,\tilde{\varepsilon}} \in \mathfrak{U}(\mathfrak{g})$, $i \in \{1, 2, \ldots, d-1\}$ be given by the induction hypothesis corresponding to $\tilde{\mathcal{A}}_{\tilde{\varepsilon}}$. Because all the actions are polynomial, we have a relation of the form

$$\mathrm{Ad}(\exp(tX))(\tilde{V}_{i,\tilde{\varepsilon}}) = \sum_{j=0}^{N_i}(-t)^j\tilde{V}_{ij}$$

with $\tilde{V}_{ij} \in \mathfrak{U}(\tilde{\mathfrak{g}})$ for all i, j. Let's now define

$$V_{i,\varepsilon} = \sum_{j=0}^{N_i}\tilde{V}_{ij}Y^jZ^{N_i-j} \in \mathfrak{U}(\tilde{\mathfrak{g}}), \quad i \in \{1, 2, \ldots, d-1\}.$$

Then

$$\mathrm{Ad}(\exp(-tX))(Y^jZ^{N_i-j}) = (Y - tZ)^jZ^{N_i-j}$$

and

$$\mathrm{Ad}\big(\exp(-tX)\big)(V_{i,\varepsilon}) = \sum_{j=0}^{N_i}\big(\mathrm{Ad}\big(\exp(-tX)\big)(\tilde{V}_{ij})\big)(Y - tZ)^jZ^{N_i-j}.$$

The element $V_{d,\varepsilon}$ will be constructed in (iii).

(ii) *Choice of a particular point on each orbit*
Let's now come back to the notations and results of (3.4) to (3.6). For every $\tilde{l} \in p(\mathcal{W}) \subset \tilde{\mathcal{W}}$, there exists exactly one $t = t(\tilde{l}) \in \mathbb{R}$ such that

$$d\pi_{\mathrm{Ad}^*(\exp(tX))(\tilde{l})}(Y) = 0,$$

with $\mathrm{Ad}^*(\exp(tX))(\tilde{l}) \in \tilde{\mathfrak{g}}^*$.
In fact,

$$d\pi_{\mathrm{Ad}^*(\exp(tX))(\tilde{l})}(Y) = \tilde{P}_Y\big(i\mathrm{Ad}^*(\exp(tX))(\tilde{l})\big)\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}}}} = [Q_0(\tilde{l}) + Q_1(\tilde{l})t]\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}}}}.$$

Hence

$$t = t(\tilde{l}) = -\frac{Q_0(\tilde{l})}{Q_1(\tilde{l})}$$

gives the desired result. In particular, let's notice that $Q_1(\tilde{l}) \neq 0$ by the definition of $\mathcal{W}$. Moreover, $t = t(\tilde{l})$ is $\tilde{G}$-invariant. By the same remark as previously, t may in fact be considered as a map defined on $\mathcal{W} \subset \mathfrak{g}^*$ that does not depend on the last coordinate. The map

$$\begin{array}{rcl} \mathcal{W} & \rightarrow & \mathbb{R} \\ l & \mapsto & t(l) = -\dfrac{Q_0(l)}{Q_1(l)} \end{array}$$

is $\mathcal{C}^\infty$, as it is a rational function defined on all of $\mathcal{W}$.
Let's now define the map

$$\begin{array}{rcl} \mathcal{W} & \rightarrow & \mathcal{W} \\ l & \mapsto & l_1 = \mathrm{Ad}^*\big(\exp(t(l)X)\big)(l). \end{array}$$

This map is of course $\mathcal{C}^\infty$. In fact, the coordinates of l_1 are rational functions of the coordinates of l, the denominators being powers of $Q_1(l)$. The image l_1 of l belongs to the orbit of l. Let's put $\tilde{l}_1 = l_1|_{\tilde{\mathfrak{g}}}$. Then, by construction,

$$\tilde{l}_1 = p(\mathrm{Ad}^*(\exp(tX))(l)) = \mathrm{Ad}^*(\exp(tX))(p(l)) = \mathrm{Ad}^*(\exp(tX))(\tilde{l})$$

and

$$d\pi_{\tilde{l}_1}(Y) = d\pi_{\mathrm{Ad}^*(\exp(tX))(\tilde{l})}(Y) = 0.$$

In particular,

$$0 = \tilde{P}_Y(i\tilde{l}_1) = Q(0, \tilde{l}_1) = Q_0(\tilde{l}_1)$$

and

$$d\tilde{\pi}_{l_1}(Y)\tilde{\xi}(t,\tilde{g}) = \tilde{P}_Y(i\mathrm{Ad}^*\big(\exp(tX)\big)(\tilde{l}_1))\tilde{\xi}(t,\cdot)(\tilde{g}) = Q_1(l_1)t \cdot \tilde{\xi}(t,\tilde{g}) \quad \forall t \in \mathbb{R}, \forall \tilde{g} \in \tilde{G},$$

as $Q_1(l_1) = Q_1(\tilde{l}_1)$ by definition. Moreover, $Q_1(l_1) = Q_1(l) \neq 0$, as Q_1 is G-invariant.

(iii) *Induction construction and proof for these particular choices*
For $\tilde{l}_1$, we have

$$\begin{aligned} d\pi_{\tilde{l}_1}\big(\mathrm{Ad}(\exp(-tX))(Y^j Z^{N_i-j})\big) &= d\pi_{\tilde{l}_1}\Big((Y - tZ)^j Z^{N_i-j}\Big) \\ &= (-t)^j d\pi_{\tilde{l}_1}(Z^{N_i}) \\ &= (-1)^{N_i} Q_1(l)^{N_i}(-t)^j \mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}_1}}} \\ &= C_i(-t)^j \mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}_1}}} \end{aligned}$$

where $C_i = (-1)^{N_i} Q_1(l)^{N_i}$, as $d\pi_{\tilde{l}_1}(Y) = 0$ and $d\pi_{\tilde{l}_1}(Z) = -Q_1(l_1)\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}_1}}} = -Q_1(l)\mathbf{1}_{\mathfrak{H}_{\pi_{\tilde{l}_1}}}$. Therefore

$$d\pi_{\tilde{l}_1}\big(\mathrm{Ad}(\exp(-tX))V_{i,\varepsilon}\big) = \sum_{j=0}^{N_i} C_i(-t)^j d\pi_{\tilde{l}_1}\big(\mathrm{Ad}(\exp(-tX))\tilde{V}_{ij}\big) = C_i d\pi_{\tilde{l}_1}(\tilde{V}_{i,\tilde{\varepsilon}}), \quad \forall t \in \mathbb{R}.$$

For any $\tilde{l} \in \tilde{\mathfrak{g}}^*$, let's note $\tilde{\mathfrak{p}}(\tilde{l})$ for the Vergne polarization, $\tilde{P}(\tilde{l}) = \exp(\tilde{\mathfrak{p}}(\tilde{l}))$ and $\xi_{\tilde{l}} \in \mathcal{S}(\mathbb{R}^{d-1}) \equiv \mathfrak{H}^\infty_{\pi_{\tilde{l}}}$ for the functions given by the induction hypothesis for point (ii) of theorem 7.2. By induction, the point (v) of theorem 7.2 gives,

$$\bigcap_{j=1}^{d-1} \mathrm{Ker} d\pi_{\tilde{l}}(\tilde{V}_{j,\tilde{\varepsilon}}) = \mathbb{C}\xi_{\tilde{l}}, \quad \forall \tilde{l} \in \tilde{\mathcal{A}}_{\tilde{\varepsilon}}$$

and

$$\bigcap_{j=1}^{d-1} \mathrm{Ker} d\pi_{\tilde{l}}(\tilde{V}_{j,\tilde{\varepsilon}}) = \{0\}, \quad \forall l \in \tilde{\mathcal{W}} \setminus \tilde{\mathcal{A}}_{\tilde{\varepsilon}}$$

where $p(\mathcal{W}) \subset \tilde{\mathcal{W}}$. For $l_1, \tilde{l}_1$ constructed here, let $\eta_{l_1} \in \mathcal{S}(\mathbb{R})$ (to be determined later) and let's define $\tilde{\xi}_{l_1} = \eta_{l_1} \otimes \xi_{\tilde{l}_1} \in \mathcal{S}(\mathbb{R}^d) \equiv \mathfrak{H}^\infty_{\tilde{\pi}_{l_1}}$. Then

$$d\tilde{\pi}_{l_1}(V_{i,\varepsilon})\tilde{\xi}_{l_1}(t,\tilde{g}) = \eta_{l_1}(t) \cdot d\pi_{\tilde{l}_1}(\mathrm{Ad}(\exp(-tX))V_{i,\varepsilon})\xi_{\tilde{l}_1}(\tilde{g}) = C_i\eta_{l_1}(t) d\pi_{\tilde{l}_1}(\tilde{V}_{i,\tilde{\varepsilon}})\xi_{\tilde{l}_1}(\tilde{g}) = 0$$

where $t \in \mathbb{R}$, $\tilde{g} \in \tilde{G}$, $i \in \{1, 2, \ldots, d-1\}$.
In case I the dimension of the orbits of maximal dimension increases by 2 and hence the dimension of $\mathfrak{g}/\mathfrak{p}$ increases by 1, when we go from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$, and therefore we have to introduce one supplementary relation. Let's consider $V_{d,\varepsilon} = X - i\varepsilon_d Y \in \mathfrak{U}(\mathfrak{g})$. Then

$$\begin{aligned} d\tilde{\pi}_{l_1}(X - i\varepsilon_d Y)\tilde{\xi}_{l_1}(t,\tilde{g}) &= \big(-\frac{\partial}{\partial t} - i\varepsilon_d Q_1(l_1)t\big)\tilde{\xi}_{l_1}(t,\cdot)(\tilde{g}) \\ &= \xi_{\tilde{l}_1}(\tilde{g})\big(-\frac{\partial}{\partial t} - \varepsilon_d R_d(l_1)t\big)\eta_{l_1}(t), \end{aligned}$$

as

$$d\tilde{\pi}_{l_1}(X)\tilde{\xi}(t,\tilde{g}) = \frac{d}{ds}\xi\big(\exp(-sX)\exp(tX)\cdot\tilde{g}\big)|_{s=0} = -\frac{\partial\tilde{\xi}}{\partial t}(t,\tilde{g}).$$

So we may take

$$\eta_{l_1}(t) = e^{-\frac{1}{2}\varepsilon_d R_d(l_1)t^2}$$

in order to get

$$d\tilde{\pi}_{l_1}(X - i\varepsilon_d Y)\tilde{\xi}_{l_1} = 0.$$

If we define $\xi_{l_1} \in \mathfrak{H}^\infty_{\pi_{l_1}}$ $(\equiv \mathcal{S}(\mathbb{R}^d))$ by $\xi_{l_1}(\exp(tX)\cdot\tilde{g}) = \tilde{\xi}_{l_1}(t,\tilde{g})$ for $t \in \mathbb{R}$ and $\tilde{g} \in \tilde{G}$, then

$$d\pi_{l_1}(V_{i,\varepsilon})\xi_{l_1} = 0 \quad \text{for } i \in \{1, \ldots, d\},$$

where $V_{d,\varepsilon} = X - i\varepsilon_d Y$.
We proceed similarly for the functions ζ_{k,l_1}, $k \in \{1, \ldots, d-1\}$ of theorem 7.2, point (iii). The functions $\zeta_{d,l_1} \in \mathfrak{H}^\infty_{\pi_{l_1}}$ and $\tilde{\zeta}_{d,l_1} \in \mathfrak{H}^\infty_{\tilde{\pi}_{l_1}}$ are obtained by

$$\tilde{\zeta}_{d,l_1} = \varphi_{l_1} \otimes \xi_{\tilde{l}_1},$$

where $\varphi_{l_1} \in \mathcal{S}(\mathbb{R})$ is any Schwartz function such that $\big(-\frac{\partial}{\partial t} - \varepsilon_d R_d(l_1)t\big)\varphi_{l_1}(t) \neq 0$.

In order to prove point (v) of theorem 7.2 for l_1 (associated to an arbitrarily chosen l, as explained previously), let's assume that $\psi_{l_1} \in \mathfrak{H}^\infty_{\pi_{l_1}} \equiv \mathcal{S}(\mathbb{R}^d)$ such that $\psi_{l_1} \in \cap_{j=1}^d \mathrm{Ker} d\pi_{l_1}(V_{j,\varepsilon})$. Let's define $\tilde{\psi}_{l_1}$ by $\tilde{\psi}_{l_1}(t,\tilde{g}) = \psi_{l_1}((\exp tX \cdot \tilde{g})$. One has in particular

$$\begin{aligned} d\pi_{l_1}(X - i\varepsilon_d Y)\psi_{l_1}((\exp tX)\cdot \tilde{g}) &= d\tilde{\pi}_{l_1}(X - i\varepsilon_d Y)\tilde{\psi}_{l_1}(t,\tilde{g}) \\ &= \Big(-\frac{\partial}{\partial t} - \varepsilon_d R_d(l_1)t\Big)\tilde{\psi}_{l_1}(t,\cdot)(\tilde{g}) \\ &= 0, \end{aligned}$$

i. e.

$$\frac{\partial}{\partial t}\tilde{\psi}_{l_1}(t,\tilde{g}) = -\varepsilon_d R_d(l_1)t\tilde{\psi}_{l_1}(t,\tilde{g}).$$

If $l \in \mathcal{A}_\varepsilon$, $l_1 \in \mathcal{A}_\varepsilon$ and $\varepsilon_d R_d(l_1) > 0$. So

$$\psi_{l_1}((\exp tX)\cdot\tilde{g}) = \tilde{\psi}_{l_1}(t,\tilde{g}) = C(l_1,\tilde{g})e^{-\frac{1}{2}\varepsilon_d R_d(l_1)t^2},$$

where $C(l_1,\tilde{g}) = C_{l_1}(\tilde{g})$ is a Schwartz function in $\tilde{g}$ that satisfies

$$d\pi_{\tilde{l}_1}(\tilde{V}_{j,\tilde{\varepsilon}})C_{l_1}(\tilde{g}) = 0, \quad j = 1,\ldots,d-1.$$

By the induction hypothesis, $C_{l_1}(\tilde{g}) = C(l_1)\xi_{\tilde{l}_1}(\tilde{g})$ and

$$\psi_{l_1} = C(l_1)\eta_{l_1}\otimes\xi_{\tilde{l}_1} = C(l_1)\xi_{l_1} \in \mathbb{C}\xi_{l_1}.$$

If $l \notin \mathcal{A}_\varepsilon$, $l_1 \notin \mathcal{A}_\varepsilon$ and one has $\varepsilon_d R_d(l_1) < 0$ or $\tilde{l}_1 = p(l_1) \notin \tilde{\mathcal{A}}_{\tilde{\varepsilon}}$.
In case $\varepsilon_d R_d(l_1) < 0$, the equation

$$\frac{\partial}{\partial t}\tilde{\psi}_{l_1}(t,\tilde{g}) = -\varepsilon_d R_d(l_1)t\tilde{\psi}_{l_1}(t,\tilde{g})$$

doesn't have a non-zero Schwartz solution and

$$\bigcap_{j=1}^d \mathrm{Ker} d\pi_{l_1}(V_{j,\varepsilon}) = \bigcap_{j=1}^d \mathrm{Ker} d\tilde{\pi}_{l_1}(V_{j,\varepsilon}) = \{0\}.$$

In case $\varepsilon_d R_d(l_1) > 0$ and $\tilde{l}_1 = p(l_1) \notin \tilde{\mathcal{A}}_{\tilde{\varepsilon}}$, $\cap_{j=1}^{d-1}\mathrm{Ker} d\pi_{\tilde{l}_1}(\tilde{V}_{j,\tilde{\varepsilon}}) = \{0\}$, by the induction hypothesis. Hence, if we start as in the case where $l \in \mathcal{A}_\varepsilon$, the system

$$d\pi_{\tilde{l}_1}(\tilde{V}_{j,\tilde{\varepsilon}})C_{l_1}(\tilde{g}) = 0, \quad j = 1,\ldots,d-1,$$

with $C_{l_1}(\tilde{g})$ Schwartz function, implies that $C_{l_1} = 0$ and so $\psi_{l_1} = 0$. This proves that if $l \notin \mathcal{A}_\varepsilon$, $\cap_{j=1}^d \mathrm{Ker} d\pi_{l_1}(V_{j,\varepsilon}) = \{0\}$.
Thus, for every $l \in \mathcal{W}$ there exists $l_1 \in \mathcal{O}_l$ for which theorem 7.2 is satisfied. It remains to complete the construction for all $l \in \mathcal{W}$.

(iv) *Construction for any linear form in a dense subset*
Let's now go back to the originally chosen $l \in \mathcal{W}$. By definition of l_1, $l = \mathrm{Ad}^*(\exp(-t(l)X)(l_1)$, where $t(l)$ is a rational function in l defined on all of $\mathcal{W}$. If we write $\mathfrak{p}(l)$ and $\mathfrak{p}(l_1)$ for the Vergne polarizations of l and l_1 respectively (for the given Jordan-Hölder sequence), then $\mathfrak{p}(l) = \mathrm{Ad}(\exp(-t(l)X))(\mathfrak{p}(l_1))$. The representations π_l and π_{l_1} are unitary equivalent, the equivalence between the representation spaces being given by

$$\begin{aligned} \mathfrak{H}_{\pi_{l_1}} &\to \mathfrak{H}_{\pi_l} \\ \varphi_{l_1} &\mapsto \varphi_l \end{aligned}$$

where

$$\varphi_l(g) = \varphi_{\mathrm{Ad}^*(\exp(-t(l)X))(l_1)}(g) = \varphi_{l_1}(g \cdot \exp(-t(l)X))$$

with $l_1 = \mathrm{Ad}^*(\exp(t(l)X)(l)$. This leads us to define ξ_l by the formula

$$\xi_l(g) := \xi_{\mathrm{Ad}^*(\exp(-t(l)X))(l_1)}(g) = \xi_{l_1}(g \cdot \exp(-t(l)X))$$

with $l_1 = \mathrm{Ad}^*(\exp(t(l)X)(l)$, where the function ξ_{l_1} is given by the construction of (iii). Hence the functions ξ_l have been constructed for every $l \in \mathcal{W}$. It is then easy to check that they really satisfy the relation

$$\xi_{\mathrm{Ad}^*(u)(l)}(g) = \xi_l(g \cdot u), \quad \forall u, g \in G.$$

Because of the unitary equivalence, we then have

$$\bigcap_{j=1}^{d} \mathrm{Ker} d\pi_l(V_{j,\varepsilon}) = \mathbb{C}\xi_l, \quad \forall l \in \mathcal{A}_\varepsilon,$$

and

$$\bigcap_{j=1}^{d} \mathrm{Ker} d\pi_l(V_{j,\varepsilon}) = \{0\}, \quad \forall l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon.$$

We proceed similarly for the functions $\zeta_{k,l}$.

(v) *Smoothness questions*

Let's note $\mathcal{F}_\varepsilon$ and $\tilde{\mathcal{F}}_{\tilde{\varepsilon}}$ for the function spaces to be considered at the level of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively (see section 6 for the definition). The mapping

$$\begin{array}{rcl} \mathcal{W} & \rightarrow & \mathbb{R} \\ l & \mapsto & t(l) \end{array}$$

is rational, defined on all of $\mathcal{W}$. Let's note that $t(l) = -\frac{Q_0(l)}{Q_1(l)}$ is real for all $l \in \mathcal{W}$, as $Q_0(l)$ and $Q_1(l)$ are both purely imaginary. So

$$\begin{array}{rcl} \mathcal{W} & \rightarrow & \mathcal{W} \\ l & \mapsto & l_1 = \mathrm{Ad}^*\Big(\exp(t(l)X)\Big)(l) \end{array}$$

and

$$\begin{array}{rcl} \mathcal{W} & \rightarrow & \tilde{\mathcal{W}} = p(\mathcal{W}) \\ l & \mapsto & \tilde{l}_1 = p(l_1) = l_1|_{\tilde{\mathfrak{g}}} \end{array}$$

are also rational.

By the induction hypothesis, $(\xi_{\tilde{l}})_{\tilde{l}} \in \tilde{\mathcal{F}}_{\tilde{\varepsilon}}$. Moreover,

$$\begin{array}{rcl} \mathcal{W} & \rightarrow & \mathcal{S}(\mathbb{R}) \\ l & \mapsto & \eta_l \end{array}$$

defined by $\eta_l(t) = e^{-\frac{1}{2}\varepsilon_d R_d(l) t^2}$ is $\mathcal{C}^\infty$ in (l, t) and Schwartz in t, constant (as a function of l) on G-orbits. So the construction of ξ_{l_1} given by

$$\xi_{l_1}(\exp(tX) \cdot \tilde{g}) = \eta_{l_1}(t) \cdot \xi_{\tilde{l}_1}(\tilde{g})$$

and the one of ξ_l defined by

$$\begin{aligned}\xi_l(\exp(sX)\cdot\tilde{g}) &= \xi_{\mathrm{Ad}^*(\exp(-t(l)X))(l_1)}(\exp(sX)\cdot\tilde{g})\\ &= \xi_{l_1}(\exp(sX)\cdot\tilde{g}\cdot\exp(-t(l)X))\\ &= \eta_{l_1}(s-t(l))\xi_{\tilde{l}_1}(\exp(t(l)X)\cdot\tilde{g}\cdot\exp(-t(l)X)),\end{aligned}$$

with $\eta_{l_1}=\eta_l$ by invariance on G-orbits, show that $(\xi_l)_l\in\mathcal{F}_\varepsilon$. Similarly for the functions $\zeta_{k,l}$ if we choose the φ_{l_1}'s such that $(l_1,t)\mapsto\varphi_{l_1}(t)$ is $\mathcal{C}^\infty$ in (l_1,t) and Schwartz in t (see the construction given in (iii)).

8.3 The case II situation

In this case the Vergne polarizations $\mathfrak{p}(l)$ (for $l\in\mathcal{W}$ in $\mathfrak{g}$) and $\tilde{\mathfrak{p}}(\tilde{l})$ (for $\tilde{l}=l|_{\tilde{\mathfrak{g}}}$ in $\tilde{\mathfrak{g}}$) satisfy $\tilde{\mathfrak{p}}(\tilde{l})=\mathfrak{p}(l)\cap\tilde{\mathfrak{g}}$. The complementary bases to $\mathfrak{p}(l)$ in $\mathfrak{g}$ and to $\tilde{\mathfrak{p}}(\tilde{l})$ in $\tilde{\mathfrak{g}}$ may be chosen to be equal (see section 4 for this choice). For this choice of the basis,

$$G/P(l)\equiv\tilde{G}/\tilde{P}(\tilde{l})\equiv\mathbb{R}^d.$$

The representation spaces $\mathfrak{H}_{\pi_l}$ and $\mathfrak{H}_{\pi_l}^\infty$ (resp. $\mathfrak{H}_{\pi_{\tilde{l}}}$ and $\mathfrak{H}_{\pi_{\tilde{l}}}^\infty$) may be identified with $L^2(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ (with the appropriate covariance conditions). For $\xi\in\mathcal{S}(\mathbb{R}^d)\equiv\mathfrak{H}_{\pi_l}^\infty\equiv\mathfrak{H}_{\pi_{\tilde{l}}}^\infty$ and for any $U\in\tilde{\mathfrak{g}}\subset\mathfrak{g}$:

$$\begin{aligned}d\pi_l(U)\xi(s_1,\ldots,s_d) &= d\pi_l(U)\xi(\exp(s_dX_{j_d})\cdots\exp(s_1X_{j_1}))\\ &= \frac{d}{dt}\xi(\exp(-tU)\exp(s_dX_{j_d})\cdots\exp(s_1X_{j_1}))|_{t=0}\\ &= d\pi_{\tilde{l}}(U)\xi(s_1,\ldots,s_d)\end{aligned}$$

In fact, as $U\in\tilde{\mathfrak{g}}\subset\mathfrak{g}$, $X_{j_1},\ldots,X_{j_d}\in\tilde{\mathfrak{g}}\subset\mathfrak{g}$, $\tilde{\mathfrak{p}}(\tilde{l})\subset\mathfrak{p}(l)$, we have in this case the same covariance relation wether we identify $\mathcal{S}(\mathbb{R}^d)$ with $\mathfrak{H}_{\pi_l}^\infty$ or with $\mathfrak{H}_{\pi_{\tilde{l}}}^\infty$. Similarly, for $U\in\mathfrak{U}(\tilde{\mathfrak{g}})\subset\mathfrak{U}(\mathfrak{g})$, $d\pi_l(U)\xi=d\pi_{\tilde{l}}(U)\xi$, for all $\xi\in\mathcal{S}(\mathbb{R}^d)$. As

$$\begin{aligned}d &= \max\{\ \dim(\mathfrak{g}/\mathfrak{p})\mid\exists l\in\mathfrak{g}^*\text{ s. t. }\mathfrak{p}=\mathfrak{p}(l)\ \text{ is a polarization for } l \text{ in } \mathfrak{g}\ \}\\ &= \max\{\ \dim(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})\mid\exists\tilde{l}\in\tilde{\mathfrak{g}}^*\text{ s. t. }\tilde{\mathfrak{p}}=\tilde{\mathfrak{p}}(\tilde{l})\ \text{ is a polarization for } \tilde{l} \text{ in } \tilde{\mathfrak{g}}\ \},\end{aligned}$$

the number of the relations of the form $d\pi(U)\xi=0$ in theorem 7.2 must be the same for G and $\tilde{G}$. Moreover, for any $\varepsilon\in\{-1,1\}^d$, the sets $\mathcal{A}_\varepsilon$, resp. $\tilde{\mathcal{A}}_\varepsilon$ are related by

$$\tilde{\mathcal{A}}_\varepsilon=p(\mathcal{A}_\varepsilon)=\{\ \tilde{l}\in\tilde{g}^*\mid\exists l\in\mathcal{A}_\varepsilon\text{ s. t. }\tilde{l}=l|_{\tilde{\mathfrak{g}}}\ \},$$

as n is not a jump index in this case. By the induction hypothesis, there exist $\tilde{V}_{1,\tilde{\varepsilon}},\ldots,\tilde{V}_{d,\tilde{\varepsilon}}\in\mathfrak{U}(\tilde{\mathfrak{g}})\subset\mathfrak{U}(\mathfrak{g})$ and a map $(\xi_{\tilde{l}})_{\tilde{l}}\in\tilde{\mathcal{F}}_{\tilde{\varepsilon}}$ such that

$$\begin{aligned}\xi_{\tilde{l}} &\neq 0\\ d\pi_{\tilde{l}}(\tilde{V}_{i,\tilde{\varepsilon}})\xi_{\tilde{l}} &= 0\quad\forall i\in\{1,\ldots,d\},\end{aligned}$$

for all $\tilde{l}\in\tilde{\mathcal{A}}_\varepsilon$. If we then put $V_{i,\varepsilon}:=\tilde{V}_{i,\tilde{\varepsilon}}$ for all i and define $\xi_l:=\xi_{\tilde{l}}$ with $\tilde{l}=p(l)$ for every $l\in\mathcal{A}_\varepsilon$, we get a map $(\xi_l)_l\in\mathcal{F}_\varepsilon$ such that

$$\begin{aligned}\xi_l &\neq 0\\ d\pi_l(V_{i,\varepsilon})\xi_l &= d\pi_{\tilde{l}}(\tilde{V}_{i,\tilde{\varepsilon}})\xi_{\tilde{l}}\ =\ 0\quad\forall i\in\{1,\ldots,d\},\end{aligned}$$

for all $l \in \mathcal{A}_\varepsilon$. The statement (v) of theorem 7.2 remains of course true. The compatibility condition

$$\xi_{\mathrm{Ad}^*(u)(l)}(g) = \xi_l(g \cdot u), \quad \forall u, g \in G$$

is again satisfied. Similarly for the functions ζ in the statement of theorem 7.2.

This completes the proof of theorem 7.2.

9 A Fourier inversion type theorem

9.1

Let's introduce the following notations: If $\mathcal{S}$ denotes the set of Pukanszky jump indices, let $\mathcal{T} = \{1, \ldots, n\} \setminus \mathcal{S}$ and $V_{\mathcal{T}} = \sum_{j \in \mathcal{T}} \mathbb{R} X_j^* \subset \mathfrak{g}^*$. Let $d\mu(l) = \big(\frac{1}{2\pi}\big)^{n-d} |Pf(l)| dl$ be the Plancherel measure, which lives on $V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}$. The Pfaffian $Pf(l)$ is defined by

$$|Pf(l)| = \big(P(l)\big)^{\frac{1}{2}} = \big(\det(< l, [X_i, X_j] >)_{i,j \in \mathcal{S}}\big)^{\frac{1}{2}}.$$

For all $\pi \in \hat{G}$, let $HS(\mathfrak{H}_\pi)$ denote the space of Hilbert-Schmidt operators on $\mathfrak{H}_\pi$ and let

$$\int_{\hat{G}}^{\oplus} HS(\mathfrak{H}_\pi) d\mu(\pi) = \big(\frac{1}{2\pi}\big)^{n-d} \int_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}}^{\oplus} HS(\mathfrak{H}_{\pi_l}) |Pf(l)| dl$$

stand for the space of all families of Hilbert-Schmidt operators $(A_{\pi_l})_l$ such that $A_{\pi_l} \in HS(\mathfrak{H}_{\pi_l})$ for all l and

$$\int_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}} \|A_{\pi_l}\|^2_{HS} \, |Pf(l)| dl < +\infty.$$

We then have the Plancherel theorem (see [Co-Gr]):

Theorem 9.2. *(i) Let G be a connected, simply connected, nilpotent Lie group, dx a fixed Haar measure on G, and $d\mu(l) = \big(\frac{1}{2\pi}\big)^{n-d} |Pf(l)| dl$ the corresponding Plancherel measure, where $n = \dim \mathfrak{g}$ and $2d$ is the maximal dimension of the coadjoint orbits. The map*

$$\begin{array}{rcl} A : L^1(G) \cap L^2(G) & \to & \displaystyle\int_{\hat{G}}^{\oplus} HS(\mathfrak{H}_\pi) d\mu(\pi) \\ f & \mapsto & \big(\pi_l(f)\big)_l \end{array}$$

extends to an isometry from all of $L^2(G)$ onto $\int_{\hat{G}}^{\oplus} HS(\mathfrak{H}_\pi) d\mu(\pi)$. The image of $f \in L^2(G)$ by this isometry will also be denoted by $\big(\pi_l(f)\big)_l$.
(ii) If $f, g \in L^2(G)$ such that $\pi_l(f)$ and $\pi_l(g)$ have as operator kernels the functions $F(l, \cdot, \cdot)$ and $G(l, \cdot, \cdot)$, then

$$\begin{aligned} < f, g > &= \big(\frac{1}{2\pi}\big)^{n-d} \int_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}} \mathrm{tr}\Big(\pi_l(f) \pi_l(g^*)\Big) |Pf(l)| dl \\ &= \big(\frac{1}{2\pi}\big)^{n-d} \int_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(l, x, y) \overline{G(l, x, y)} |Pf(l)| dx dy dl \end{aligned}$$

and

$$\begin{aligned}\|f\|_2^2 &= \left(\frac{1}{2\pi}\right)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \|\pi_l(f)\|^2_{HS}\ |Pf(l)|dl\\ &= \left(\frac{1}{2\pi}\right)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|F(l,x,y)|^2\ |Pf(l)|dxdydl.\end{aligned}$$

9.3

In order to state a Fourier inversion type theorem that will be used in this paper, let's first give the following definition:
Let $\mathcal{U}\subset\mathfrak{g}^*_{Puk}\cap V_{\mathcal{T}}$ be open. Let $\mathfrak{p}(l)$ be the Vergne polarization for l with respect to the given basis $\{X_1,\dots,X_n\}$. Let $\mathcal{R}=\{i_1,\dots,i_d\}$ be the set of local jump indices (the same for all $l\in\mathfrak{g}^*$) and $\{X_k \mid k\in\mathcal{R}\}$ the corresponding coexponential basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$. Then $G/P(l)$ may be identified with $\mathbb{R}^d$ by

$$X=(x_1,\dots,x_d)\equiv\exp(x_1X_{k_1})\cdots\exp(x_dX_{k_d}).$$

The space of kernels $\mathcal{N}_c(G,\mathcal{U})$ is defined to be the set of all $\mathcal{C}^\infty$-functions

$$F: V_{\mathcal{T}}\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$$

such that the following conditions are satisfied:
(i)

$$F(l,\cdot,\cdot)\equiv 0 \quad \text{if } l\notin\mathcal{U}$$

(ii)

$$\begin{aligned}\|F\|_{K,C,A_1,A_2,B_1,B_2} &= \sup_{l\in K;x,y\in\mathbb{R}^d}\left[\sup_{|a_1|\le A_1;|b_1|\le B_1;|c|\le C;|a_2|\le A_2;|b_2|\le B_2}\left|x^{a_1}y^{b_1}\frac{\partial^c}{\partial l^c}\frac{\partial^{a_2}}{\partial x^{a_2}}\frac{\partial^{b_2}}{\partial y^{b_2}}F(l,x,y)\right|\right]\\ &< +\infty\end{aligned}$$

(iii) There exists a compact set K in $\mathcal{U}$ such that $F(l,\cdot,\cdot)\equiv 0$ if $l\notin K$.

The kernel function $F(l,\cdot,\cdot)$ may be considered as a function on $G/P(l)\times G/P(l)$ by writing

$$F(l;\exp(x_1X_{k_1})\cdots\exp(x_dX_{k_d});\exp(y_1X_{k_1})\cdots\exp(y_dX_{k_d})) := F(l;x_1,\dots,x_d;y_1,\dots,y_d),$$

or even as a function on $G\times G$, if we introduce the covariance relation

$$F(l;x\cdot h;y\cdot h') := \overline{\chi_l(h)}\chi_l(h')F(l;x;y), \quad \forall x,y\in G;\forall h,h'\in P(l).$$

We may even extend F as a function in l to the whole orbit of l by setting

$$F((Ad^*g)(l);x;y) := F(l;x\cdot g;y\cdot g), \quad \forall x,y,g\in G.$$

This is done in order to reflect correctly the unitary equivalence between π_l and $\pi_{(Ad^*g)(l)}$, and its incidence on the corresponding operator kernels.

We then have the following "Fourier inversion type" theorem, which was established in ([Lu-Mo-Sc]):

Theorem 9.4. *Let G be a connected, simply connected, nilpotent Lie group with a fixed Jordan-Hölder basis $\{X_1, \dots, X_n\}$. Then there exists a Zariski open subset $\mathfrak{g}^*_{gen}$ of $\mathfrak{g}^*$ (set of generic elements in the sense of Ludwig-Zahir), contained in $\mathfrak{g}^*_{Puk}$, such that all the elements of $\mathfrak{g}^*_{gen}$ have the same set of local jump indices $\mathcal{R}$ and such that for every function $F \in \mathcal{N}_c(G, \mathfrak{g}^*_{gen})$ there exists a unique function $f \in \mathcal{S}(G)$ satisfying: For every $l \in \mathfrak{g}^*_{gen}$, the operator $\pi_l(f)$ has $F(l, \cdot, \cdot)$ as a kernel, provided the polarization associated to l is the Vergne polarization $\mathfrak{p}(l)$ with respect to the given basis and the coexponential basis to $\mathfrak{p}(l)$ in $\mathfrak{g}$ is given by $\{X_k \mid k \in \mathcal{R}\}$. Moreover, $\pi_l(f) = 0$ for every $l \in \mathfrak{g}^* \setminus \mathfrak{g}^*_{gen}$. The map $F \mapsto f$ is continuous in the given topologies.*

10 Main result on the Lie group level

10.1

We start with the following definition:

Definition: Let $G = \exp(\mathfrak{g})$ be a connected, simply connected, non-abelian, nilpotent Lie group.
(i) For every $U \in \mathfrak{g}$ and every $\varphi \in \mathcal{S}(G)$, we define

$$\varphi * U(g) := \frac{d}{ds}\varphi\big(g \cdot \exp(sU)\big)|_{s=0}.$$

We then extend this definition to all $U \in \mathfrak{U}(\mathfrak{g})$.
(ii) Let $U_1, \dots, U_r \in \mathfrak{U}(\mathfrak{g})$. We say that the U_i's are *Schwartz independent* if, for every $k \in \{1, \dots, r\}$, there exists $\varphi_k \in \mathcal{S}(G)$ such that

$$\begin{aligned} \varphi_k * U_k &\neq 0 \\ \varphi_k * U_j &= 0 \;\text{ if } j \neq k \end{aligned}$$

This means that $U_1, \dots, U_r$ are independent if considered as operators acting from the right on the Schwartz space $\mathcal{S}(G)$.

We then have the following theorem:

Theorem 10.2. *Let $G = \exp(\mathfrak{g})$ be a connected, simply connected, non-abelian, nilpotent Lie group, equipped with a fixed Jordan-Hölder basis. Let*

$$d := \max\{\dim(\mathfrak{g}/\mathfrak{p}) \mid \exists l \in \mathfrak{g}^* \text{ s. t. } \mathfrak{p} = \mathfrak{p}(l) \text{ is a polarization for } l \text{ in } \mathfrak{g}\}.$$

We then have the following results:
There exists a Zariski open, dense subset $\mathcal{W}$ of $\mathfrak{g}^*_{Puk}$ and open subsets $\mathcal{A}_\varepsilon$ of $\mathfrak{g}^*$, for every $\varepsilon \in \{-1, 1\}^d$, such that

$$\mathcal{W} = \dot{\cup}_{\varepsilon \in \{-1,1\}^d} \mathcal{A}_\varepsilon$$

(disjoint union). Some of the sets $\mathcal{A}_\varepsilon$ may be empty.
Given a fixed $\varepsilon \in \{-1, 1\}^d$ such that $\mathcal{A}_\varepsilon \neq \emptyset$, there exist $U_{1,\varepsilon}, \dots, U_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$, given by a fixed algorithm, and $(\xi_l)_l \in \mathcal{F}_\varepsilon$ satisfying $\xi_l \neq 0$ for all $l \in \mathcal{A}_\varepsilon$ such that the following properties are satisfied:
(i) $U_{1,\varepsilon}, \dots, U_{d,\varepsilon}$ are Schwartz independent.
(ii) For every $l_0 \in \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}$, there exists $0 \neq \varphi \in \mathcal{S}(G)$ such that $\pi_{l_0}(\varphi) \neq 0$ and $\varphi * U_{k,\varepsilon} = 0$ for all $k \in \{1, \dots, d\}$.

(iii) If $0 \neq \varphi \in \mathcal{S}(G)$ is any solution of $\varphi * U_{k,\varepsilon} = 0$, $1 \leq k \leq d$, then $\pi_l(\varphi) = 0$ if $l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon$ and $\mathfrak{Im}\pi_l(\varphi^*) \subset \mathbb{C}\xi_l$ if $l \in \mathcal{A}_\varepsilon$. In that case there exists $\eta_l \in L^2(\mathbb{R}^d) \equiv \mathfrak{H}_{\pi_l}$ such that

$$\pi_l(\varphi^*) = P_{\xi_l,\eta_l}$$

where P_{ξ_l,η_l} is a rank one operator defined by

$$P_{\xi_l,\eta_l}(\xi) =< \xi, \eta_l > \xi_l, \quad \forall \xi \in L^2(\mathbb{R}^d) \equiv \mathfrak{H}_{\pi_l}.$$

(iv) The set $\{U_{1,\varepsilon}, \dots, U_{d,\varepsilon}\}$ is maximal among all sets $\{V_1, \dots, V_r\}$ of elements of $\mathfrak{U}(\mathfrak{g})$ which are Schwartz independent and which satisfy

$$\forall l \in \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}, \exists\ 0 \neq \varphi \in \mathcal{S}(G) \text{ such that } \pi_l(\varphi) \neq 0 \text{ and } \varphi * V_j = 0 \text{ for } 1 \leq j \leq r.$$

Proof: Let $V_{1,\varepsilon}, \dots, V_{d,\varepsilon}$ be given as in theorem 7.2 and let's take $U_{k,\varepsilon} = V^*_{k,\varepsilon}$ for all $k \in \{1, \dots, d\}$.
(ii) For the given $l_0 \in \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}$, let $\gamma = (\gamma_l)_l \in \mathcal{F}_\varepsilon$ with $\gamma_l \equiv 0$ if $l \in \mathcal{A}_{\varepsilon,\mathcal{T}}$ outside a compact subset K of $\mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen}$ and such that $\gamma_{\tilde{l}_0} \neq 0$, where $\{\tilde{l}_0\} = \mathcal{O}_{l_0} \cap V_{\mathcal{T}}$, $\mathcal{O}_{l_0}$ being the orbit of l_0. Let's consider $F(l, \cdot, \cdot) = F_l = \xi_l \otimes \overline{\gamma}_l$ for $l \in \mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen}$ and $F(l, \cdot, \cdot) = 0$ for $l \in (V_{\mathcal{T}} \cap \mathfrak{g}^*_{gen}) \setminus (\mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen})$. Then $F \in \mathcal{N}_c(G, \mathfrak{g}^*_{gen})$ and there exists, by theorem 9.4, $\varphi \in \mathcal{S}(G)$ such that $\pi_l(\varphi^*)$ has $F(l, \cdot, \cdot)$ as a kernel for all $l \in \mathfrak{g}^*_{gen}$. Moreover, $\pi_l(\varphi^*) = 0$ if $l \in \mathfrak{g}^* \setminus \mathfrak{g}^*_{gen}$. By construction, $\pi_l(\varphi^*) = P_{\xi_l,\gamma_l}$ if $l \in \mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen}$ and $\pi_l(\varphi^*) = \pi_l(\varphi) = 0$ if $l \notin \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}$. In particular, as $\gamma_{\tilde{l}_0} \neq 0$, $\pi_{\tilde{l}_0}(\varphi) \neq 0$ and $\pi_{l_0}(\varphi) \neq 0$ because $\pi_{\tilde{l}_0}$ and π_{l_0} are unitary equivalent. Moreover

$$\pi_l(\varphi * U_{j,\varepsilon})^* = d\pi_l(U^*_{j,\varepsilon})\pi_l(\varphi^*) = d\pi_l(V_{j,\varepsilon})\pi_l(\varphi^*) = 0, \quad \forall l \in \mathfrak{g}^*.$$

In fact, if $l \in \mathfrak{g}^* \setminus (\mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen})$, this is true because $\pi_l(\varphi^*) = 0$. If $l \in \mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen}$,

$$\pi_l(\varphi * U_{j,\varepsilon})^* = d\pi_l(U^*_{j,\varepsilon})\pi_l(\varphi^*) = d\pi_l(V_{j,\varepsilon})P_{\xi_l,\gamma_l} = \Big(d\pi_l(V_{j,\varepsilon})\xi_l\Big) < \cdot, \gamma_l >= 0,$$

by the construction of $V_{j,\varepsilon}$ and ξ_l. Hence $\varphi * U_{j,\varepsilon} = 0$ for all $j \in \{1, \dots, d\}$.
(i) We proceed similarly to prove the Schwartz independence. In fact, for any $k \in \{1, \dots, d\}$, let's replace ξ_l by $\zeta_{k,l}$ of theorem 7.2 to define the kernel $F_k(l, \cdot, \cdot)$ and the function $\varphi_k \in \mathcal{S}(G)$. By the same computation as previously, $\pi_l(\varphi_k * U_{j,\varepsilon}) = 0$ for all l and $\varphi_k * U_{j,\varepsilon} = 0$, if $j \neq k$. For $j = k$, $\gamma_{\tilde{l}_0} \neq 0$ and $d\pi_{\tilde{l}_0}(V_{k,\varepsilon})\zeta_{\tilde{l}_0,k} \neq 0$. So

$$\pi_{\tilde{l}_0}(\varphi_k * U_{k,\varepsilon})^* = \Big(d\pi_{\tilde{l}_0}(V_{k,\varepsilon})\zeta_{\tilde{l}_0,k}\Big) < \cdot, \gamma_{\tilde{l}_0} >\neq 0.$$

Hence $\varphi_k * U_{k,\varepsilon} \neq 0$.
(iii) Let now $0 \neq \varphi \in \mathcal{S}(G)$ be any solution of $\varphi * U_{k,\varepsilon} = 0$, $1 \leq k \leq d$. Then

$$d\pi_l(U^*_{j,\varepsilon})\pi_l(\varphi^*) = d\pi_l(V_{j,\varepsilon})\pi_l(\varphi^*) = 0, \quad 1 \leq j \leq d,$$

for all $l \in \mathfrak{g}^*$ and

$$\mathfrak{Im}\pi_l(\varphi^*) \cap \mathcal{S}(\mathbb{R}^d) = \{0\} \quad \text{if } l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon$$

$$\mathfrak{Im}\pi_l(\varphi^*) \cap \mathcal{S}(\mathbb{R}^d) \subset \bigcap_{j=1}^{d} \mathrm{Ker} d\pi_l(V_{j,\varepsilon}) = \mathbb{C}\xi_l \quad \text{if } l \in \mathcal{A}_\varepsilon$$

by theorem 7.2.

Let's assume that $\mathfrak{Im}\pi_l(\varphi^*) = \mathbb{C}\xi_l$. For every $\psi \in \mathcal{S}(\mathbb{R}^d)$, there exists $C(l,\psi) \in \mathbb{C}$ such that

$$\pi_l(\varphi^*)\psi = C(l,\psi)\xi_l.$$

The map

$$\begin{array}{rcl} \mathcal{S}(\mathbb{R}^d) \equiv \mathfrak{H}_{\pi_l}^\infty & \to & \mathbb{C} \\ \psi & \mapsto & C(l,\psi) \end{array}$$

is linear and, for $l \in \mathcal{A}_\varepsilon$ fixed,

$$|C(l,\psi)| = \frac{\|\pi_l(\varphi^*)\psi\|_2}{\|\xi_l\|_2} \leq \left(\frac{\|\pi_l(\varphi^*)\|_{op}}{\|\xi_l\|_2}\right)\|\psi\|_2.$$

This proves that the linear map $\psi \mapsto C(l,\psi)$ may be extended to a bounded linear map from $L^2(\mathbb{R}^d)$ to $\mathbb{C}$, and hence that there exists $\eta_l \in L^2(\mathbb{R}^d) \equiv \mathfrak{H}_{\pi_l}$ such that $C(l,\psi) =< \psi, \eta_l >$ for every $\psi \in L^2(\mathbb{R}^d) \equiv \mathfrak{H}_{\pi_l}$. In particular,

$$\pi_l(\varphi^*)\psi =< \psi, \eta_l > \xi_l = P_{\xi_l,\eta_l}\psi$$

and $\pi_l(\varphi^*) = P_{\xi_l,\eta_l}$.
(iv) To prove the maximality of $\{U_{1,\varepsilon}, \ldots, U_{d,\varepsilon}\}$, let's now assume that there exists $U \in \mathfrak{U}(\mathfrak{g})$ such that for every $l_0 \in \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}$, there exists $\varphi \in \mathcal{S}(G)$ satisfying $\pi_{l_0}(\varphi) \neq 0$ and

$$\begin{array}{rcl} \varphi * U_{1,\varepsilon} & = & 0 \\ & \cdots & \\ \varphi * U_{d,\varepsilon} & = & 0 \\ \varphi * U & = & 0 \end{array}$$

Hence $\pi_{l_0}(\varphi^*) = P_{\xi_{l_0},\eta_{l_0}}$ with $\eta_{l_0} \neq 0$, by the previous arguments. But then

$$0 = \pi_{l_0}(\varphi * U)^* = d\pi_{l_0}(U^*)P_{\xi_{l_o},\eta_{l_0}} =< \cdot, \eta_{l_0} > d\pi_{l_0}(U^*)\xi_{l_0}.$$

As $\eta_{l_0} \neq 0$, $d\pi_{l_0}(U^*)\xi_{l_0} = 0$. This may be done for all $l_0 \in \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}$, i. e. $d\pi_{l_0}(U^*)\xi_{l_0} = 0$ for every $l_0 \in \mathcal{A}_\varepsilon \cap \mathfrak{g}^*_{gen}$. We shall now show that in this case $U_{1,\varepsilon}, \ldots, U_{d,\varepsilon}, U$ cannot be Schwartz independent. In fact, let $0 \neq \zeta \in \mathcal{S}(G)$ be such that $\zeta * U_{k,\varepsilon} = 0$, for $k = 1, \ldots, d$. Then $\pi_l(\zeta) = 0$ if $l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon$ and $\pi_l(\zeta^*) = P_{\xi_l,\eta_l}$ for some $\eta_l \in L^2(\mathbb{R}^d)$ if $l \in \mathcal{A}_\varepsilon$. Finally, if $l \in \mathcal{A}_\varepsilon$,

$$\big(\pi_l(\zeta * U)\big)^* = d\pi_l(U^*)P_{\xi_l,\eta_l} =< \cdot, \eta_l > d\pi_l(U^*)\xi_l = 0.$$

This implies that $\pi_l(\zeta * U) = 0$ for all $l \in \mathcal{W}$ and $\zeta * U = 0$, as $\mathcal{W}$ is dense in $\mathfrak{g}^*$. So $U_{1,\varepsilon}, \ldots, U_{d,\varepsilon}, U$ cannot be Schwartz independent. Hence $\{U_{1,\varepsilon}, \ldots, U_{d,\varepsilon}\}$ is maximal with the required properties. □

11 Weak solutions

11.1

Let's first give the following definition.

Definition: For any fixed $l \in \mathcal{A}_\varepsilon$ and any $V_1, \ldots, V_r \in \mathfrak{U}(\mathfrak{g})$, we say that $\eta \in L^2(\mathbb{R}^d)$ is a weak solution of

$$d\pi_l(V_j)\xi = 0, \quad j \in \{1, \ldots, r\}$$

if

$$< \eta, d\pi_l(V_j^*)\varphi >= 0, \quad \forall j \in \{1, \ldots, r\}, \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The weak solutions of the system of equations of theorem 7.2 are then characterized by the following theorem:

Theorem 11.2. *Let $G = \exp(\mathfrak{g})$ be a connected, simply connected nilpotent Lie group. Let's use the notations d, $\mathcal{W}$, $\mathcal{A}_\varepsilon$ as in theorem 7.2. For a fixed ε, let $V_{1,\varepsilon}, \ldots, V_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ and ξ_l be as in theorem 7.2. Then the weak solutions of the systems*

$$\begin{cases} d\pi_l(V_{1,\varepsilon})\xi = 0 \\ \quad \ldots \\ d\pi_l(V_{d,\varepsilon})\xi = 0 \end{cases}$$

coincide with the functions on $\mathbb{R}^d$ given by

$$\alpha_l = C(l)\xi_l \quad \text{almost everywhere if } l \in \mathcal{A}_\varepsilon,$$

where $C(l)$ is a constant depending on l, and

$$\alpha_l = 0 \quad \text{almost everywhere if } l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon,$$

i.e. the weak solutions coincide with the strong solutions almost everywhere.

Proof: The functions ξ_l are of course weak solutions. The converse is proven by induction. For the Heisenberg group $G = H_1$, see ([Lu-Mo]).
If $n = j_d$ is a jump index, then $V_{d,\varepsilon} = X_{j_d} - i\varepsilon_d Y_{j_d}$. For any $l \in \mathcal{W}$, let's take $l_1 \in \mathcal{O}_l$ as in the proof of theorem 7.2. We shall first make the proof for this l_1. Let α be a weak solution and let $\varphi \in \mathcal{S}(\mathbb{R})$, $\psi \in \mathcal{S}(\mathbb{R}^{d-1})$ be arbitrary. By assumption and as Y_{j_d} is anti-hermitian,

$$< \alpha, d\pi_{l_1}(X_{j_d} - i\varepsilon_d Y_{j_d})^* \varphi \otimes \psi >=< \alpha, d\pi_{l_1}(-X_{j_d} - i\varepsilon_d Y_{j_d})\varphi \otimes \psi >= 0,$$

i.e.

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \alpha(s_1, \ldots, s_d) \overline{\left[\frac{\partial}{\partial s_d}\varphi(s_d) - \varepsilon_d R_d(l_1) s_d \varphi(s_d)\right]} \overline{\psi(s_1, \ldots, s_{d-1})} ds_1 \ldots ds_{d-1} ds_d = 0.$$

As $\psi \in \mathcal{S}(\mathbb{R}^{d-1})$ is arbitrary,

$$\int_{\mathbb{R}} \alpha(s_1, \ldots, s_{d-1}, s_d) \overline{\left[\frac{\partial}{\partial s_d}\varphi(s_d) - \varepsilon_d R_d(l_1) s_d \varphi(s_d)\right]} ds_d = 0$$

for almost all $s_1, \ldots, s_{d-1}$. Then, by the result of the Heisenberg case ([Lu-Mo]),

$$\alpha(s_1, \ldots, s_{d-1}, s_d) = C(l_1, s_1, \ldots, s_{d-1}) e^{-\frac{1}{2}\varepsilon_d R_d(l_1) s_d^2}$$

almost everywhere if $\varepsilon_d R_d(l_1) > 0$. Here, for l_1 fixed but arbitrary,

$$C_{l_1}(s_1, \ldots, s_{d-1}) = C(l_1, s_1, \ldots, s_{d-1}) \in L^2(\mathbb{R}^{d-1}).$$

Then, for every $j \in \{1, \dots, d\}$ and every $\varphi \in \mathcal{S}(\mathbb{R})$, $\psi \in \mathcal{S}(\mathbb{R}^{d-1})$, for $\tilde{V}_{j,\varepsilon}, C_j$ as in the proof of theorem 7.2,

$$\begin{aligned}
&< \alpha, d\pi_{l_1}(V^*_{j,\varepsilon})\varphi \otimes \psi >= 0 \\
&< C_{l_1}(s_1, \dots, s_{d-1}) e^{-\frac{1}{2}\varepsilon_d R_d(l_1) s_d^2}, d\tilde{\pi}_{l_1}(V^*_{j,\varepsilon})[\varphi(s_d)\psi(s_1, \dots, s_{d-1})] >= 0 \\
&\overline{C}_j < e^{-\frac{1}{2}\varepsilon_d R_d(l_1) s_d^2}, \varphi(s_d) >_{\mathbb{R}} < C_{l_1}(s_1, \dots, s_{d-1}), d\pi_{\tilde{l}_1}(\tilde{V}^*_{j,\varepsilon})\psi(s_1, \dots, s_{d-1}) >_{\mathbb{R}^{d-1}} = 0 \\
&< C_{l_1}(s_1, \dots, s_{d-1}), d\pi_{\tilde{l}_1}(\tilde{V}^*_{j,\varepsilon})\psi(s_1, \dots, s_{d-1}) >_{\mathbb{R}^{d-1}} = 0
\end{aligned}$$

as φ is arbitrary. Hence $C_{l_1}(s_1, \dots, s_{d-1})$ is a weak solution in $L^2(\mathbb{R}^{d-1})$ of

$$d\pi_{\tilde{l}_1}(\tilde{V}_{j,\varepsilon})\xi = 0, \quad j = 1, \dots, d-1.$$

By induction, $C_{l_1}(s_1, \dots, s_{d-1}) = C(l_1)\xi_{\tilde{l}_1}(s_1, \dots, s_{d-1})$ almost everywhere and $\alpha = C(l_1)\eta_{l_1} \otimes \xi_{\tilde{l}_1} = C(l_1)\tilde{\xi}_{l_1} \equiv C(l_1)\xi_{l_1}$ almost everywhere (with the notations of theorem 7.2). As a matter of fact, the previous reasoning is justified by the following: As $X \in \mathfrak{g}$, $X^* = -X$, $(\mathrm{ad}(X)V)^* = \mathrm{ad}(X)(V^*)$ and $[\mathrm{Ad}(\exp(-tX))V]^* = \mathrm{Ad}(\exp(-tX))(V^*)$. Hence

$$\begin{aligned}
d\tilde{\pi}_{l_1}(V^*_{j,\varepsilon})\tilde{\xi}(t, \tilde{g}) &= d\pi_{\tilde{l}_1}\Big(\mathrm{Ad}(\exp(-tX))V^*_{j,\varepsilon}\Big)\tilde{\xi}(t, \cdot)(\tilde{g}) \\
&= \Big[d\pi_{\tilde{l}_1}\Big(\mathrm{Ad}(\exp(-tX))V_{j,\varepsilon}\Big)\Big]^* \tilde{\xi}(t, \cdot)(\tilde{g}) \\
&= \Big[C_j d\pi_{\tilde{l}_1}(\tilde{V}_{j,\varepsilon})\Big]^* \tilde{\xi}(t, \cdot)(\tilde{g}) \\
&= \overline{C}_j d\pi_{\tilde{l}_1}(\tilde{V}^*_{j,\varepsilon})\tilde{\xi}(t, \cdot)(\tilde{g})
\end{aligned}$$

If $l \in \mathcal{W} \setminus \mathcal{A}_\varepsilon$, $l_1 \in \mathcal{W} \setminus \mathcal{A}_\varepsilon$, either $\varepsilon_d R_d(l_1) < 0$ and $\alpha(s_1, \dots, s_{d-1}, s_d) = 0$, or $\varepsilon_d R_d(l_1) > 0$ and $p(l_1) = \tilde{l}_1 \notin \tilde{\mathcal{A}}_\varepsilon$. By induction this implies $C_{l_1}(s_1, \dots, s_{d-1}) = 0$. In all cases, $\alpha = 0$.
In order to show the result for the original l, let's recall that $l_1 \in \mathcal{O}_l$, that π_{l_1} and π_l are unitary equivalent and that if ξ_{l_1} and ξ_l respectively are the solutions given by theorem 7.2, then these solutions transform one into the other by this unitary equivalence. As moreover this unitary equivalence respects the scalar product, the result on the weak solutions remains correct for the original l.
In case II, the representation spaces and the representations don't change when we go from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$. Nothing has to be proven in the induction step. □

11.3

We may now study the weak solutions of the main system of differential equations considered in this paper. Let's use the following definition:

Definition: For any fixed $l \in \mathcal{A}_\varepsilon$ and any $U_1, \dots, U_r \in \mathfrak{U}(\mathfrak{g})$, we say that $f \in L^2(\mathbb{R}^d)$ is a weak solution of the system $f * U_j = 0$, $j = 1, \dots, r$, if, for every $\varphi \in \mathcal{S}(G)$,

$$< f, \varphi * U_j^* >= 0, \quad j = 1, \dots, r.$$

11.4

Let now $V_{1,\varepsilon},\ldots,V_{d,\varepsilon}$ be as in theorem 7.2 and let's put $U_{k,\varepsilon}=V^*_{k,\varepsilon}$ $(k=1,\ldots,d)$. By the Plancherel theorem, we have for every weak solution f of $f*U_{k,\varepsilon}=0$ $(k\in\{1,\ldots,d\})$ and every $\varphi\in\mathcal{S}(G)$, for every $k\in\{1,\ldots,d\}$,

$$\begin{aligned} 0 &= <f,\varphi*U^*_{k,\varepsilon}> \\ &= (\frac{1}{2\pi})^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \operatorname{tr}\Big(\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi^*)\Big)|Pf(l)|dl. \end{aligned}$$

Let's now choose a sequence of functions $f_n\in\mathcal{S}(G)$ which converge to f in $L^2(G)$ and, for $k\in\{1,\ldots,d\}$ fixed but arbitrary, let's replace φ by $f_n*U_{k,\varepsilon}*\varphi*\varphi^*$. Hence

$$\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \operatorname{tr}\Big(\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)\Big[\pi_l(f_n)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)\Big]^*\Big)|Pf(l)|dl=0.$$

But (f_n) converges to f in $L^2(G)$. Hence

$$\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \operatorname{tr}\Big(\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)\Big[\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)\Big]^*\Big)|Pf(l)|dl=0$$

and

$$\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)=0 \quad \text{for almost all } l\in V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}.$$

This may be done for every $k\in\{1,\ldots,d\}$. Let's now choose the functions φ in a countable dense subset $\mathcal{C}$ of $\mathcal{S}(G)$. For every $\varphi\in\mathcal{C}$, there exists a set of measure zero $N_\varphi\subset V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}$ such that

$$\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)=0, \quad \forall\varphi\in\mathcal{C},\forall l\in(V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})\setminus N_\varphi,\forall k\in\{1,\ldots,d\}.$$

The set $N=\cup_{\varphi\in\mathcal{C}}N_\varphi$ has again measure zero and

$$\pi_l(f)d\pi_l(U_{k,\varepsilon})\pi_l(\varphi)\eta=0, \quad \forall l\in(V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})\setminus N,\forall\varphi\in\mathcal{C},\forall\eta\in\mathcal{S}(G)\equiv\mathfrak{H}^\infty_{\pi_l},\forall k\in\{1,\ldots,d\}.$$

As $\pi_l(\mathcal{C})\mathfrak{H}^\infty_{\pi_l}$ is dense in $\mathfrak{H}^\infty_{\pi_l}$, $\pi_l(f)=0$ on $\mathfrak{Im}(d\pi_l(U_{k,\varepsilon}))$ for all $k\in\{1,\ldots,d\}$ and, for all $\varphi,\psi\in\mathfrak{H}^\infty_{\pi_l}$,

$$<d\pi_l(U_{k,\varepsilon})\varphi,\pi_l(f^*)\psi>=<\pi_l(f)d\pi_l(U_{k,\varepsilon})\varphi,\psi>=0, \quad \forall k\in\{1,\ldots,d\},$$

i.e. for all $\psi\in\mathfrak{H}^\infty_{\pi_l}$ and for all $l\in(V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})\setminus N$, $\pi_l(f^*)\psi$ is a weak solution of the system of equations $d\pi_l(U^*_{k,\varepsilon})\xi=d\pi_l(V_{k,\varepsilon})\xi=0$ $(k\in\{1,\ldots,d\})$. By theorem 11.2 and by the density of $\mathfrak{H}^\infty_{\pi_l}$ in $\mathfrak{H}_{\pi_l}$, we may then conclude that

$$\pi_l(f^*)\mathfrak{H}_{\pi_l}\subset\mathbb{C}\xi_l, \quad \text{for almost all } l\in V_{\mathcal{T}}\cap\mathcal{A}_\varepsilon$$

where the ξ_l's are as in theorem 7.2, and

$$\pi_l(f^*)\mathfrak{H}_{\pi_l}=\{0\}, \quad \text{for almost all } l\in V_{\mathcal{T}}\setminus(V_{\mathcal{T}}\cap\mathcal{A}_\varepsilon).$$

11.5

Let's still assume that f is a weak solution of $f * U_{k,\varepsilon} = 0$ $(k = 1, \dots d)$. Let's fix $l \in V_{\mathcal{T}} \cap \mathcal{A}_\varepsilon \setminus (N \cap \mathcal{A}_\varepsilon)$. By the same reasoning as in theorem 10.2 (iii), there exists $\eta_l \in L^2(\mathbb{R}^d)$ such that

$$\pi_l(f^*) = P_{\xi_l,\eta_l} \quad \text{and} \quad \pi_l(f) = P^*_{\xi_l,\eta_l} = P_{\eta_l,\xi_l}.$$

So $\pi_l(f)$ is a kernel operator whose kernel is equal to $\eta_l \otimes \overline{\xi_l}$, for almost all $l \in V_{\mathcal{T}} \cap \mathcal{A}_\varepsilon$. Hence

$$\|\pi_l(f)\|_{HS} = \|\eta_l \otimes \overline{\xi}_l\|_2 = \|\eta_l\|_2 \; \|\xi_l\|_2$$

for almost all $l \in V_{\mathcal{T}} \cap \mathcal{A}_\varepsilon$ and $\pi_l(f) = 0$ for almost all $l \in V_{\mathcal{T}} \setminus (V_{\mathcal{T}} \cap \mathcal{A}_\varepsilon)$. We may hence put $\eta_l \equiv 0$ if $l \notin \mathcal{A}_\varepsilon$ and get

$$\begin{aligned} \|f\|_2^2 &= \left(\frac{1}{2\pi}\right)^{n-d} \int_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}} \|\pi_l(f)\|^2_{HS} |Pf(l)| dl \\ &= \left(\frac{1}{2\pi}\right)^{n-d} \int_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}} \|\eta_l\|_2^2 \; \|\xi_l\|_2^2 |Pf(l)| dl \\ &< +\infty \end{aligned}$$

11.6

Let ε fixed but arbitrary. In order to characterize the weak solutions of the system $f * U_{k,\varepsilon} = 0$ $(k = 1, \dots, d$ and ε fixed), let's introduce the following notations:

Definition: For any fixed $\varepsilon \in \{-1, 1\}^d$ such that $\mathcal{A}_\varepsilon \neq \emptyset$, the subspaces $\mathcal{S}_\varepsilon(G)$ of $\mathcal{S}(G)$ and $L^2_\varepsilon(G)$ of $L^2(G)$ are defined by

$$\begin{aligned} \mathcal{S}_\varepsilon(G) &= \{f \in \mathcal{S}(G) \mid f * U_{k,\varepsilon} = 0, \; k = 1, \dots, d\} \\ L^2_\varepsilon(G) &= \overline{\mathcal{S}_\varepsilon(G)} \end{aligned}$$

The weak solutions of our system of equations are then characterized by the following theorem:

Theorem 11.7. *Let $G = \exp(\mathfrak{g})$ be a connected, simply connected nilpotent Lie group equipped with a fixed Jordan-Hölder basis. Let's use the notations d, $\mathcal{A}_\varepsilon$ as in theorem 7.2. For a fixed $\varepsilon \in \{-1, 1\}^d$, let $V_{1,\varepsilon}, \dots, V_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ be as in theorem 7.2. Let $U_{k,\varepsilon} = V^*_{k,\varepsilon}$ for $k = 1, \dots, d$. Then the set of weak solutions of the system of equations $f * U_{k,\varepsilon} = 0$, $k = 1, \dots, d$, is equal to $L^2_\varepsilon(G)$, i. e. every weak solution is the limit, in $L^2(G)$, of a sequence of strong solutions of the same equations. We have a similar result for the system $f * U^u_{k,\varepsilon} = 0$, $k = 1, \dots, d$, $u \in G$, where $U_{k,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ is obtained by the action of G on $\mathfrak{U}(\mathfrak{g})$.*

Proof: Let $f \in L^2(G)$ be a weak solution such that $\pi_l(f) = P_{\eta_l,\xi_l}$ for almost all $l \in \mathcal{A}_{\varepsilon,\mathcal{T}}$ and $\pi_l(f) = 0$ if $l \notin \mathcal{A}_\varepsilon$. Let $\gamma = (\gamma_l)_l \in \mathcal{F}_\varepsilon$ with $\gamma_l \equiv 0$ if $l \in \mathcal{A}_{\varepsilon,\mathcal{T}}$ outside a compact subset K of $\mathcal{A}_{\varepsilon,\mathcal{T}}$. The function γ and the compact set K will be determined later more precisely. Let $g \in \mathcal{S}(G)$ be a strong solution constructed as in theorem 10.2 such that $\pi_l(g) = P_{\gamma_l,\xi_l}$ for all

$l \in \mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen}$. Then

$$\begin{aligned}
\|g-f\|_2^2 &= \big(\frac{1}{2\pi}\big)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \|\pi_l(g)-\pi_l(f)\|^2_{HS}|Pf(l)|dl \\
&= \big(\frac{1}{2\pi}\big)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{gen}} \|P_{(\gamma_l-\eta_l),\xi_l}\|^2_{HS}|Pf(l)|dl \\
&= \big(\frac{1}{2\pi}\big)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{gen}} \|\gamma_l-\eta_l\|^2_2\ \|\xi_l\|^2_2|Pf(l)|dl \\
&= \big(\frac{1}{2\pi}\big)^{n-d}\|\gamma(\cdot,\cdot)-\eta(\cdot,\cdot)\|^2_{L^2((V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})\times\mathbb{R}^d,\|\xi_l\|_2^2|Pf(l)|dlds_1\cdots ds_d)}
\end{aligned}$$

Let $M \in \mathbb{N}$ be arbitrary and let's determine K and γ such that $\|g-f\|_2 < \frac{1}{M}$. By 10.5 we know that

$$\big(\frac{1}{2\pi}\big)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \|\eta_l\|^2_2\ \|\xi_l\|^2_2|Pf(l)|dl = \|f\|_2^2 < +\infty,$$

i. e. the map $l \mapsto \|\eta_l\|_2$ belongs to $L^2(V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}, \|\xi_l\|_2^2|Pf(l)|dl)$ and this map is zero outside $\mathcal{A}_{\varepsilon,\mathcal{T}}$. As moreover $\mathcal{A}_{\varepsilon,\mathcal{T}}\cap\mathfrak{g}^*_{gen}$ is an open dense subset of $\mathcal{A}_{\varepsilon,\mathcal{T}}$, there exists a compact subset K_1 of $\mathcal{A}_{\varepsilon,\mathcal{T}}\cap\mathfrak{g}^*_{gen}$ with non-empty interior $\mathrm{int}(K_1)$ in $V_{\mathcal{T}}$ such that

$$\Big(\big(\frac{1}{2\pi}\big)^{n-d}\int_{\mathcal{A}_{\varepsilon,\mathcal{T}}\setminus K_1} \|\eta_l\|^2_2\ \|\xi_l\|^2_2|Pf(l)|dl\Big)^{\frac{1}{2}} < \frac{1}{M}.$$

Let now K be a compact subset of $V_{\mathcal{T}}$ with non-empty interior $\mathrm{int}(K)$ in $V_{\mathcal{T}}$ such that

$$\emptyset \neq \mathrm{int}(K_1) \subset K_1 \subset \mathrm{int}(K) \subset K \subset \mathcal{A}_{\varepsilon,\mathcal{T}}\cap\mathfrak{g}^*_{gen}.$$

Let's construct $\varphi_K \in \mathcal{C}^\infty_c(V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})$ such that

$$\begin{aligned}
&0 \le \varphi_K \le 1 \\
&\varphi_K \equiv 1 \text{ on } K_1 \\
&\mathrm{supp}\varphi_K \subset K
\end{aligned}$$

and let's put

$$\gamma_l(s_1,\ldots,s_d) = \varphi_K(l)\cdot\eta_l(s_1,\ldots,s_d) \in L^2((V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})\times\mathbb{R}^d, \|\xi_l\|_2^2\ |Pf(l)|dlds_1\cdots ds_d).$$

Then

$$\begin{aligned}
&\big(\frac{1}{2\pi}\big)^{n-d}\|\eta_\cdot(\cdot)-\gamma_\cdot(\cdot)\|^2_{L^2((V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk})\times\mathbb{R}^d,\|\xi_l\|_2^2|Pf(l)|dlds_1\cdots ds_d)} \\
&\quad= \big(\frac{1}{2\pi}\big)^{n-d}\int_{V_{\mathcal{T}}\cap\mathfrak{g}^*_{Puk}} \|\eta_l(\cdot)\|^2_2\Big(1-\varphi_K(l)\Big)^2\|\xi_l\|^2_2|Pf(l)|dl \qquad (11.1)\\
&\quad\le \big(\frac{1}{2\pi}\big)^{n-d}\int_{(\mathcal{A}_{\varepsilon,\mathcal{T}}\cap\mathfrak{g}^*_{gen})\setminus K_1} \|\eta_l(\cdot)\|^2_2\ \|\xi_l\|^2_2|Pf(l)|dl \\
&\quad< \big(\frac{1}{M}\big)^2.
\end{aligned}$$

As M is arbitrarily large, this proves the theorem. □

12 Generation of the L^2 space

12.1

For every $\varepsilon \in \{-1,1\}^d$, we have defined an open subset $\mathcal{A}_\varepsilon$ of $\mathfrak{g}^*$ such that $\mathcal{W} = \dot{\cup}_{\varepsilon\in\{-1,1\}}\mathcal{A}_\varepsilon$ is a dense open subset of $\mathfrak{g}^*$. For each ε such that $\mathcal{A}_\varepsilon \neq \emptyset$ we have built a well defined set of elements $U_{1,\varepsilon},\dots,U_{d,\varepsilon}$ of $\mathfrak{U}(\mathfrak{g})$ and defined $\mathcal{S}_\varepsilon(G)$ and $L^2_\varepsilon(G)$ by

$$\begin{aligned} \mathcal{S}_\varepsilon(G) &= \{f \in \mathcal{S}(G) \mid f * U_{k,\varepsilon} = 0,\ k = 1,\dots,d\ \} \\ L^2_\varepsilon(G) &= \overline{\mathcal{S}_\varepsilon(G)} \end{aligned}$$

where $L^2_\varepsilon(G)$ coincides also with the set of weak solutions of the system of equations $f * U_{k,\varepsilon} = 0$, $k = 1,\dots,d$.
Let's also recall that G acts on $\mathfrak{g}$ by Ad and that this action may then be extended to all of $\mathfrak{U}(\mathfrak{g})$. If $u \in G$ and $V \in \mathfrak{U}(\mathfrak{g})$, let's write V^u for this action of u on V. Let's then define the following subspaces:

Definition: For $u \in G$, we define

$$\mathcal{S}_{u,\varepsilon}(G) = \{\varphi \in \mathcal{S}(G) \mid \varphi * U^u_{j,\varepsilon} = 0,\ j = 1,\dots,d\}$$

and

$$L^2_{u,\varepsilon}(G) = \overline{\mathcal{S}_{u,\varepsilon}(G)}.$$

For $u = e$ (identity element of the group), we continue to write $\mathcal{S}_\varepsilon(G)$ and $L^2_\varepsilon(G)$ instead of $\mathcal{S}_{e,\varepsilon}(G)$ and $L^2_{e,\varepsilon}(G)$.

As $\psi * U_{j,\varepsilon} = 0$ if and only if $\psi^u * U^u_{j,\varepsilon} = 0$, we have $\mathcal{S}_{u,\varepsilon}(G) = \Big(\mathcal{S}_\varepsilon(G)\Big)^u$ and $L^2_{u,\varepsilon}(G) = \Big(L^2_\varepsilon(G)\Big)^u$.

Let now $\psi \in L^2_{u,\varepsilon}(G)$, i. e. $\varphi = \psi^{u^{-1}} \in L^2_\varepsilon(G)$. As $\pi_l(\varphi^u) = \pi_l(u^{-1})\pi_l(\varphi)\pi_l(u)$, $\pi_l(\varphi) = 0$ for all $l \notin \mathcal{A}_\varepsilon$ implies $\pi_l(\psi) = 0$ for all $l \notin \mathcal{A}_\varepsilon$ and conversely. Similarly, $\pi_l(\varphi) = P_{\eta_l,\xi_l}$ for almost all $l \in \mathcal{A}_\varepsilon$ implies $\pi_l(\psi) = P_{\pi_l(u)\eta_l,\pi_l(u)\xi_l}$ for almost all $l \in \mathcal{A}_\varepsilon$, i. e. $\pi_l(\psi)$ is also a rank one operator in this case.

12.2

We have the following orthogonality relations:

Proposition 12.3. *Let $u, u' \in G$ and $\varepsilon, \varepsilon' \in \{-1,1\}^d$ such that $\mathcal{A}_\varepsilon \neq \emptyset$, $\mathcal{A}_{\varepsilon'} \neq \emptyset$ and $\varepsilon \neq \varepsilon'$. Then the subspaces $L^2_{u,\varepsilon}(G)$ and $L^2_{u',\varepsilon'}(G)$ are orthogonal.*

Proof: In fact, $\varphi \in L^2_{u,\varepsilon}(G)$ implies $\pi_l(\varphi) = 0$ for all $l \notin \mathcal{A}_\varepsilon$ and $\psi \in L^2_{u',\varepsilon'}(G)$ implies $\pi_l(\psi) = 0$ for all $l \notin \mathcal{A}_{\varepsilon'}$. As $\mathcal{A}_\varepsilon \cap \mathcal{A}_{\varepsilon'} = \emptyset$ if $\varepsilon \neq \varepsilon'$, the Plancherel theorem gives the result. □

Remark: For the same ε, the spaces $L^2_{u,\varepsilon}(G)$ and $L^2_{u',\varepsilon}(G)$ are not necessarily orthogonal, probably their intersection is not even trivial. If u and u' differ only by a central element, the spaces $L^2_{u,\varepsilon}(G)$ and $L^2_{u',\varepsilon}(G)$ even coincide.

12.4

In order to show that these subspaces generate all of $L^2(G)$, we need a Wiener type result for $L^2(G)$. It uses the following definition:

Definition: Let G be a locally compact, second countable, type I, unimodular group. Let $\mathcal{V} \subset L^2(G)$ be a closed subset of $L^2(G)$. Let $\{\xi_i \mid i \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{V}$. We call *support* of $\mathcal{V}$ and write $\mathrm{Supp}\mathcal{V}$ the set

$$\mathrm{Supp}\mathcal{V} = \bigcup_{i\in\mathbb{N}} \{ \pi \in \hat{G} \mid \pi(\xi_i) \neq 0 \},$$

up to a set of measure zero. Here $\pi(\xi_i)$ denotes the operator in the L^2-sense given by the Plancherel theorem. The set $\mathrm{Supp}\mathcal{V}$ is defined up to a set of measure zero, is measurable and is independent of the choice of the countable dense subset $\{ \xi_i \mid i \in \mathbb{N} \}$.

We then have the following "Wiener type" result:

Theorem 12.5. *Let G be a locally compact, second countable, type I, unimodular group. Let $\mathcal{V} \subset L^2(G)$ be a closed subspace that is invariant by right and left translations by elements of G. If $\mathrm{Supp}\mathcal{V} = \hat{G}$ (up to a set of measure zero), then $\mathcal{V} = L^2(G)$.*

Proof: This is a consequence of a more general result by Sutherland ([Su]). □

Corollary 12.6. *Let G be a locally compact, second countable, type I, unimodular group. Let $\mathcal{V} \subset L^2(G)$ be a closed subspace that is invariant by right and left translations by elements of G. Assume that there is a set of measure zero N in $\hat{G}$ such that for all $\pi \in \hat{G} \setminus N$ there exists $\xi \in \mathcal{V}$ such that $\pi(\xi) \neq 0$. Then $\mathcal{V} = L^2(G)$.*

We may then write $L^2(G)$ in the following way:

Theorem 12.7. *Let G be a connected, simply connected, non-abelian, nilpotent Lie group. Let's use the notations of (12.1). Then*

$$\overline{\bigoplus_{\varepsilon\in\{-1,1\}^d}^{\perp} \left(\sum_{u\in G} L^2_{u,\varepsilon}(G) \right)}^{L^2(G)} = L^2(G).$$

Here the exterior sum is an orthogonal direct sum, whereas the interior sum isn't.

Proof: Let's call the space on the left hand side $\mathcal{V}$. Then $\mathcal{V}$ is invariant by left translations, as all the $L^2_{u,\varepsilon}(G)$'s are so. Because $\mathcal{V}$ is also invariant by conjugations, by construction, it is invariant by right translations. For almost all $\pi_l \in \hat{G}$, there exists $\varepsilon \in \{-1,1\}^d$ such that $l \in \mathcal{A}_\varepsilon$. For this ε and for the $U_{1,\varepsilon}, \ldots, U_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ given by theorem 10.2, it is possible to construct $\varphi \in \mathcal{S}_\varepsilon(G) \subset L^2_\varepsilon(G)$ such that $\pi_l(\varphi)$ is a non-zero rank one operator, by theorem 10.2. Hence $\mathcal{V} = L^2(G)$, by corollary 12.6. □

13 Disintegration of the left regular representation on $L^2_\varepsilon(G)$

13.1

Let $\varepsilon \in \{-1,1\}^d$ be arbitrary such that $\mathcal{A}_\varepsilon \neq \emptyset$. Let ρ be the left regular representation of $L^1(G)$ on $L^2(G)$ defined by

$$\rho(f)g = f * g, \quad \forall f \in L^1(G), \forall g \in L^2(G).$$

Let's notice that the closed subspace $L^2_\varepsilon(G)$ of $L^2(G)$ is invariant for this representation. Let $\rho_0 = \rho|_{L^2_\varepsilon(G)}$ be the restriction of ρ to this subspace. We shall now disintegrate ρ_0 into irreducible representations. Let's first recall that for every $g \in L^2_\varepsilon(G)$ and almost every $l \in \mathcal{A}_\varepsilon$, there exists $\eta_l = \eta_l(g) \in L^2(\mathbb{R}^d) \equiv \mathfrak{H}_{\pi_l}$ such that $\pi_l(\mathfrak{g}) = P_{\eta_l,\xi_l}$. Then, for every $f \in L^1(G)$,

$$\pi_l(f * g) = \pi_l(f)P_{\eta_l,\xi_l} = P_{\pi_l(f)\eta_l,\xi_l},$$

i. e. $\eta_l(f * g) = \pi_l(f)\eta_l(g)$.
On the other hand, if $g \in L^2_\varepsilon(G)$ and $l \notin \mathcal{A}_\varepsilon$, then $\pi_l(g) = 0$. We may hence take $\eta_l \equiv 0$, if $l \notin \mathcal{A}_\varepsilon$.

13.2

Let's now consider the space

$$\mathfrak{H} = \Big(\frac{1}{2\pi}\Big)^{n-d} \int^{\oplus}_{V_{\mathcal{T}} \cap \mathfrak{g}^*_{Puk}} n_l \mathfrak{H}_{\pi_l} \|\xi_l\|_2^2 \; |Pf(l)| dl$$

with $n_l = 0$ if $l \in V_{\mathcal{T}} \backslash (\mathcal{A}_{\varepsilon,\mathcal{T}})$ and $n_l = 1$ for almost all $l \in \mathcal{A}_{\varepsilon,\mathcal{T}}$ and let's define the representation Π of $L^1(G)$ on $\mathfrak{H}$ by

$$(\Pi(f)\zeta)_l = \pi_l(f)\zeta_l, \quad \forall f \in L^1(G), \forall \zeta = (\zeta_l)_l \in \mathfrak{H}.$$

Then the representations $(L^2_\varepsilon(G), \rho_0)$ and $(\mathfrak{H}, \Pi)$ are equivalent. In fact, let's define

$$U : L^2_\varepsilon(G) \to \mathfrak{H}$$

by

$$(Ug)_l = \eta_l(g), \quad \text{for almost all } l \in V_T.$$

By 11.5, U is an isometry of $L^2_\varepsilon(G)$ into $\mathfrak{H}$, as

$$\|Ug\|^2 = \Big(\frac{1}{2\pi}\Big)^{n-d} \int_{\mathcal{A}_{\varepsilon,\mathcal{T}}} \|\eta_l(g)\|_2^2 \; \|\xi_l\|_2^2 \; |Pf(l)| dl = \|g\|_2^2.$$

As this map is also linear, by construction, it is one-to-one. It is onto, as, by the proof of theorem 10.2, every $\gamma \in \mathcal{F}_\varepsilon$ such that supp$\gamma(l,s) \subset K \times \mathbb{R}^d$ where K is a compact subset of $\mathcal{A}_{\varepsilon,\mathcal{T}} \cap \mathfrak{g}^*_{gen}$ is in the image of $L^2_\varepsilon(G)$ by U and as these functions are dense in $\mathfrak{H}$. The map U intertwines the representations ρ_0 and Π, as, for every $g \in L^2_\varepsilon(G)$ and almost every $l \in V_\mathcal{T}$,

$$\begin{aligned}
\Big[(U \circ \rho_0(f))g\Big]_l &= \Big[U(\rho_0(f)g)\Big]_l \\
&= \eta_l\Big(\rho_0(f)g\Big) \\
&= \eta_l(f * g) \\
&= \pi_l(f)\eta_l(g) \\
&= \pi_l(f)\Big(Ug\Big)_l \\
&= \Big[\Pi(f)(Ug)\Big]_l .
\end{aligned}$$

So ρ_0 is unitary equivalent to

$$\Pi = \big(\frac{1}{2\pi}\big)^{n-d} \int^{\oplus}_{V_\mathcal{T} \cap \mathfrak{g}^*_{Puk}} n_l \pi_l \|\xi_l\|_2^2 \ |Pf(l)| dl$$

with $n_l = 0$ if $l \notin \mathcal{A}_\varepsilon$ and $n_l = 1$ for almost all $l \in \mathcal{A}_{\varepsilon,\mathcal{T}}$, and we have a disintegration of $\rho_0 = \rho|_{L^2_\varepsilon(G)}$ into the irreducible representations π_l with multiplicities 1 (for almost all $l \in \mathcal{A}_{\varepsilon,\mathcal{T}}$) and 0 (if $l \notin \mathcal{A}_\varepsilon$). We have similar results for all the $L^2_{u,\varepsilon}(G)$, $u \in G$ and $\rho|_{L^2_{u,\varepsilon}(G)}$.

13.3 Final remarks

The previous theorem gives us a generation of the space $L^2(G)$ as a sum of kernels of well chosen left invariant partial differential operators. Unfortunately, this sum is not direct, even if it is formed by bigger orthogonal blocks. For any ε such that $\mathcal{A}_\varepsilon \neq \emptyset$, the space $L^2_\varepsilon(G)$ is invariant by the left regular representation ρ and the restriction of the left regular representation to the subspace $L^2_\varepsilon(G)$ disintegrates into irreducible representations π_l with multiplicities 1 for almost all $l \in \mathcal{A}_\varepsilon$ (mod $\mathrm{Ad}^*(G)$) and 0 otherwise. This remains of course true for all $L^2_{u,\varepsilon}(G)$, as $L^2_{u,\varepsilon}(G)$ is deduced from $L^2_\varepsilon(G)$ by conjugation.

14 Examples

14.1 The Heisenberg group

The case of the $2n+1$ dimensional Heisenberg group H_n has been studied in ([Lu-Mo]). In this case, $d = n$, $\mathcal{A}_\varepsilon \neq \emptyset$ if and only if $\varepsilon = (1, 1, \dots, 1)$ or $\varepsilon = (-1, -1, \dots, -1)$. Let's write $\mathrm{sgn}(\varepsilon) = +$ if $\varepsilon = (1, 1, \dots, 1)$ and $\mathrm{sgn}(\varepsilon) = -$ if $\varepsilon = (-1, -1, \dots, -1)$. The operators $U_{k,\varepsilon}$ are given by $U_{k,\varepsilon} = X_k + i\ \mathrm{sgn}(\varepsilon) Y_k$. The sum of theorem 12.7 may be replaced by a direct sum if we restrict the choice of the elements $u \in H_n$. In fact, for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) = a + ib \in \mathbb{C}^n$, $a, b \in \mathbb{R}^n$, let's define $u_{(\alpha,\varepsilon)} = \exp(\sum_{k=1}^n (a_k X_k + b_k Y_k)) \in H_n$ if $\mathrm{sgn}(\varepsilon) = +$ and $u_{(\alpha,\varepsilon)} = \exp(\sum_{k=1}^n (-a_k X_k + b_k Y_k))$ if $\mathrm{sgn}(\varepsilon) = -$. Then $(X_k + i\ \mathrm{sgn}(\varepsilon) Y_k)^{u_{\alpha,\varepsilon}} = X_k + i\ \mathrm{sgn}(\varepsilon) Y_k + i\alpha_k Z$, for $k = 1, \dots, n$ and

$$L^2(H_n) = \overline{\bigoplus_{\alpha \in \mathbb{C}^n} \Big(L^2_{\alpha,+}(H_n) \oplus L^2_{\alpha,-}(H_n) \Big)}^{L^2(H_n)} \qquad (*)$$

where $L^2_{\alpha,+}(H_n) = L^2_{u_{(\alpha,\varepsilon)},\varepsilon}(H_n)$ with $\varepsilon = (1, 1, \ldots, 1)$ and $L^2_{\alpha,-}(H_n) = L^2_{u_{(\alpha,\varepsilon)},\varepsilon}(H_n)$ with $\varepsilon = (-1, -1, \ldots, -1)$. As a matter of fact, in ([Lu-Mo]) an even richer decomposition is proven: For $\alpha, \beta \in \mathbb{C}^n$, let

$$L^2_{(\alpha,\beta),\pm}(H_n) = \overline{\{\ f \in \mathcal{S}(H_n) \mid f * (X_k \pm iY_k + i\alpha_k Z) = -i\beta_k f,\ k = 1, \ldots, n\ \}}^{L^2(H_n)}.$$

Then

$$L^2(H_n) = \overline{\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} \left(L^2_{(\alpha,\beta),+}(H_n) \oplus L^2_{(\alpha,\beta),-}(H_n) \right)}^{L^2(H_n)} .$$

By the same arguments than those developed in ([Lu-Mo]), we show that the decomposition of $(*)$ is sufficient. In particular, for $\beta \neq 0$,

$$L^2_{(\alpha,\beta),\pm}(H_n) \subset \overline{\bigoplus_{\alpha'\in\mathbb{C}^{n}} \left(L^2_{\alpha',+}(H_n) \oplus L^2_{\alpha',-}(H_n) \right)}^{L^2(H_n)} .$$

So, in the case of the Heisenberg group, we have a decomposition of $L^2(H_n)$ as a direct sum (or, more precisely, a closure of a direct sum) and a complete disintegration of the left regular representation with multiplicities 0 and 1 on each summand of the direct sum.

14.2

Let G be the connected, simply connected, nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is spanned by the generators $X_1, X_2, \ldots, X_{11}$ satisfying

$$\begin{array}{ccc}
[X_{11}, X_9] = X_7 & [X_{11}, X_8] = X_5 & [X_{11}, X_2] = X_1 \\
[X_{10}, X_9] = -X_4 & [X_{10}, X_8] = -X_6 & [X_{10}, X_3] = -X_1 \\
[X_9, X_8] = X_1 & [X_9, X_7] = -X_3 & [X_9, X_4] = X_2 \\
[X_8, X_6] = X_2 & [X_8, X_5] = -X_3 & \\
[X_7, X_4] = X_1 & & \\
[X_6, X_5] = -X_1 & &
\end{array}$$

This example is found in ([Co-Gr]) with other notations. The computations show that $d = 5$, $\mathcal{R} = \{6, 7, 9, 10, 11\}$ and, for every $\varepsilon = (\varepsilon_6, \varepsilon_7, \varepsilon_9, \varepsilon_{10}, \varepsilon_{11}) \in \{-1, 1\}^5$ such that $\mathcal{A}_\varepsilon \neq \emptyset$,

$$\begin{array}{rcl}
V_{6,\varepsilon} & = & X_6 - i\varepsilon_6 X_5 \\
V_{7,\varepsilon} & = & X_1^6 X_7 - i\varepsilon_7 X_1^6 X_4 + i\varepsilon_7 X_1^5 X_2 X_8 + 2i\varepsilon_7 X_1^5 X_2 X_3 - 3i\varepsilon_7 X_1^4 X_2^2 X_5 \\
V_{9,\varepsilon} & = & X_1^2 X_9 - i\varepsilon_9 X_1^3 X_8 + i\varepsilon_9 X_1^2 X - 2 + i\varepsilon_9 X_1^2 X_3 X_6 + X_1 X_3 X_4 + X_1 X_2 X_7 \\
V_{10,\varepsilon} & = & X_{10} + i\varepsilon_{10} X_3 \\
V_{11,\varepsilon} & = & X_{11} - i\varepsilon_{11} X_2
\end{array}$$

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