Fine Disintegration of the Left Regular Representation of a nilpotent Lie Group

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Abstract

Let H_n be the $(2n+1)$-dimensional Heisenberg group. We decompose $L^2(H_n)$ as the closure of a direct sum of infinitely many left translation invariant eigenspaces (for certain systems of partial differential equations). The restriction of the left regular representation to each one of these eigenspaces disintegrates into a direct integral of unitary irreducible representations, such that each infinite dimensional unitary irreducible representation appears with multiplicity 0 or 1 in this disintegration.

1 Introduction

A fundamental problem raised by a lot of areas of mathematics is the question of how to decompose a given mathematical object into its "elementary constituencies". In representation theory this is the question of the decomposition of a given representation into irreducible ones. In this paper we shall solve this problem for the Heisenberg group H_n and for the left regular representation ρ of $L^1(H_n)$ on the Hilbert space $L^2(H_n)$ defined by

$$\rho(f)(\xi) = f * \xi, \quad \forall f \in L^1(H_n), \forall \xi \in L^2(H_n). \tag{1.1}$$

How can this representation be disintegrated into irreducible ones?
To be more precise, let's first recall some fundamental definitions: Let G be a locally compact group. We say that $(\pi, \mathfrak{H})$ is a (unitary) representation of G, if $\mathfrak{H}$ is a Hilbert space, $\pi : G \to \mathcal{L}(\mathfrak{H})$ a map of G into the set of bounded linear operators on $\mathfrak{H}$, such that

$$\pi(x \cdot y) = \pi(x) \circ \pi(y), \quad \forall x, y \in G \tag{1.2}$$

$$\pi(x^{-1}) = \pi(x)^*, \quad \forall x \in G \tag{1.3}$$

(i. e. the operators $\pi(x)$ are unitary) and if the map $x \to \pi(x)$ is strongly continuous. We say that $(\pi, \mathfrak{H})$ is (topologically) irreducible if there is no non-trivial closed invariant

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(under the action of π) subspace in $\mathfrak{H}$. If $(\pi, \mathfrak{H})$ is a unitary representation of G, we get a $*$-representation of the group algebra $L^1(G)$ on $\mathfrak{H}$ by

$$\pi(f)(\xi) = \int_G f(x)\pi(x)(\xi)dx, \quad \forall f \in L^1(G), \forall \xi \in \mathfrak{H}, \tag{1.4}$$

where dx is the Haar measure on the group G. This representation of $L^1(G)$ is irreducible if and only if this is the case for the corresponding representation of the group G.

Let now H_n be the $(2n+1)$-dimensional Heisenberg group. This is a connected, simply connected, nilpotent Lie group, whose Lie algebra admits $2n+1$ generators $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ satisfying $[X_i, Y_j] = \delta_{ij}Z$ where δ_{ij} equals 1 if $i = j$ and 0 otherwise. The unitary irreducible representations of H_n and $L^1(H_n)$ are (except for the characters) induced representations and are characterized by the Kirillov orbit method for nilpotent Lie groups.

It is known, as a consequence of the Plancherel theorem, that every infinite dimensional irreducible representation has to appear with infinite multiplicity in the disintegration of the left regular representation. How can these infinitely many copies of a same irreducible representation be realized within the left regular representation $(\rho, L^2(H_n))$? What does such a disintegration concretely mean? One such realization has been obtained in ([1]). In this paper we construct another one from a totally different point of view. This new decomposition is not a disintegration in the classical sense, but a decomposition of $L^2(H_n)$ into the closure of a direct sum of eigenspaces of a family of left invariant differential operators. On each one of these eigenspaces the left regular representation disintegrates into irreducible representations with multiplicities 0 and 1. More precisely, this may be achieved in the following way: For every $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}^n$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, let $L^2_{(\alpha,\beta),+}(H_n)$ be the set of weak solutions, in $L^2(H_n)$, of the system of equations

$$\varphi * (X_k + iY_k + i\alpha_k Z) = -i\beta_k \varphi, \quad \forall k \in \{1, \ldots, n\}, \tag{1.5}$$

resp. $L^2_{(\alpha,\beta),-}(H_n)$ the set of weak solutions of

$$\varphi * (X_k - iY_k + i\alpha_k Z) = -i\beta_k \varphi, \quad \forall k \in \{1, \ldots, n\}. \tag{1.6}$$

Then

$$L^2(H_n) = \overline{\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} \left(L^2_{(\alpha,\beta),+}(H_n) \oplus L^2_{(\alpha,\beta),-}(H_n) \right)}^{L^2(H_n)}. \tag{1.7}$$

The spaces $L^2_{(\alpha,\beta),\pm}(H_n)$ are non-trivial, isomorphic (as linear spaces) and isometric. Hence we have obtained a decomposition of $L^2(H_n)$ as a direct sum of eigenspaces that is already very interesting for its own sake. Moreover, as these subspaces are all invariant under the left regular representation, we may consider the restriction of this representation to any one of these eigenspaces. This restriction may be disintegrated into irreducible ones, such that each infinite dimensional irreducible representation appears with the multiplicity 0 or 1 in the disintegration. This shows how the infinitely

many copies of a particular irreducible representation π are distributed in the disintegration of the left regular representation and gives a lot of insight into the fine structure of this disintegration.
Let's just say a few words about the methods used in this paper. To do this, let's recall the following facts: For every $\lambda \in \mathbb{R}^*$, let's write χ_λ for the character defined on $P_\lambda = \exp(\sum_{k=1}^n \mathbb{R}Y_k + \mathbb{R}Z)$ by $\chi_\lambda(y,z) = e^{-i\lambda z}$, with $y \in \mathbb{R}^n$, $z \in \mathbb{R}$. Let π_λ be the induced representation $\pi_\lambda = \mathrm{ind}_{P_\lambda}^{H_n} \chi_\lambda$ and $d\pi_\lambda$ the corresponding representation of the enveloping algebra $\mathfrak{U}(\mathfrak{h}_n)$. Then

$$f * (X_k + iY_k) = 0 \Leftrightarrow d\pi_\lambda(X_k - iY_k)\pi_\lambda(f^*) = 0, \quad \forall \lambda \in \mathbb{R}^*. \tag{1.8}$$

This leads us to solve the system of equations

$$d\pi_\lambda(X_k - iY_k)(\xi) = 0, \quad \forall k \in \{1, \dots, n\}, \tag{1.9}$$

(weak solutions) in the representation space $\mathfrak{H}_\lambda$ (which may be identified with $L^2(\mathbb{R}^n)$). The set of these weak solutions is $\{0\}$ if $\lambda > 0$ and $\mathbb{C} \cdot \alpha_\lambda$ almost everywhere, where $\alpha_\lambda(s_1, \dots, s_n) = e^{\frac{\lambda}{2}\sum_{k=1}^n s_k^2}$ if $\lambda < 0$. One then shows that f is a solution of $f * (X_k + iY_k) = 0$ $(k = 1, \dots, n)$ if and only if $\pi_\lambda(f^*)$ is, almost everywhere, a rank one operator onto $\mathbb{C} \cdot \alpha_\lambda$ if $\lambda < 0$, resp. zero if $\lambda > 0$. Conversely, given a smooth family of such rank one operators, one may, under suitable hypotheses, construct a solution f via the Plancherel theorem. These methods also show that the set of weak solutions of the system of equations $f * (X_k + iY_k) = 0$ $(k = 1, \dots, n)$ coincides with the closure, in $L^2(H_n)$, of the set of strong solutions of the same equation, and they are useful in the disintegration problem.

2 Generalities on the Heisenberg group

2.1

The Heisenberg group H_n may be defined as follows: As a manifold, H_n is identified with $\mathbb{R}^{2n+1}$. The group product is then defined by

$$(x,y,z) \cdot (x',y',z') = (x + x', y + y', z + z' + \frac{1}{2}(x \cdot y' - x' \cdot y)), \tag{2.10}$$

where $x, x', y, y' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}$ and where $x \cdot y'$ and $x' \cdot y$ denote the scalar product in $\mathbb{R}^n$. In the following, (x,y,z) will always denote an element of $H_n = \mathbb{R}^{2n+1}$, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$. The Lie algebra $\mathfrak{h}_n$ of H_n is spanned by $2n+1$ generators $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ satisfying the relations

$$[X_k, Y_k] = Z, \quad k \in \{1, \dots, n\}. \tag{2.11}$$

It may also be identified with $\mathbb{R}^{2n+1}$ by taking as basis vectors X_k, Y_k, Z $(k \in \{1, \dots, n\})$ and $[(p,q,r),(p',q',r')] = (0, 0, p \cdot q' - q \cdot p')$. The exponential mapping then coincides with the identity map on $\mathbb{R}^{2n+1}$. Hence we may identify, in H_n,

$$\exp(x_k X_k) = \big((0, \dots, x_k, \dots, 0), (0, \dots, 0), 0\big) \tag{2.12}$$

$$\exp(y_k Y_k) = \big((0, \dots, 0), (0, \dots, y_k, \dots, 0), 0\big) \tag{2.13}$$

$$\exp(zZ) = \big((0, \dots, 0), (0, \dots, 0), z\big), \tag{2.14}$$

where x_k is the kth coordinate of the first bracket and y_k the kth coordinate of the second bracket.

2.2

Let's recall that if G is a connected, simply connected, nilpotent Lie group and if $\mathfrak{g}$ is its Lie algebra, then $\mathfrak{g}$ acts on the space of smooth functions $\mathcal{C}^\infty(G)$ by

$$f * U(g) = \frac{d}{ds} f\big(g \cdot \exp(sU)\big)|_{s=0} \tag{2.15}$$

$$U * f(g) = \frac{d}{ds} f\big(\exp(-sU) \cdot g\big)|_{s=0}, \tag{2.16}$$

for all $g \in G$, for all $U \in \mathfrak{g}$.
For the Heisenberg group H_n we then get the following formulas:

$$f * X_k(x,y,z) = \frac{\partial f}{\partial x_k}(x,y,z) - \frac{1}{2} y_k \frac{\partial f}{\partial z}(x,y,z) \tag{2.17}$$

$$f * Y_k(x,y,z) = \frac{\partial f}{\partial y_k}(x,y,z) + \frac{1}{2} x_k \frac{\partial f}{\partial z}(x,y,z) \tag{2.18}$$

$$f * Z(x,y,z) = \frac{\partial f}{\partial z}(x,y,z) \tag{2.19}$$

$$f * (X_k + iY_k)(x,y,z) = (\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k}) f(x,y,z) + \frac{i}{2}(x_k + i y_k) \frac{\partial f}{\partial z}(x,y,z) \tag{2.20}$$

$$f * (X_k - iY_k)(x,y,z) = (\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k}) f(x,y,z) - \frac{i}{2}(x_k - i y_k) \frac{\partial f}{\partial z}(x,y,z). \tag{2.21}$$

2.3 Irreducible unitary representations

2.3.1

The unitary characters of H_n are parametrized by $\mathbb{R}^n \times \mathbb{R}^n \times \{0\}$ and are given by

$$\chi_{(a,b,0)}(x,y,z) = e^{-i(a\cdot x + b \cdot y)}, \quad \forall (x,y,z) \in H_n. \tag{2.22}$$

2.3.2

The infinite dimensional irreducible unitary representations of H_n are obtained in the following way: For every $\lambda \in \mathbb{R}^*$, let $l_\lambda = \lambda Z^* \in \mathfrak{h}_n^*$ and $\mathfrak{p}_\lambda = \sum_{k=1}^n \mathbb{R} Y_k + \mathbb{R} Z$ a polarization associated to l_λ. Let $P_\lambda = \exp(\mathfrak{p}_\lambda) \equiv \sum_{k=1}^n \mathbb{R} Y_k + \mathbb{R} Z$ and $\pi_\lambda = \mathrm{ind}_{P_\lambda}^{H_n} \chi_\lambda$, with $\chi_\lambda(x,y,z) = e^{-i\lambda z}$, the corresponding induced representation. This representation is realized on the space $\mathfrak{H}_\lambda$ which is the space of measurable functions $\xi : H_n \to \mathbb{C}$ such that

$$\xi \ \big((x,y,z)(0,q,r)\big) = e^{i\lambda r} \xi(x,y,z), \quad (x,y,z) \in H_n, (0,q,r) \in P_\lambda \text{ a. e.,} \tag{2.23}$$

$$\| \ \xi \|^2 = \int_{\mathbb{R}^n} |\xi(x,0,0)|^2 dx < +\infty. \tag{2.24}$$

The representation π_λ is then defined on $\mathfrak{H}_\lambda$ by

$$\big(\pi_\lambda(p,q,r)\xi\big)(x,y,z) = \xi((-p,-q,-r)(x,y,z)). \tag{2.25}$$

As

$$\xi(x,y,z) = \xi\big((x,0,0)(0,y,z-\frac{1}{2}x\cdot y)\big) = e^{i\lambda z}e^{-\frac{1}{2}i\lambda x\cdot y}\xi(x,0,0), \tag{2.26}$$

one may identify $(x,0,0)$ with $x \in \mathbb{R}^n$ and say that π_λ acts on the space $L^2(\mathbb{R}^n)$ by

$$\big(\pi_\lambda(p,q,r)\xi\big)(x) = e^{-i\lambda r}e^{i\lambda x\cdot q}e^{-\frac{1}{2}i\lambda p\cdot q}\xi(x-p), \quad \forall x \in \mathbb{R}^n. \tag{2.27}$$

The representations π_λ correspond to the generic orbits in the sense of Pukanszky ([3], [2]).

2.3.3

The corresponding unitary representations of the Lie algebra $\mathfrak{h}_n$ act on the space of $\mathcal{C}^\infty$-vectors $\mathfrak{H}_\lambda^\infty$ and are defined by

$$\big(d\pi_\lambda(U)\xi\big)(s) = \frac{d}{dt}\xi\big(\exp(-tU)(s,0,0)\big)|_{t=0}, \quad s=(s_1,\dots,s_n)\in\mathbb{R}^n. \tag{2.28}$$

The space $\mathfrak{H}_\lambda^\infty$ may be identified with $\mathcal{S}(\mathbb{R}^n)$ endowed with the appropriate covariance relation, and an easy computation shows that

$$\begin{aligned}
\big(d\pi_\lambda(X_k)\xi\big)(s) &= -\frac{\partial\xi}{\partial s_k}(s) && (2.29)\\
\big(d\pi_\lambda(Y_k)\xi\big)(s) &= i\lambda s_k\xi(s) && (2.30)\\
\big(d\pi_\lambda(X_k+iY_k)\xi\big)(s) &= -\frac{\partial\xi}{\partial s_k}(s) - \lambda s_k\xi(s) && (2.31)\\
\big(d\pi_\lambda(X_k-iY_k)\xi\big)(s) &= -\frac{\partial\xi}{\partial s_k}(s) + \lambda s_k\xi(s). && (2.32)
\end{aligned}$$

2.3.4

Let's recall that the associated irreducible $*$-representation of $L^1(H_n)$ is then defined by

$$\pi_\lambda(f)(\xi) = \int_{\mathbb{R}^{2n+1}} f(x,y,z)\pi_\lambda(x,y,z)(\xi)dxdydz \quad \forall f\in L^1(H_n), \forall\xi\in\mathfrak{H}_\lambda. \tag{2.33}$$

2.4 Plancherel theorem

Let's finally recall some results about the Plancherel theorem for the Heisenberg group: Let's write $HS(\mathfrak{H}_\lambda)$ for the Hilbert-Schmidt operators on $\mathfrak{H}_\lambda$. Let's consider the space given by the direct integral

$$\mathcal{H} = \int_{\mathbb{R}^*}^{\oplus} HS(\mathfrak{H}_\lambda)\ |\lambda|d\lambda, \tag{2.34}$$

where $|\lambda|d\lambda$ is the Plancherel measure. For all $f \in L^1(H_n) \cap L^2(H_n)$, we have

$$\|f\|_2^2 = \int_{\mathbb{R}^*} \|\pi_\lambda(f)\|_{HS}^2 |\lambda| d\lambda, \tag{2.35}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. The statement of the Plancherel theorem is that the map

$$f \in L^1(H_n) \cap L^2(H_n) \mapsto \big(\pi_\lambda(f)\big)_{\lambda \in \mathbb{R}^*} \in \mathcal{H} \tag{2.36}$$

extends to an isometry from $L^2(H_n)$ onto $\mathcal{H}$. For every $f \in L^2(H_n)$, we shall again write $\pi_\lambda(f)$ for the Hilbert-Schmidt operators associated to f in this isometry. In particular, for any fixed $f \in L^2(H_n)$, the map $\lambda \mapsto \|\pi_\lambda(f)\|_{HS}$ is measurable and

$$\int_{\mathbb{R}^*} \|\pi_\lambda(f)\|_{HS}^2 |\lambda| d\lambda < +\infty. \tag{2.37}$$

Moreover,

$$< f, g > = \int_{\mathbb{R}^*} \operatorname{tr}\big(\pi_\lambda(f)\pi_\lambda(g^*)\big) |\lambda| d\lambda, \quad \forall f, g \in L^2(H_n). \tag{2.38}$$

See ([2], [3]) for more details on the abstract Plancherel theorem.

3 Strong solutions of a particular system of differential equations

3.1

In this section we shall study the solutions of the system of equations

$$f * (X_k + iY_k) = 0, \quad \forall k \in \{1, \ldots, n\}, \tag{3.39}$$

in the Schwartz space $\mathcal{S}(H_n)$. This is done by studying their image under the representations π_λ. In fact,

$$\begin{aligned} f * (X_k + iY_k) = 0 \quad &\Leftrightarrow \quad (X_k - iY_k) * f^* = 0 \\ &\Leftrightarrow \quad d\pi_\lambda(X_k - iY_k)\pi_\lambda(f^*) = 0, \quad \forall \lambda \neq 0, \\ &\Leftrightarrow \quad d\pi_\lambda(X_k - iY_k)(\pi_\lambda(f^*)\xi_\lambda) = 0, \quad \forall \xi_\lambda \in \mathfrak{H}_\lambda^\infty, \forall \lambda \neq 0 \\ &\Leftrightarrow \quad \pi_\lambda(f^*)(\mathfrak{H}_\lambda^\infty) \subset \operatorname{Ker} d\pi_\lambda(X_k - iY_k), \quad \forall \lambda \neq 0, \end{aligned} \tag{3.40}$$

where $\operatorname{Ker} d\pi_\lambda(X_k - iY_k)$ stands for the kernel of this operator in the Schwartz space $\mathcal{S}(\mathbb{R}^n) \equiv \mathfrak{H}_\lambda^\infty$. But

$$\begin{aligned} \xi_\lambda \in \bigcap_{k=1}^{n} \operatorname{Ker} d\pi_\lambda(X_k - iY_k) \quad &\Leftrightarrow \quad -\frac{\partial \xi_\lambda}{\partial s_k}(s) + \lambda s_k \xi_\lambda(s) = 0 \ \forall k \in \{1, \ldots, n\} \\ &\Leftrightarrow \quad \xi_\lambda(s) = C(\lambda) e^{\frac{\lambda}{2}\sum_{k=1}^n s_k^2} = C(\lambda)\alpha_\lambda(s), \end{aligned} \tag{3.41}$$

where $\alpha_\lambda(s) = e^{\frac{\lambda}{2}\sum_{k=1}^n s_k^2}$ and where $C(\lambda)$ is a constant depending on λ. As ξ_λ has to be a Schwartz function, we have to take $C(\lambda) = 0$ if $\lambda \in]0, +\infty[$. Hence, if f is a solution of $f * (X_k + iY_k) = 0$ for $k = 1, \ldots, n$, then

$$\pi_\lambda(f^*)(\mathfrak{H}_\lambda^\infty) = \pi_\lambda(f^*)(\mathcal{S}(\mathbb{R}^n)) \subset \begin{cases} \{0\} & \text{if } \lambda > 0 \\ \mathbb{C} \cdot \alpha_\lambda & \text{if } \lambda < 0 \end{cases} \tag{3.42}$$

In particular, $\pi_\lambda(f^*) = \pi_\lambda(f) = 0$ if $\lambda > 0$.

3.2

Let $f \in \mathcal{S}(H_n)$ be a solution of the system $f * (X_k + iY_k) = 0$ for $k = 1, \ldots, n$. Then, for every $\lambda \in \mathbb{R}^*$, there exists $\eta_\lambda \in L^2(\mathbb{R}^n) \equiv \mathfrak{H}_\lambda$ such that $\eta_\lambda = 0$ if $\lambda > 0$ and $\pi_\lambda(f^*) = P_{\alpha_\lambda, \eta_\lambda}$ where

$$\pi_\lambda(f^*)(\xi) = P_{\alpha_\lambda,\eta_\lambda}(\xi) = <\xi, \eta_\lambda> \alpha_\lambda, \quad \forall \xi \in L^2(\mathbb{R}^n), \forall \lambda \in \mathbb{R}^*. \tag{3.43}$$

Moreover, by the Plancherel theorem,

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbb{R}^*} \|\pi_\lambda(f)\|_{HS}^2 |\lambda| d\lambda \\ &= \pi^{\frac{n}{2}} \int_{\mathbb{R}^*} \|\eta_\lambda\|_2^2 |\lambda|^{1-\frac{n}{2}} d\lambda \\ &< +\infty. \end{aligned} \tag{3.44}$$

In fact, by eq. (3.42), there exists, for every $\xi \in L^2(\mathbb{R}^n) \equiv \mathfrak{H}_\lambda$, a constant $C_\lambda(\xi) \in \mathbb{C}$ such that $\pi_\lambda(f^*)(\xi) = C_\lambda(\xi)\alpha_\lambda$ and $C_\lambda(\xi) = 0$ if $\lambda > 0$. For fixed λ, this C_λ defines a bounded linear map from $L^2(\mathbb{R}^n)$ into $\mathbb{C}$. In fact, it is easy to check that for any $\lambda < 0$, $\|\alpha_\lambda\|_2 = \pi^{\frac{n}{4}}|\lambda|^{-\frac{n}{4}}$ and

$$\pi^{\frac{n}{4}}|\lambda|^{-\frac{n}{4}}|C_\lambda(\xi)| = \|\pi_\lambda(f^*)\xi\|_2 \leq \|\pi_\lambda(f^*)\|_{op}\|\xi\|_2. \tag{3.45}$$

So there exists $\eta_\lambda \in L^2(\mathbb{R}^n)$ such that $C_\lambda(\xi) = <\xi, \eta_\lambda>$ and

$$\pi_\lambda(f^*)(\xi) = <\xi, \eta_\lambda> \alpha_\lambda, \quad \forall \xi \in L^2(\mathbb{R}^n). \tag{3.46}$$

For $\lambda > 0$ we just take $\eta_\lambda = 0$. Hence

$$\pi_\lambda(f^*) = P_{\alpha_\lambda,\eta_\lambda} \quad \text{and} \quad \pi_\lambda(f) = P_{\eta_\lambda,\alpha_\lambda}. \tag{3.47}$$

So $\pi_\lambda(f)$ is a kernel operator whose kernel is equal to $\eta_\lambda \otimes \overline{\alpha}_\lambda$ and

$$\|\pi_\lambda(f)\|_{HS} = \|\eta_\lambda \otimes \overline{\alpha}_\lambda\|_2 = \pi^{\frac{n}{4}}|\lambda|^{-\frac{n}{4}}\|\eta_\lambda\|_2. \tag{3.48}$$

Remark 1. *What precedes remains true if we only have*

$$\pi_\lambda(f^*)(\mathfrak{H}_\lambda) = \pi_\lambda(f^*)(L^2(\mathbb{R}^n)) \subset \begin{cases} \{0\} & a.\ e.\ \text{if } \lambda > 0 \\ \mathbb{C} \cdot \alpha_\lambda & a.\ e.\ \text{if } \lambda < 0 \end{cases} \tag{3.49}$$

for some $f \in L^2(H_n)$ (where $\pi_\lambda(f^*)$ is taken in the sense of the Plancherel theorem). Then η_λ is only defined for almost all λ.

3.3

Conversely, the previous considerations allow us to construct solutions to the system $f * (X_k + iY_k) = 0$ for $k = 1, \dots, n$ via the Plancherel formula. This is done in the following proposition.

Proposition 3.1. *Let* $0 < \varepsilon < K$ *and* $\gamma \in \mathcal{S}(\mathbb{R}^{n+1})$ *such that* $\operatorname{supp}\gamma(\lambda, s) \subset [-K, -\varepsilon] \times \mathbb{R}^n$. *Let's write* $\gamma_\lambda(s) = \gamma(\lambda, s)$ *for all* $(\lambda, s) \in \mathbb{R}^{n+1}$. *Then there exists* $f \in \mathcal{S}(H_n)$ *such that*

$$\pi_\lambda(f^*) = P_{\alpha_\lambda, \gamma_\lambda}, \quad \forall \lambda \in \mathbb{R}^*, \tag{3.50}$$

where $P_{\alpha_\lambda,\gamma_\lambda}$ *is the rank one operator defined by*

$$P_{\alpha_\lambda,\gamma_\lambda}\xi =< \xi, \gamma_\lambda > \alpha_\lambda, \quad \forall \xi \in \mathfrak{H}_\lambda. \tag{3.51}$$

Hence, for this $0 \neq f \in \mathcal{S}(H_n)$,

$$d\pi_\lambda(X_k - iY_k)\pi_\lambda(f^*) = 0, \quad \forall \lambda \in \mathbb{R}^*, \forall k \in \{1, \dots, n\} \tag{3.52}$$

and

$$f * (X_k + iY_k) = 0, \quad \forall k \in \{1, \dots, n\}. \tag{3.53}$$

Proof. By the Plancherel formula, we must have

$$\begin{aligned} f^*(a) &= \big(\frac{1}{2\pi}\big)^{2n+1} \int_{\mathbb{R}} \operatorname{tr}\big(\pi_\lambda(a^{-1})\pi_\lambda(f^*)\big)|\lambda| d\lambda \\ &= \big(\frac{1}{2\pi}\big)^{2n+1} \int_{\mathbb{R}} \operatorname{tr}\big(\pi_\lambda(a^{-1})P_{\alpha_\lambda,\gamma_\lambda}\big)|\lambda| d\lambda, \end{aligned} \tag{3.54}$$

for every $a \in H_n$, where $|\lambda|d\lambda$ is the Plancherel measure. But, for every $\xi \in \mathfrak{H}_\lambda$, we have

$$\begin{aligned} &\Big(\pi_\lambda(\exp(\sum_{j=1}^n x_jX_j) \quad \exp \quad (\sum_{j=1}^n y_jY_j + zZ))^{-1}P_{\alpha_\lambda,\gamma_\lambda}\Big)\xi(\exp(\sum_{j=1}^n s_jX_j)) \\ &\qquad = \pi_\lambda(\exp(\sum_{j=1}^n x_jX_j)\exp(\sum_{j=1}^n y_jY_j + zZ))^{-1} \\ &\qquad\qquad \cdot < \xi, \gamma_\lambda > \alpha_\lambda(\exp(\sum_{j=1}^n s_jX_j)) \\ &\qquad = < \xi, \gamma_\lambda > \alpha_\lambda(\exp(\sum_{j=1}^n x_jX_j)\exp(\sum_{j=1}^n y_jY_j + zZ)\exp(\sum_{j=1}^n s_jX_j)) \\ &\qquad = \int_{\mathbb{R}^n} \alpha_\lambda(\exp(\sum_{j=1}^n x_jX_j)\exp(\sum_{j=1}^n y_jY_j + zZ)\exp(\sum_{j=1}^n s_jX_j)) \\ &\qquad\qquad \cdot\xi(\exp(\sum_{j=1}^n t_jX_j))\overline{\gamma_\lambda}(t_1, \dots, t_n)dt_1 \cdots dt_n. \end{aligned} \tag{3.55}$$

As the kernel of this operator is equal to

$$K(s,t)=\alpha_\lambda(\exp(\sum_{j=1}^n x_jX_j)\exp(\sum_{j=1}^n y_jY_j+zZ)\exp(\sum_{j=1}^n s_jX_j))\overline{\gamma_\lambda}(t_1,\dots,t_n), \tag{3.56}$$

the function f^* has to be defined by

$$\begin{aligned} f^*(\exp(\sum_{j=1}^n x_jX_j) \quad \exp \quad (\sum_{j=1}^n y_jY_j+zZ)) &= (\frac{1}{2\pi})^{2n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^n} K(s,s)ds|\lambda|d\lambda \\ &= (\frac{1}{2\pi})^{2n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \alpha_\lambda(\exp(\sum_{j=1}^n x_jX_j)\exp(\sum_{j=1}^n y_jY_j+zZ) \\ &\qquad \cdot\exp(\sum_{j=1}^n s_jX_j))\overline{\gamma_\lambda}(s)ds|\lambda|d\lambda \\ &= (\frac{1}{2\pi})^{2n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \alpha_\lambda(\exp(\sum_{j=1}^n (x_j+s_j)X_j) \\ &\qquad \cdot\exp(\sum_{j=1}^n y_jY_j+zZ-s\cdot yZ))\overline{\gamma_\lambda}(s)ds|\lambda|d\lambda \\ &= (\frac{1}{2\pi})^{2n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i\lambda z}e^{-i\lambda s\cdot y}\alpha_\lambda(x+s)\overline{\gamma}(\lambda,s)ds|\lambda|d\lambda \\ &= -(\frac{1}{2\pi})^{2n+1}\int_{-K}^{-\varepsilon}\int_{\mathbb{R}^n} e^{\frac{\lambda}{2}\sum_{j=1}^n(x_j+s_j)^2}\overline{\gamma}(\lambda,s)\lambda e^{i\lambda z}e^{-i\lambda s\cdot y}dsd\lambda. \end{aligned} \tag{3.57}$$

Let's show that this last formula really defines a Schwartz function. In fact, the function

$$F(x,s,\lambda):=e^{\frac{\lambda}{2}\sum_{j=1}^n(x_j+s_j)^2}\overline{\gamma}(\lambda,s)\lambda \tag{3.58}$$

is a Schwartz function that is zero if $\lambda\in\mathbb{R}\setminus[-K,-\varepsilon]$. So the function

$$G(x,y,\lambda):=\hat{F}^2(x,\lambda y,\lambda)=\int_{\mathbb{R}^n} F(x,s,\lambda)e^{-i\lambda y\cdot s}ds \tag{3.59}$$

is again a Schwartz function, as non-zero values are obtained only if $\lambda\in[-K,-\varepsilon]$. Finally,

$$f^*(\exp(\sum_{j=1}^n x_jX_j)\exp(\sum_{j=1}^n y_jY_j+zZ))=-(\frac{1}{2\pi})^{2n+1}\hat{G}^3(x,y,-z), \tag{3.60}$$

with $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$, is also a Schwartz function. By construction,

$$\pi_\lambda(f^*)=P_{\alpha_\lambda,\gamma_\lambda} \tag{3.61}$$

and this completes the proof. □

3.4

Let's finish this section by several remarks:

Remark 2. **a)** *Given any* $\lambda_0 < 0$*, let's choose* ε, K *such that* $-K < \lambda_0 < -\varepsilon$ *and* γ *such that* $\gamma_{\lambda_0} \neq 0$*. Then the* $f \in \mathcal{S}(H_n)$ *constructed in the previous proposition satisfies* $f * (X_k + iY_k) = 0$ *for* $k = 1, \ldots, n$ *and* $\pi_{\lambda_0}(f) \neq 0$*. Of course this may also be done simultaneously for every* λ_0 *in a given interval* $[a, b] \subset]-\infty, 0[$*.*
b) *If we consider the system* $f * (X_k - iY_k) = 0$ *for* $k = 1, \ldots, n$ *instead of* $f * (X_k + iY_k) = 0$*, then a similar construction leads to solutions in the Schwartz space satisfying* $\pi_\lambda(f) = 0$ *if* $\lambda < 0$ *and* $\pi_{\lambda_0}(f) \neq 0$ *for any given* $\lambda_0 > 0$*, resp. for* λ_0 *in any given interval* $[a, b] \subset]0, +\infty[$*.*

4 Weak solutions

4.1

Definition 1. *A function* $\eta \in L^2(\mathbb{R}^n)$ *is said to be a weak solution of the system* $d\pi_\lambda(X_k - iY_k)\xi = 0$ *(*$k \in \{1, \ldots, n\}$*), if*

$$< \eta, d\pi_\lambda(X_k + iY_k)\varphi >= 0, \quad \forall k \in \{1, \ldots, n\}, \tag{4.62}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$*, i. e. if*

$$\int_{\mathbb{R}^n} \eta(s_1, \ldots, s_n)\overline{[\frac{\partial}{\partial s_k}\varphi(s_1, \ldots, s_n) + \lambda s_k \varphi(s_1, \ldots, s_n)]} ds_1 \cdots ds_n = 0, \tag{4.63}$$

for all $k \in \{1, \ldots, n\}$*, for every* $\varphi \in \mathcal{S}(\mathbb{R}^n)$*. If* $\eta \in \mathcal{S}(\mathbb{R}^n)$*, then this system is of course equivalent to* $< d\pi_\lambda(X_k - iY_k)\eta, \varphi >= 0$ *(*$k \in \{1, \ldots, n\}$*).*

4.2

The following arguments will show that the weak solutions coincide, in fact, with the strong solutions, if these are considered as L^2-functions. First we shall show the result for the three-dimensional Heisenberg group H_1, and then we shall generalize to H_n.

Lemma 4.1. *On* $\mathbb{R}$ *we have*

$$\{\varphi' \mid \varphi \in \mathcal{C}_c^\infty(\mathbb{R})\} = \{\chi \in \mathcal{C}_c^\infty(\mathbb{R}) \mid \int_{\mathbb{R}} \chi(s) ds = 0\}. \tag{4.64}$$

Proof. If $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, then $\int_{\mathbb{R}} \varphi'(s) ds = 0$. Conversely, if $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\mathrm{supp}\chi \subset [a, b]$ and $\int_{\mathbb{R}} \chi(s) ds = 0$, then the function φ defined by $\varphi(x) = \int_{-\infty}^x \chi(s) ds$ is in $\mathcal{C}_c^\infty(\mathbb{R})$ (as $\mathrm{supp}\varphi \subset [a, b]$) and $\varphi'(x) = \chi(x)$ for all $x \in \mathbb{R}$. □

Lemma 4.2. *Let* $\varphi, \varphi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ *be such that* $\int_{\mathbb{R}} \varphi_0(s) ds = 1$*. Then the function*

$$\chi = \varphi - \big(\int_{\mathbb{R}} \varphi(s) ds\big)\varphi_0 \tag{4.65}$$

is in $\mathcal{C}_c^\infty(\mathbb{R})$ *and satisfies* $\int_{\mathbb{R}} \chi(s) ds = 0$*.*

Proof. Easy computation. □

4.3

Let now $H_1 = \exp \mathfrak{h}_1$ with $\mathfrak{h}_1 =< X, Y, Z >$ and $[X, Y] = Z$ be the three-dimensional Heisenberg group. We then have the following result:

Proposition 4.1. *The weak solutions of*

$$d\pi_\lambda(X - iY)\xi = 0 \tag{4.66}$$

coincide with the functions of the form

$$\eta_\lambda(s) = C(\lambda)e^{\frac{\lambda}{2}s^2} \quad \text{almost everywhere if } \lambda < 0, \tag{4.67}$$

where $C(\lambda)$ is a constant, and

$$\eta_\lambda(s) \equiv 0 \quad \text{almost everywhere if } \lambda > 0. \tag{4.68}$$

Proof. It is easy to check that the functions η_λ as described above are weak solutions. Conversely, let's assume that $\eta \in L^2(\mathbb{R})$ is a weak solution. Let's define $\psi(s) = \eta(s)e^{-\frac{\lambda}{2}s^2}$. Then, for every $\xi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned} < \psi, \frac{d}{ds}\xi > &= \int_{\mathbb{R}} \psi(s)\frac{d}{ds}\overline{\xi}(s)ds \\ &= \int_{\mathbb{R}} \eta(s)\big[\frac{d}{ds}\big(e^{-\frac{\lambda}{2}s^2}\overline{\xi}(s)\big) + \lambda s e^{-\frac{\lambda}{2}s^2}\overline{\xi}(s)\big]ds \\ &= - < \eta, d\pi_\lambda(X + iY)\phi > \\ &= 0, \end{aligned} \tag{4.69}$$

where $\phi(s) = e^{-\frac{\lambda}{2}s^2}\xi(s) \in \mathcal{C}_c^\infty(\mathbb{R})$, because η is a weak solution. Let now $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be defined as in Lemma 4.2 for an arbitrary real-valued $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then, by Lemma 4.1 and Lemma 4.2,

$$0 =< \psi, \varphi - \big(\int_{\mathbb{R}} \varphi(s)ds\big)\varphi_0 >=< \psi, \varphi > - < \psi, \varphi_0 >< 1, \varphi > \tag{4.70}$$

and

$$< \psi - C \cdot 1, \varphi >= 0, \tag{4.71}$$

where $C =< \psi, \varphi_0 >$. As $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ may be chosen arbitrarily (but real-valued), this proves that $\psi = C \cdot 1$ almost everywhere, i. e. that

$$\eta(s) = C \cdot e^{\frac{\lambda}{2}s^2} \tag{4.72}$$

almost everywhere. As η has to be in $L^2(\mathbb{R})$, this implies that $C = 0$ if $\lambda > 0$. □

4.4

Finally, let's generalize the previous result to H_n:

Proposition 4.2. *Let H_n be the $(2n+1)$-dimensional Heisenberg group, for n arbitrary. The weak solutions of the system*

$$d\pi_\lambda(X_k - iY_k)\xi = 0, \quad \forall k \in \{1,\dots,n\}, \tag{4.73}$$

coincide with the functions on $\mathbb{R}^n$ given by

$$\eta_\lambda(s_1,\dots,s_n) = C(\lambda)e^{\frac{\lambda}{2}\sum_{k=1}^{n} s_k^2} \quad \text{almost everywhere if } \lambda < 0, \tag{4.74}$$

where $C(\lambda)$ is a constant depending on λ, and

$$\eta_\lambda(s_1,\dots,s_n) \equiv 0 \quad \text{almost everywhere if } \lambda > 0. \tag{4.75}$$

Proof. The result is true for $n = 1$ by Proposition 4.1. Let's assume that it is true for $n-1$ and let's prove it for n. The functions $\eta_\lambda(s_1,\dots,s_n) = C(\lambda)e^{\frac{\lambda}{2}\sum_{k=1}^{n} s_k^2}$ almost everywhere (for $\lambda < 0$) are of course weak solutions. Conversely, let $\eta \in L^2(\mathbb{R}^n)$ be a weak solution. For almost all $s_n \in \mathbb{R}$, $\eta(\cdot,\cdots,s_n) \in L^2(\mathbb{R}^{n-1})$. For every $\varphi \in \mathcal{S}(\mathbb{R})$, $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$, for every $k \in \{1,\dots,n-1\}$, we have:

$$< \eta, d\pi_\lambda(X_k + iY_k)\psi \otimes \varphi >= 0 \tag{4.76}$$

$$\int_{\mathbb{R}}[\int_{\mathbb{R}^{n-1}} \eta(s_1,\dots,s_{n-1},s_n)\overline{[\frac{\partial}{\partial s_k}\psi(s_1,\dots,s_{n-1}) + \lambda s_k\psi(s_1,\dots,s_{n-1})]}ds_1\cdots ds_{n-1}]\overline{\varphi(s_n)}ds_n = 0. \tag{4.77}$$

Hence, for almost every s_n,

$$\int_{\mathbb{R}^{n-1}} \eta(s_1,\dots,s_{n-1},s_n)\overline{[\frac{\partial}{\partial s_k}\psi(s_1,\dots,s_{n-1}) + \lambda s_k\psi(s_1,\dots,s_{n-1})]}ds_1\cdots ds_{n-1} = 0 \tag{4.78}$$

for every $k \in \{1,\dots,n-1\}$. By recurrence hypothesis,

$$\eta(s_1,\dots,s_{n-1},s_n) = C(\lambda,s_n)e^{\frac{\lambda}{2}\sum_{k=1}^{n-1} s_k^2} \quad \text{for almost every } \lambda < 0, \tag{4.79}$$

where $C(\lambda,s_n)$ depends on λ and s_n, and

$$\eta(s_1,\dots,s_{n-1},s_n) = 0 \quad \text{for almost every } \lambda > 0. \tag{4.80}$$

As $\eta \in L^2(\mathbb{R}^n)$, $C(\lambda,s_n)$ is an L^2-function in s_n for almost all $\lambda < 0$. Finally, for every

$\varphi \in \mathcal{S}(\mathbb{R})$, $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$, for almost all $\lambda < 0$,

$$< \eta, d\pi_\lambda(X_n + iY_n)\psi \otimes \varphi >= 0 \tag{4.81}$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} C(\lambda, s_n) \cdot e^{\frac{\lambda}{2}\sum_{k=1}^{n-1} s_k^2} \cdot \overline{\psi(s_1, \ldots, s_{n-1})} \cdot \overline{[\frac{\partial}{\partial s_n}\varphi(s_n) + \lambda s_n \varphi(s_n)]} ds_1 \cdots ds_{n-1} ds_n = 0 \tag{4.82}$$

$$\Big(\int_{\mathbb{R}} C(\lambda, s_n)\overline{[\frac{\partial}{\partial s_n}\varphi(s_n) + \lambda s_n \varphi(s_n)]} ds_n\Big) \cdot \Big(\int_{\mathbb{R}^{n-1}} e^{\frac{\lambda}{2}\sum_{k=1}^{n-1} s_k^2}\overline{\psi(s_1, \ldots, s_{n-1})} ds_1 \cdots ds_{n-1}\Big) = 0 \tag{4.83}$$

$$\int_{\mathbb{R}} C(\lambda, s_n)\overline{[\frac{\partial}{\partial s_n}\varphi(s_n) + \lambda s_n \varphi(s_n)]} ds_n = 0. \tag{4.84}$$

Hence, by the result for $n = 1$,

$$C(\lambda, s_n) = C(\lambda)e^{\frac{\lambda}{2}s_n^2} \tag{4.85}$$

and

$$\eta(s_1, \ldots, s_n) = C(\lambda)e^{\frac{\lambda}{2}\sum_{k=1}^{n} s_k^2} \tag{4.86}$$

almost everywhere, for almost all $\lambda < 0$. □

4.5

The Plancherel theorem will now be used to study the weak solutions of our system of differential equations $f * (X_k + iY_k) = 0$. These weak solutions are defined by:

Definition 2. *A function $f \in L^2(H_n)$ is said to be a weak solution of the system of equations $f * (X_k + iY_k) = 0$ ($k \in \{1, \ldots, n\}$), if, for every $\varphi \in \mathcal{S}(H_n)$,*

$$< f, \varphi * (X_k - iY_k) >= 0, \quad \forall k \in \{1, \ldots, n\}. \tag{4.87}$$

4.6

By the Plancherel theorem, we have for every weak solution f of $f * (X_k + iY_k) = 0$ ($k \in \{1, \ldots, n\}$) and every $\varphi \in \mathcal{S}(H_n)$, for every $k \in \{1, \ldots, n\}$,

$$\begin{aligned} 0 &= < f, \varphi * (X_k - iY_k) > \\ &= -\int_{\mathbb{R}^*} \operatorname{tr}\big(\pi_\lambda(f) d\pi_\lambda(X_k + iY_k)\pi_\lambda(\varphi^*)\big)|\lambda| d\lambda. \end{aligned} \tag{4.88}$$

Let's now choose a sequence of functions $f_n \in \mathcal{S}(H_n)$ which converge to f in $L^2(H_n)$ and, for $k \in \{1, \ldots, n\}$ fixed but arbitrary, let's replace φ by $f_n * (X_k + iY_k) * \varphi * \varphi^*$. Hence

$$\int_{\mathbb{R}^*} \operatorname{tr}\big(\pi_\lambda(f) d\pi_\lambda(X_k + iY_k)\pi_\lambda(\varphi)\big[\pi_\lambda(f_n) d\pi_\lambda(X_k + iY_k)\pi_\lambda(\varphi)\big]^*\big)|\lambda| d\lambda = 0. \tag{4.89}$$

But (f_n) converges to f in $L^2(H_n)$. Hence

$$\int_{\mathbb{R}^*} \operatorname{tr}\big(\pi_\lambda(f) d\pi_\lambda(X_k+iY_k)\pi_\lambda(\varphi)\big[\pi_\lambda(f) d\pi_\lambda(X_k+iY_k)\pi_\lambda(\varphi)\big]^*\big)|\lambda| d\lambda = 0 \tag{4.90}$$

and

$$\pi_\lambda(f) d\pi_\lambda(X_k+iY_k)\pi_\lambda(\varphi) = 0 \quad \text{for almost all } \lambda. \tag{4.91}$$

This may be done for every $k \in \{1,\ldots,n\}$. Let's now choose the functions φ in a countable dense subset $\mathcal{C}$ of $\mathcal{S}(H_n)$. For every $\varphi \in \mathcal{C}$, there exists a set of measure zero $N_\varphi \subset \mathbb{R}^*$ such that

$$\pi_\lambda(f) d\pi_\lambda(X_k+iY_k)\pi_\lambda(\varphi) = 0, \quad \forall\varphi\in\mathcal{C}, \forall\lambda\in\mathbb{R}^*\setminus N_\varphi, \forall k\in\{1,\ldots,n\}. \tag{4.92}$$

The set $N = \cup_{\varphi\in\mathcal{C}} N_\varphi$ has again measure zero and

$$\pi_\lambda(f) d\pi_\lambda(X_k+iY_k)\pi_\lambda(\varphi)\eta = 0, \quad \forall\lambda\in\mathbb{R}^*\setminus N, \forall\varphi\in\mathcal{C}, \forall\eta\in\mathcal{S}(\mathbb{R}^n)\equiv\mathfrak{H}_\lambda^\infty, \tag{4.93}$$

for all $k \in \{1,\ldots,n\}$. As $\pi_\lambda(\mathcal{C})\mathfrak{H}_\lambda^\infty$ is dense in $\mathfrak{H}_\lambda^\infty$, $\pi_\lambda(f) = 0$ on $\operatorname{Im}(d\pi_\lambda(X_k+iY_k))$ for all $k \in \{1,\ldots,n\}$ and, for all $\varphi, \psi \in \mathfrak{H}_\lambda^\infty$,

$$< d\pi_\lambda(X_k+iY_k)\varphi, \pi_\lambda(f^*)\psi > = < \pi_\lambda(f) d\pi_\lambda(X_k+iY_k)\varphi, \psi > = 0, \tag{4.94}$$

for all $k \in \{1,\ldots,n\}$, i.e. for all $\psi \in \mathfrak{H}_\lambda^\infty$ and for all $\lambda \in \mathbb{R}^*\setminus N$, $\pi_\lambda(f^*)\psi$ is a weak solution of the system of equations $d\pi_\lambda(X_k-iY_k)\xi = 0$ ($k \in \{1,\ldots,n\}$). By Proposition 4.2 and by the density of $\mathfrak{H}_\lambda^\infty$ in $\mathfrak{H}_\lambda$, we may then conclude that, almost everywhere,

$$\pi_\lambda(f^*)\mathfrak{H}_\lambda \subset \mathbb{C}\alpha_\lambda, \tag{4.95}$$

where $\alpha_\lambda(s) = e^{\frac{\lambda}{2}\sum_{k=1}^n s_k^2}$ if $\lambda < 0$ and zero otherwise.

Finally, the arguments of (3.2) and Remark 1 show that if $f \in L^2(H_n)$ is a weak solution of the system $f * (X_k+iY_k) = 0$ for $k = 1,\ldots,n$, then $\pi_\lambda(f) = P_{\eta_\lambda,\alpha_\lambda}$ for some $\eta_\lambda \in L^2(\mathbb{R}^n)$ for almost all $\lambda < 0$ and $\pi_\lambda(f) = 0$ for almost all $\lambda > 0$. Moreover,

$$\|f\|_2^2 = \pi^{\frac{n}{2}} \int_{\mathbb{R}^*} \|\eta_\lambda\|_2^2 |\lambda|^{1-\frac{n}{2}} d\lambda < +\infty. \tag{4.96}$$

4.7

The set of weak solutions is characterized by the following theorem:

Theorem 4.1. *Every weak solution of the system of equations $f * (X_k+iY_k) = 0$ for $k = 1,\ldots,n$ coincides with a limit, in $L^2(H_n)$, of a sequence of strong solutions of the same equations, and conversely.*

Proof. Let $f \in L^2(H_n)$ be a weak solution such that $\pi_\lambda(f) = P_{\eta_\lambda,\alpha_\lambda}$ for all $\lambda \in (\mathbb{R}^* \setminus N) \cap \mathbb{R}^*_-$ and $\pi_\lambda(f) = 0$ if $\lambda > 0$, as in (4.6). Let $g \in \mathcal{S}(H_n)$ be any strong solution constructed as in Proposition 3.1, such that $\pi_\lambda(g) = P_{\gamma_\lambda,\alpha_\lambda}$ for some $\gamma \in \mathcal{S}(\mathbb{R}^{n+1})$ with $\text{supp}\gamma(\lambda, s) \subset [-K, -\varepsilon] \times \mathbb{R}^n$, for $K, \varepsilon \in \mathbb{R}^*_+$ and γ to be determined. Then

$$\begin{aligned}
\|g - f\|_2^2 &= \int_{\mathbb{R}^*} \|\pi_\lambda(g) - \pi_\lambda(f)\|^2_{HS}|\lambda|d\lambda \\
&= \int_{\mathbb{R}^*} \|P_{(\gamma_\lambda-\eta_\lambda),\alpha_\lambda}\|^2_{HS}|\lambda|d\lambda \\
&= \pi^{\frac{n}{2}} \int_{\mathbb{R}^*_-} \|\gamma_\lambda - \eta_\lambda\|_2^2|\lambda|^{1-\frac{n}{2}}d\lambda \\
&= \pi^{\frac{n}{2}} \|\gamma(\cdot,\cdot) - \eta_\cdot(\cdot)\|^2_{L^2(\mathbb{R}^*_-\times\mathbb{R}^n,|\lambda|^{1-\frac{n}{2}}d\lambda ds)} \qquad (4.97)
\end{aligned}$$

Let now $N \in \mathbb{N}^*$ be arbitrary. Because $\eta_\cdot(\cdot) \in L^2(\mathbb{R}^*_- \times \mathbb{R}^n, |\lambda|^{1-\frac{n}{2}}d\lambda ds)$, there exists $N_1 \in \mathbb{N}^*$ (to be taken greater than N) such that

$$\int_{-\infty}^{-N_1} \|\eta_\lambda\|_2^2|\lambda|^{1-\frac{n}{2}}d\lambda + \int_{-\frac{1}{N_1}}^{0} \|\eta_\lambda\|_2^2|\lambda|^{1-\frac{n}{2}}d\lambda < \frac{1}{N^2}. \qquad (4.98)$$

Let's now build a function $\varphi_N \in \mathcal{C}^\infty_c(\mathbb{R})$ such that

$$0 \leq \varphi_N \leq 1 \qquad (4.99)$$

$$\varphi_N \equiv 1 \text{ on } [-N_1, -\frac{1}{N_1}] \qquad (4.100)$$

$$\text{supp}\varphi_N \subset [-2N_1, -\frac{1}{2N_1}] \qquad (4.101)$$

and let's put $\delta_\lambda(s) = \eta_\lambda(s)\varphi_N(\lambda) \in L^2(\mathbb{R}^*_- \times \mathbb{R}^n, |\lambda|^{1-\frac{n}{2}}d\lambda ds)$. Then $\text{supp}\delta_\cdot(\cdot) \subset [-2N_1, -\frac{1}{2N_1}] \times \mathbb{R}^n$ and

$$\begin{aligned}
\|\eta_\cdot(\cdot) &- \delta_\cdot(\cdot)\|^2_{L^2(\mathbb{R}^*_-\times\mathbb{R},|\lambda|^{1-\frac{n}{2}}d\lambda ds)} \\
&= \int_{-\infty}^{-N_1} \|\eta_\lambda\|_2^2\big(1 - \varphi_N(\lambda)\big)^2|\lambda|^{1-\frac{n}{2}}d\lambda + \int_{-\frac{1}{N_1}}^{0} \|\eta_\lambda\|_2^2\big(1 - \varphi_N(\lambda)\big)^2|\lambda|^{1-\frac{n}{2}}d\lambda \\
&\leq \int_{-\infty}^{-N_1} \|\eta_\lambda\|_2^2|\lambda|^{1-\frac{n}{2}}d\lambda + \int_{-\frac{1}{N_1}}^{0} \|\eta_\lambda\|_2^2|\lambda|^{1-\frac{n}{2}}d\lambda \\
&< \frac{1}{N^2}. \qquad (4.102)
\end{aligned}$$

Let N_2 such that $0 < N < N_1 < N_2$ be arbitrary. As one may approach δ in L^2 by Schwartz functions, there exists $\gamma \in \mathcal{S}(\mathbb{R}^{n+1})$ such that $\text{supp}\gamma \subset [-2N_2, -\frac{1}{2N_2}] \times \mathbb{R}^n$ and

$$\|\delta_\cdot(\cdot) - \gamma(\cdot,\cdot)\|_2 \leq \big[\sup\big((2N_2)^{1-\frac{n}{2}}, (\frac{1}{2N_2})^{1-\frac{n}{2}}\big)\big]^{-\frac{1}{2}}\big(\frac{1}{N}\big). \qquad (4.103)$$

Finally,

$$\begin{aligned}
\|\eta_{\cdot}(\cdot) &- \gamma(\cdot,\cdot)\|_{L^2(\mathbb{R}^*_-\times\mathbb{R}^n,|\lambda|^{1-\frac{n}{2}}d\lambda ds)} \\
&\le \|\eta_{\cdot}(\cdot)-\delta_{\cdot}(\cdot)\|_{L^2(\mathbb{R}^*_-\times\mathbb{R}^n,|\lambda|^{1-\frac{n}{2}}d\lambda ds)} + \|\delta_{\cdot}(\cdot)-\gamma(\cdot,\cdot)\|_{L^2(\mathbb{R}^*_-\times\mathbb{R}^n,|\lambda|^{1-\frac{n}{2}}d\lambda ds)} \\
&\le \frac{1}{N} + \left[\sup\left((2N_2)^{1-\frac{n}{2}},(\frac{1}{2N_2})^{1-\frac{n}{2}}\right)\right]^{\frac{1}{2}}\|\delta-\gamma\|_2 \\
&\le \frac{2}{N}. \qquad (4.104)
\end{aligned}$$

As N has been chosen arbitrarily, this completes the proof. □

5 Eigenspaces

5.1

Instead of $X_k + iY_k$ we may of course study the operators $X_k - iY_k$, or, more generally, the operators $X_k \pm iY_k + i\alpha_k Z$, for $\alpha_k \in \mathbb{C}$. This leads to the following definitions:

Definition 3. *Let* $\alpha = (\alpha_1, \dots, \alpha_n) = a + ib \in \mathbb{C}^n$ *(*$a, b \in \mathbb{R}^n$*) and* $\beta = (\beta_1, \dots, \beta_n) = c + id \in \mathbb{C}^n$ *(*$c, d \in \mathbb{R}^n$*) be arbitrary. We define the following spaces:*

$$\begin{aligned}
\mathcal{S}_{(\alpha,\beta),+}(H_n) &= \{f \in \mathcal{S}(H_n) \mid f * (X_k + iY_k + i\alpha_k Z) = -i\beta_k f,\ k = 1,\dots,n\} \\
L^2_{(\alpha,\beta),+}(H_n) &= \overline{\mathcal{S}_{(\alpha,\beta),+}(H_n)}^{L^2(H_n)} \\
\mathcal{S}_{(\alpha,\beta),-}(H_n) &= \{f \in \mathcal{S}(H_n) \mid f * (X_k - iY_k + i\alpha_k Z) = -i\beta_k f,\ k = 1,\dots,n\} \\
L^2_{(\alpha,\beta),-}(H_n) &= \overline{\mathcal{S}_{(\alpha,\beta),-}(H_n)}^{L^2(H_n)} \qquad (5.105)
\end{aligned}$$

It follows immediately from this definition, that all these spaces are invariant by left translations in $\mathcal{S}(H_n)$, resp. $L^2(H_n)$. The spaces $L^2_{(\alpha,\beta),+}(H_n)$ and $L^2_{(\alpha,\beta),-}(H_n)$ are invariant by the left regular representation.

5.2

Up to now we have in fact shown that the spaces $\mathcal{S}_{(0,0),+}(H_n)$ and $L^2_{(0,0),+}(H_n)$ are non trivial and that $L^2_{(0,0),+}(H_n)$ coincides with the set of weak solutions of the system $f * (X_k + iY_k) = 0$ for $k = 1, \dots, n$. One could show similarly the corresponding results for $\mathcal{S}_{(0,0),-}(H_n)$ and $L^2_{(0,0),-}(H_n)$. But these are more easily obtained by the following result:

Proposition 5.1. *The map*

$$\begin{aligned}
\Psi &: L^2_{(0,0),+}(H_n) \to L^2_{(0,0),-}(H_n) \\
(\Psi\varphi)(x,y,z) &= \varphi(y,x,-z) \qquad (5.106)
\end{aligned}$$

is a linear isometry between $L^2_{(0,0),+}(H_n)$ *and* $L^2_{(0,0),-}(H_n)$ *and maps* $\mathcal{S}_{(0,0),+}(H_n)$ *onto* $\mathcal{S}_{(0,0)_-}(H_n)$*.*

Proof. Simple calculation for the Schwartz functions. ☐

Corollary 5.1. *(i) The space* $L^2_{(0,0),-}(H_n)$ *coincides with the set of weak solutions of the system of equations* $f * (X_k - iY_k) = 0$.
(ii) For every $\lambda_0 > 0$ *there exists* $f \in \mathcal{S}_{(0,0),-}(H_n) \subset L^2_{(0,0),-}(H_n)$ *such that* $\pi_{\lambda_0}(f) \neq 0$. *More generally, for every* $[a,b] \subset]0,+\infty[$, *there exists* $f \in \mathcal{S}_{(0,0),-}(H_n) \subset L^2_{(0,0),-}(H_n)$ *such that* $\pi_\lambda(f) \neq 0$ *for all* $\lambda \in [a,b]$.

5.3

As a matter of fact, all the spaces $L^2_{(\alpha,\beta),\pm}(H_n)$ are isometric and isomorphic (as linear spaces). This is shown in details in the following proposition:

Proposition 5.2. *(i) Let* $\alpha = a + ib \in \mathbb{C}^n$ *and* $u = \exp(\sum_{k=1}^n (a_k X_k + b_k Y_k)) = \prod_{k=1}^n \exp(a_k X_k + b_k Y_k) \in H_n$. *Then, for any fixed* $\beta = c + id \in \mathbb{C}^n$, *the map*

$$\begin{aligned} &\Psi_{(\alpha,\beta),+} : L^2_{(0,\beta),+}(H_n) \to L^2_{(\alpha,\beta),+}(H_n) \\ &(\Psi_{(\alpha,\beta),+}\varphi) = {}^u\varphi \end{aligned} \tag{5.107}$$

where

$$ {}^u\varphi(g) = \varphi(ugu^{-1}) \tag{5.108}$$

is a linear isometry. It maps $\mathcal{S}_{(0,\beta),+}(H_n)$ *onto* $\mathcal{S}_{(\alpha,\beta),+}(H_n)$.
(ii) Let $\beta = c + id \in \mathbb{C}^n$ *and let* $\chi_\beta(x,y,z) = e^{-i(c\cdot x + d\cdot y)}$ *(where* $c \cdot x$ *and* $d \cdot y$ *denote the scalar product in* $\mathbb{R}^n$*) be the corresponding unitary character defined on* H_n. *Then the map*

$$\begin{aligned} &\Lambda_{(0,\beta),+} : L^2_{(0,0),+}(H_n) \to L^2_{(0,\beta),+}(H_n) \\ &(\Lambda_{(0,\beta),+}\varphi) = \chi_\beta \cdot \varphi \end{aligned} \tag{5.109}$$

is a linear isometry. It maps $\mathcal{S}_{(0,0),+}(H_n)$ *onto* $\mathcal{S}_{(0,\beta),+}(H_n)$.
(iii) The map $\Phi_{(\alpha,\beta),+} = \Psi_{(\alpha,\beta),+} \circ \Lambda_{(0,\beta),+}$ *is a linear isometry between* $L^2_{(0,0),+}(H_n)$ *and* $L^2_{(\alpha,\beta),+}(H_n)$. *It maps* $\mathcal{S}_{(0,0),+}(H_n)$ *onto* $\mathcal{S}_{(\alpha,\beta),+}(H_n)$.
(iv) Let $\alpha = a + ib \in \mathbb{C}^n$ *and* $v = \exp(\sum_{k=1}^n (-a_k X_k + b_k Y_k)) = \prod_{k=1}^n \exp(-a_k X_k + b_k Y_k) \in H_n$. *Then, for any* $\beta = c + id \in \mathbb{C}^n$, *the map*

$$\begin{aligned} &\Psi_{(\alpha,\beta),-} : L^2_{(0,\beta),-}(H_n) \to L^2_{(\alpha,\beta),-}(H_n) \\ &(\Psi_{(\alpha,\beta),-}\varphi) = {}^v\varphi \end{aligned} \tag{5.110}$$

where

$$ {}^v\varphi(g) = \varphi(vgv^{-1}) \tag{5.111}$$

is a linear isometry. It maps $\mathcal{S}_{(0,\beta),-}(H_n)$ *onto* $\mathcal{S}_{(\alpha,\beta),-}(H_n)$
(v) Let $\beta = c + id \in \mathbb{C}^n$ *and let* $\rho_\beta(x,y,z) = e^{-i(c\cdot x - d\cdot y)}$. *Then the map*

$$\begin{aligned} &\Lambda_{(0,\beta),-} : L^2_{(0,0),-}(H_n) \to L^2_{(0,\beta),-}(H_n) \\ &(\Lambda_{(0,\beta),-}\varphi) = \rho_\beta \cdot \varphi \end{aligned} \tag{5.112}$$

is a linear isometry. It maps $\mathcal{S}_{(0,0),-}(H_n)$ *onto* $\mathcal{S}_{(0,\beta),-}(H_n)$.
(vi) The map $\Phi_{(\alpha,\beta),-} = \Psi_{(\alpha,\beta),-} \circ \Lambda_{(0,\beta),-}$ *is a linear isometry between* $L^2_{(0,0),-}(H_n)$ *and* $L^2_{(\alpha,\beta),-}(H_n)$. *It maps* $\mathcal{S}_{(0,0),-}(H_n)$ *onto* $\mathcal{S}_{(\alpha,\beta),-}(H_n)$.

Proof. Easy calculation for the Schwartz functions. □

Corollary 5.2. *(i) The spaces* $L^2_{(\alpha,\beta),\pm}(H_n)$ *are non-trivial.*
(ii) They coincide with the set of weak solutions of the corresponding equations.

6 Eigenspace decomposition

6.1

Proposition 6.1. *For any* $\alpha, \alpha', \beta, \beta' \in \mathbb{C}^n$, *the spaces* $L^2_{(\alpha,\beta),+}(H_n)$ *and* $L^2_{(\alpha',\beta'),-}(H_n)$ *are orthogonal.*

Proof. By the Plancherel theorem applied to $f \in L^2_{(\alpha,\beta),+}(H_n)$ and to $g \in L^2_{(\alpha',\beta'),-}(H_n)$ we have

$$< f, g >= \int_{\mathbb{R}^*} \mathrm{tr} < \big(\pi_\lambda(f) \circ \pi_\lambda(g^*)\big)|\lambda| d\lambda = 0 \tag{6.113}$$

as $\pi_\lambda(f) = 0$ if $\lambda > 0$ and $\pi_\lambda(g) = 0$ if $\lambda < 0$. □

Lemma 6.1. *Let* $f \in \mathcal{C}^\infty(H_n) \cap L^2(H_n)$. *If* f *is a weak solution of* $f*(X_k+iY_k+i\alpha_k Z) = -i\beta_k f$ *for* $k = 1, \dots, n$, *then* f *is a strong solution of the same system of equations.*

Proof. For every $\varphi \in \mathcal{S}(H_n)$ we have by assumption

$$< f, \varphi * (X_k + iY_k + i\alpha_k Z)^* >= -i\beta_k < f, \varphi >, \quad k \in \{1, \dots, n\} \tag{6.114}$$

and hence that

$$< f * (X_k + iY_k + i\alpha_k Z), \varphi >= -i\beta_k < f, \varphi >, \quad k \in \{1, \dots, n\}. \tag{6.115}$$

As all the functions are continuous and as $\varphi \in \mathcal{S}(H_n)$ is arbitrary, this proves that

$$f * (X_k + iY_k + i\alpha_k Z) = -i\beta_k f, \quad k \in \{1, \dots, n\}. \tag{6.116}$$

□

Proposition 6.2. *The sums*

$$\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),+}(H_n) \quad \text{and} \quad \bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),-}(H_n) \tag{6.117}$$

are direct sums.

Proof. (i) We have to show that for any $d \in \mathbb{N}^*$, for any distinct $(\alpha^{(j)}, \beta^{(j)}) \in \mathbb{C}^{2n}$, for $j \in \{1, \ldots, d\}$, for any $f_j \in L^2_{(\alpha^{(j)},\beta^{(j)}),+}(H_n)$,

$$\sum_{j=1}^{d} f_j = 0 \Rightarrow f_i = 0 \ \forall i. \tag{6.118}$$

The proof will be a proof by recurrence. The statement is obvious if $d = 1$. Let's assume that it is true for $d-1$ functions and let's take d such functions f_j with $\sum_{j=1}^{d} f_j = 0$. Let's also write $\alpha^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_n^{(j)}) \in \mathbb{C}^n$, $\beta^{(j)} = (\beta_1^{(j)}, \ldots, \beta_n^{(j)}) \in \mathbb{C}^n$ and $D_k^{(j)} = X_k + iY_k + i\alpha_k^{(j)} Z$. Note that the operators $D_k^{(j)}$ commute for fixed k and distinct j's.
(ii) Let's first assume that we have the additional assumption $f_j, \frac{\partial f_j}{\partial z} \in \mathcal{C}^\infty(H_n) \cap L^2(H_n)$. By Lemma 6.1, f_j is a strong solution of the corresponding system, i. e.

$$f_j * (X_k + iY_k + i\alpha_k^{(j)} Z) = -i\beta_k^{(j)} f_j, \quad \forall j \in \{1, \ldots, d\}, \forall k \in \{1, \ldots, n\}. \tag{6.119}$$

Let $r \in \{1, \ldots, d-1\}$ be arbitrary. As the $(\alpha^{(j)}, \beta^{(j)})$'s are all distinct, there exists $k \in \{1, \ldots, n\}$ such that $(\alpha_k^{(r)}, \beta_k^{(r)}) \neq (\alpha_k^{(d)}, \beta_k^{(d)})$. Let's act with $D_k^{(d)} + i\beta_k^{(d)}\delta_e$ from the right, where δ_e is the Dirac measure, i. e. $f * \delta_e = f$. As $f_d * D_k^{(d)} = -i\beta_k^{(d)} f_d$,

$$\sum_{j=1}^{d-1} f_j * (D_k^{(d)} + i\beta_k^{(d)}\delta_e) = 0. \tag{6.120}$$

Moreover,

$$\begin{aligned} f_j * (D_k^{(d)} + i\beta_k^{(d)}\delta_e) &= f_j * (X_k + iY_k + i\alpha_k^{(d)} Z + i\beta_k^{(d)}\delta_e) \\ &= f_j * \big(i(\alpha_k^{(d)} - \alpha_k^{(j)})Z + i(\beta_k^{(d)} - \beta_k^{(j)})\delta_e\big) \\ &= i(\alpha_k^{(d)} - \alpha_k^{(j)})\frac{\partial f_j}{\partial z} + i(\beta_k^{(d)} - \beta_k^{(j)}) f_j \in \mathcal{C}^\infty(H_n) \cap L^2(H_n) \end{aligned} \tag{6.121}$$

by assumption. Finally, $f_j * (D_k^{(d)} + i\beta_k^{(d)}\delta_e) \in L^2_{(\alpha^{(j)},\beta^{(j)}),+}(H_n)$, as the operators

$$\big(i(\alpha_k^{(d)} - \alpha_k^{(j)})Z + i(\beta_k^{(d)} - \beta_k^{(j)}\delta_e)\big) \quad \text{and} \quad D_r^{(j)} + i\beta_r^{(j)}\delta_e \tag{6.122}$$

commute for all $r \in \{1, \ldots, n\}$ and as $f_j \in L^2_{(\alpha^{(j)},\beta^{(j)}),+}(H_n)$. By the recurrence hypothesis,

$$f_j * (D_k^{(d)} + i\beta_k^{(d)}\delta_e) = f_j * \big(i(\alpha_k^{(d)} - \alpha_k^{(j)})Z + i(\beta_k^{(d)} - \beta_k^{(j)})\delta_e\big) = 0, \tag{6.123}$$

i. e.

$$\big(\alpha_k^{(d)} - \alpha_k^{(j)}\big)\frac{\partial(f_j)}{\partial z}(x, y, z) = -\big(\beta_k^{(d)} - \beta_k^{(j)}\big)(f_j) \tag{6.124}$$

for $j \in \{1, \ldots, d-1\}$. As $f_j \in L^2(H_n) \cap \mathcal{C}^\infty(H_n)$ for all j and as $(\alpha_k^{(r)}, \beta_k^{(r)}) \neq (\alpha_k^{(d)}, \beta_k^{(d)})$, this implies that $f_r = 0$. This was done for arbitrary r and hence $f_j = 0$ for all $j \in \{1, \ldots, d-1\}$, and so also $f_d = 0$.

(iii) Let now $f_j \in L^2_{(\alpha^{(j)},\beta^{(j)}),+}(H_n)$, $(\alpha^{(j)},\beta^{(j)})$ distinct for $j \in \{1,\dots,d\}$ and $\sum_{j=1}^d f_j = 0$. For arbitrary $\varphi \in \mathcal{C}_c^\infty(H_n)$, $\sum_{j=1}^d(\varphi * f_j) = 0$, $\varphi * f_j \in L^2_{(\alpha^{(j)},\beta^{(j)}),+}(H_n)$, as $L^2_{(\alpha^{(j)},\beta^{(j)}),+}(H_n)$ is invariant by the left regular representation, and $\varphi * f_j, \frac{\partial(\varphi * f_j)}{\partial z} \in \mathcal{C}^\infty(H_n) \cap L^2(H_n)$. So, by (ii), $\varphi * f_j = 0$ for all $j \in \{1,\dots d\}$. As we may take φ in an approximate identity, we have $f_j = 0$ for all j.
(iv) This completes the proof for

$$\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),+}(H_n) \tag{6.125}$$

A similar argument deals with the case

$$\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),-}(H_n) \tag{6.126}$$

□

Corollary 6.1. *We have*

$$\begin{aligned} &\Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),+}(H_n)\Big) \oplus_\perp \Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),-}(H_n)\Big) \\ &= \bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} \Big(L^2_{(\alpha,\beta),+}(H_n) \oplus_\perp L^2_{(\alpha,\beta),-}(H_n)\Big) \end{aligned} \tag{6.127}$$

where $\oplus$ denotes a direct sum and $\oplus_\perp$ denotes an orthogonal direct sum.

6.2

In order to show that the closure of the subspace given in Corollary 6.1 is all of $L^2(H_n)$, we need a Wiener type result for L^2. It uses the following definition:

Definition 4. *Let G be a locally compact, second countable, type I, unimodular group. Let $\mathcal{V} \subset L^2(G)$ be a closed subset of $L^2(G)$. Let $\{ \xi_i \mid i \in \mathbb{N} \}$ be a countable dense subset of $\mathcal{V}$. We call support of $\mathcal{V}$ and write* $\mathrm{Supp}\mathcal{V}$ *for the set*

$$\mathrm{Supp}\mathcal{V} = \bigcup_{i\in\mathbb{N}} \{ \pi \in \hat{G} \mid \pi(\xi_i) \neq 0 \}, \tag{6.128}$$

up to a set of measure zero. Here $\pi(\xi_i)$ denotes the operator in the L^2-sense given by the Plancherel theorem. The set $\mathrm{Supp}\mathcal{V}$ *is defined up to a set of measure zero, is measurable and is independent of the countable dense subset $\{\xi_i \mid i \in \mathbb{N}\}$.*

We then have the following "Wiener type" result:

Theorem 6.1. *Let G be a locally compact, second countable, type I, unimodular group. Let $\mathcal{V} \subset L^2(G)$ be a closed subspace that is invariant by right and left translations by elements of G. If* $\mathrm{Supp}\mathcal{V} = \hat{G}$ *(up to a set of measure zero), then $\mathcal{V} = L^2(G)$.*

Proof. This is a consequence of a more general result by Sutherland ([4]). □

Corollary 6.2. *Let G be a locally compact, second countable, type I, unimodular group. Let $\mathcal{V} \subset L^2(G)$ be a closed subspace that is invariant by right and left translations by elements of G. Assume that there is a set of measure zero N in $\hat{G}$ such that for all $\pi \in \hat{G} \setminus N$ there exists $\xi \in \mathcal{V}$ such that $\pi(\xi) \neq 0$. Then $\mathcal{V} = L^2(G)$.*

We now get the main decomposition theorem of $L^2(H_n)$ into a direct sum of eigenspaces.

Theorem 6.2. *With the notations of Definition 3, we have*

$$\begin{aligned} L^2(H_n) &= \overline{\Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),+}(H_n) \Big) \oplus_\perp \Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),-}(H_n) \Big)}^{L^2(H_n)} \\ &= \overline{\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} \Big(L^2_{(\alpha,\beta),+}(H_n) \oplus_\perp L^2_{(\alpha,\beta),-}(H_n) \Big)}^{L^2(H_n)} \qquad (6.129) \end{aligned}$$

Proof. As a matter of fact, the subspace

$$\mathcal{V} = \overline{\Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),+}(H_n) \Big) \oplus_\perp \Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),-}(H_n) \Big)}^{L^2(H_n)} \qquad (6.130)$$

is invariant by left and right translations, as it is invariant by left translations and invariant by conjugations. Moreover, Supp$\mathcal{V} = \hat{H}_n$ (up to a set of measure zero) by Remark 1. We then apply Theorem 6.1. □

7 Decomposition of the left regular representation

7.1

Let ρ be the left regular representation of $L^1(H_n)$ on $L^2(H_n)$ defined by

$$\rho(f)(\xi) = f * \xi, \quad \forall f \in L^1(H_n), \forall \xi \in L^2(H_n). \qquad (7.131)$$

Let's recall that the closed subspace $L^2_{(0,0),+}(H_n)$ of $L^2(H_n)$ is stable for this representation. Let $\rho_0 = \rho|_{L^2_{(0,0),+}(H_n)}$ be the restriction of ρ to this subspace. We shall now disintegrate ρ_0 into irreducible representations. Let's first recall that for every $\xi \in L^2_{(0,0),+}(H_n)$ and almost every $\lambda < 0$, there exists $\eta_\lambda = \eta_\lambda(\xi) \in L^2(\mathbb{R}^n) \equiv \mathfrak{H}_\lambda$ such that $\pi_\lambda(\xi) = P_{\eta_\lambda,\alpha_\lambda}$, by (4.6). Then, for every $f \in L^1(H_n)$,

$$\pi_\lambda(f * \xi) = \pi_\lambda(f) P_{\eta_\lambda,\alpha_\lambda} = P_{\pi_\lambda(f)\eta_\lambda,\alpha_\lambda}, \qquad (7.132)$$

i. e. $\eta_\lambda(f * \xi) = \pi_\lambda(f)\eta_\lambda(\xi)$.
On the other hand, if $\xi \in L^2_{(0,0),+}(H_n)$ and $\lambda > 0$, then $\pi_\lambda(\xi) = 0$ almost everywhere, by (4.6).

7.2

Let's now consider the space

$$\mathfrak{H} = \int_{\mathbb{R}^*}^{\oplus} n_\lambda \mathfrak{H}_\lambda \pi^{\frac{n}{2}} |\lambda|^{1-\frac{n}{2}} d\lambda, \quad \text{with } n_\lambda = \left\{ \begin{array}{ll} 0 & \text{if } \lambda > 0 \\ 1 & \text{if } \lambda < 0 \end{array} \right. \tag{7.133}$$

and let's define the representation Π of $L^1(H_n)$ on $\mathfrak{H}$ by

$$(\Pi(f)\zeta)_\lambda = \pi_\lambda(f)\zeta_\lambda, \quad \forall f \in L^1(H_n), \forall \zeta = (\zeta_\lambda)_\lambda \in \mathfrak{H}. \tag{7.134}$$

Then the representations $(L^2_{(0,0),+}(H_n), \rho_0)$ and $(\mathfrak{H}, \Pi)$ are unitary equivalent. In fact, let's define

$$U : L^2_{(0,0),+}(H_n) \to \mathfrak{H} \tag{7.135}$$

by

$$(U\xi)_\lambda = \eta_\lambda(\xi), \tag{7.136}$$

for almost all $\lambda \in \mathbb{R}^*$, where $\eta_\lambda(\xi) = 0$ for almost all $\lambda > 0$. By eq. (4.96), U is an isometry of $L^2_{(0,0),+}(H_n)$ into $\mathfrak{H}$, as

$$\|U\xi\|_2^2 = \int_{\mathbb{R}^*} \|\eta_\lambda(\xi)\|_2^2 \pi^{\frac{n}{2}} |\lambda|^{1-\frac{n}{2}} d\lambda = \|\xi\|_2^2. \tag{7.137}$$

As this map is also linear, by construction, it is one-to-one. It is onto, as, by Proposition 3.1, every $\gamma \in \mathcal{S}(\mathbb{R}^{n+1})$ such that $\text{supp}\gamma(\lambda, s) \subset K \times \mathbb{R}^n$ where K is a compact subset of $\mathbb{R}^*_-$ is in the image of $L^2_{(0,0),+}(H_n)$ by U and as these functions are dense in $\mathfrak{H}$. The map U intertwines the representations ρ_0 and Π, as, for every $\xi \in L^2_{(0,0),+}(H_n)$ and almost every $\lambda \in \mathbb{R}^*$,

$$\begin{aligned} \big[(U \circ \rho_0(f))\xi\big]_\lambda &= \big[U(\rho_0(f)\xi)\big]_\lambda \\ &= \eta_\lambda\big(\rho_0(f)\xi\big) \\ &= \eta_\lambda(f * \xi) \\ &= \pi_\lambda(f)\eta_\lambda(\xi) \\ &= \pi_\lambda(f)\big(U\xi\big)_\lambda \\ &= \big[\Pi(f)(U\xi)\big]_\lambda. \end{aligned} \tag{7.138}$$

So ρ_0 is unitary equivalent to

$$\Pi = \int_{\mathbb{R}^*}^{\oplus} n_\lambda \pi_\lambda \pi^{\frac{n}{2}} |\lambda|^{1-\frac{n}{2}} d\lambda, \quad \text{with } n_\lambda = \left\{ \begin{array}{ll} 0 & \text{if } \lambda > 0 \\ 1 & \text{if } \lambda < 0 \end{array} \right. \tag{7.139}$$

and we have a disintegration of $\rho_0 = \rho|_{L^2_{(0,0),+}(H_n)}$ into the irreducible representations π_λ with multiplicities 1 (for almost all $\lambda < 0$) and 0 (for almost all $\lambda > 0$).

Remark 3. **a)** *If we disintegrate similarly* $\rho|_{L^2_{(0,0),-}(H_n)}$, *we get multiplicities* 1 *if* $\lambda > 0$ *and* 0 *if* $\lambda < 0$ *almost everywhere.*
b) *We have similar results for all the spaces* $L^2_{(\alpha,\beta),\pm}(H_n)$ *thanks to Proposition 5.2.*

This may be summarized in the final theorem:

Theorem 7.1. *The left regular representation* ρ *of* $L^1(H_n)$ *on* $L^2(H_n)$ *admits the following decomposition:*

$$\begin{aligned} L^2(H_n) &= \overline{\Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),+}(H_n)\Big) \oplus_\perp \Big(\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} L^2_{(\alpha,\beta),-}(H_n)\Big)}^{L^2(H_n)} \\ &= \overline{\bigoplus_{(\alpha,\beta)\in\mathbb{C}^{2n}} \Big(L^2_{(\alpha,\beta),+}(H_n) \oplus_\perp L^2_{(\alpha,\beta),-}(H_n)\Big)}^{L^2(H_n)} \end{aligned} \tag{7.140}$$

and the restriction of ρ *to any one of the subspaces* $L^2_{(\alpha,\beta),\pm}(H_n)$ *is given by*

$$\rho|_{L^2_{(\alpha,\beta),+}(H_n)} \cong \int_{\mathbb{R}^*}^{\oplus} n_\lambda \pi_\lambda \pi^{\frac{n}{2}} |\lambda|^{1-\frac{n}{2}} d\lambda, \quad \text{with } n_\lambda = \begin{cases} 0 & \textit{if } \lambda > 0 \\ 1 & \textit{if } \lambda < 0 \end{cases} \tag{7.141}$$

and

$$\rho|_{L^2_{(\alpha,\beta),-}(H_n)} \cong \int_{\mathbb{R}^*}^{\oplus} n_\lambda \pi_\lambda \pi^{\frac{n}{2}} |\lambda|^{1-\frac{n}{2}} d\lambda, \quad \text{with } n_\lambda = \begin{cases} 1 & \textit{if } \lambda > 0 \\ 0 & \textit{if } \lambda < 0 \end{cases} \tag{7.142}$$

This gives us a concrete realization of the "disintegration" (see introduction) of the left regular representation by showing on what subspaces of $L^2(H_n)$ the infinitely many copies of one given unitary irreducible representation may be realized.

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