Holomorphic L^p-type for sub-Laplacians on connected Lie groups

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Abstract

We study the problem of determining all connected Lie groups G which have the following property (hlp): every sub-Laplacian L on G is of holomorphic L^p-type for $1 \leq p < \infty$, $p \neq 2$. First we show that semi-simple non-compact Lie groups with finite center have this property, by using holomorphic families of representations in the class one principal series of G and the Kunze-Stein phenomenon. We then apply an L^p-transference principle, essentially due to Anker, to show that every connected Lie group G whose semi-simple quotient by its radical is non-compact has property (hlp). For the convenience of the reader, we give a self-contained proof of this transference principle, which generalizes the well-known Coifman-Weiss principle. One is thus reduced to studying those groups for which the semi-simple quotient is compact, i.e. to compact extensions of solvable Lie groups. In this article, we consider semi-direct extensions of exponential solvable Lie groups by connected compact Lie groups. It had been proved in [8] that every exponential solvable Lie group S, which has a non-$*$ regular co-adjoint orbit whose restriction to the nilradical is closed, has property (hlp), and we show here that (hlp) remains valid for compact extensions of these groups.

Contents

1 Introduction **2**

2 The semi-simple case **6**
- 2.1 Preliminaries 6
- 2.2 A holomorphic family of representations of G on mixed L^p- spaces . . 12
- 2.3 A holomorphic family of compact operators 14
- 2.4 Some consequences of the Kunze-Stein phenomenon 17
- 2.5 Proof of Theorem 1.1 18

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[*] This work has been supported by the the IHP network HARP “Harmonic Analysis and Related Problems” of the European Union

keywords: connected Lie group, sub-Laplacian, functional calculus, L^p-spectral multiplier, symmetry, holomorphic L^p-type, L^p-transference by induced representations

2001 Mathematics Subject Classification: 22E30, 22E27, 43A20

3 Transference for p-induced representations 21
3.1 p-induced representations 21
3.2 A transference principle 24

4 The case of a non-compact semi-simple factor 29

5 Compact extensions of exponential solvable Lie groups 30
5.1 Compact operators arizing in induced representations 30
5.2 Proof of Theorem 1.3 35

1 Introduction

A comprehensive discussion of the problem studied in this article, background information and references to further literature can be found in [14]. We shall therefore content ourselves in this introduction by recalling some notation and results from [14].

Let $(X, d\mu)$ be a measure space. If T is a self-adjoint linear operator on a L^2-space $L^2(X, d\mu)$, with spectral resolution $T = \int_{\mathbb{R}} \lambda dE_\lambda$, and if F is a bounded Borel function on $\mathbb{R}$, then we call F an *L^p-multiplier for* T $(1 \leq p < \infty)$, if $F(T) := \int_{\mathbb{R}} F(\lambda) dE_\lambda$ extends from $L^p \cap L^2(X, d\mu)$ to a bounded operator on $L^p(X, d\mu)$. We shall denote by $\mathcal{M}_p(T)$ the space of all L^p-multipliers for T, and by $\sigma_p(T)$ the L^p- spectrum of T. We say that T is of *holomorphic L^p-type*, if there exist some non-isolated point λ_0 in the L^2-spectrum $\sigma_2(T)$ and an open complex neighborhood $\mathcal{U}$ of λ_0 in $\mathbb{C}$, such that every $F \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})$ extends holomorphically to $\mathcal{U}$. Here, $C_\infty(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ vanishing at infinity.

Assume in addition that there exists a linear subspace $\mathcal{D}$ of $L^2(X)$ which is T-invariant and dense in $L^p(X)$ for every $p \in [1, \infty[$, and that T coincides with the closure of its restriction to $\mathcal{D}$. Then, if T is of holomorphic L^p-type, the set $\mathcal{U}$ belongs to the L^p-spectrum of T, i.e.

$$\overline{\mathcal{U}} \subset \sigma_p(T).$$

In particular,

$$\sigma_2(T) \subsetneq \sigma_p(T).$$

Throughout this article, G will denote a connected Lie group.

Let dg be a *left-invariant* Haar measure on G. If $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G on the Hilbert space $\mathcal{H} = \mathcal{H}_\pi$, then we denote the integrated representation of $L^1(G) = L^1(G, dg)$ again by π, i.e. $\pi(f)\xi := \int_G f(g)\pi(g)\xi \, dg$ for every $f \in L^1(G)$, $\xi \in \mathcal{H}$. For $X \in \mathfrak{g}$, we denote by $d\pi(X)$ the infinitesimal generator of the one-parameter group of unitary operators $t \mapsto \pi(\exp tX)$. Moreover, we shall often identify X with the corresponding *right-invariant* vector field $X^r f(g) := \lim_{t \to 0} \frac{1}{t}[f((\exp tX)g) - f(g)]$ on G and write $X = X^r$.

Let X_1, ..., X_k be elements of $\mathfrak{g}$ which generate $\mathfrak{g}$ as a Lie algebra, which just means that the corresponding right-invariant vector fields satisfy Hörmander's condition. The corresponding sub-Laplacian $L := -\sum_{j=1}^k X_j^2$ is then essentially self-adjoint on $\mathcal{D}(G) \subset$

$L^2(G)$ and hypoelliptic. Denote by $\{e^{-tL}\}_{t>0}$ the heat semigroup generated by L. Since L is right G-invariant, for every $t > 0$, e^{-tL} admits a convolution kernel h_t such that

$$e^{-tL} f = h_t * f,$$

where $*$ denotes the usual convolution product in $L^1(G)$. The function $(t, g) \mapsto h_t(g)$ is smooth on $\mathbb{R}_{>0} \times G$, since the differential operator $\frac{\partial}{\partial t} + L$ is hypoelliptic. Moreover, by [20, Theorem VIII.4.3 and Theorem V.4.2], the heat kernel h_t as well as its right-invariant derivatives admit Gaussian type estimates in terms of the Carnot-Carathéodory distance δ associated to the Hörmander system $X_1, \dots X_k$.
In particular, for every right-invariant differential operator D on G, there exist constants $c_{D,t}, C_{D,t} > 0$, such that

$$|Dh_t(g)| \le C_{D,t} e^{-c_{D,t}\delta(g,e)^2}, \qquad \text{for all } g \in G, t > 0. \tag{1.1}$$

Let now $F_0 \in \mathcal{M}_p(L)$. By duality, we may assume that $1 \le p \le 2$. With F_0, also the function $\lambda \mapsto F(\lambda) := e^{-\lambda} F_0(\lambda)$ lies in $\mathcal{M}_p(L)$, since $F(L) = e^{-L} F_0(L)$, where the heat operator e^{-L} is bounded on every $L^p(G)\,(1 \le p < \infty)$. Now by [14, Lemma 6.1], the operator $F_0(L)$ is bounded also on all the spaces $L^q(G)$, $p \le q \le p'$. Hence for every test function f on G,

$$\begin{aligned} F(L)(f) &= F_0(L)(e^{-L}(f)) = F_0(L)(h_1 * f) \\ &= F_0(L)(h_{1/2} * h_{1/2} * f) = (F_0(L)h_{1/2}) * h_{1/2} * f, \end{aligned}$$

by the right invariance of the operator $F_0(L)$. Since $h_{1/2}$ is contained in every $L^q(G)$, $1 \le q \le \infty$, in particular in $L^1(G)$, we see that the operator $F(L)$ acts by convolution from the left with the function $(F_0(L)h_{1/2}) * h_{1/2}$ which is contained in every $L^q(G)$, $p \le q \le p'$, and so are all its derivates from the right. We can thus identify the operator $F(L)$ with the C^∞-function $F(L)\delta := (F_0(L)h_{1/2}) * h_{1/2}$, i.e.

$$F(L)(f) = (F(L)\delta) * f, \quad f \in \bigcup_{p \le q \le p'} L^q(G).$$

Recall that the *modular function* Δ_G on G is defined by the equation

$$\int_G f(xg)dx = \Delta_G(g)^{-1} \int_G f(x)dx, \qquad g \in G.$$

We put:

$$\begin{aligned} \check{f}(g) &:= f(g^{-1}), \\ f^*(g) &:= \Delta_G^{-1}(g)\overline{f(g^{-1})}. \end{aligned}$$

Then $f \mapsto f^*$ is an isometric involution on $L^1(G)$, and for any unitary representation π of G, we have:

$$\pi(f)^* = \pi(f^*)\ . \tag{1.2}$$

The group G is said to be *symmetric*, if the associated group algebra $L^1(G)$ is symmetric, i.e. if every element $f \in L^1(G)$ with $f^* = f$ has a real spectrum with respect to the

involutive Banach algebra $L^1(G)$.
In this paper we consider connected Lie groups for which every sub-Laplacian is of holomorphic L^p-type. First, in the Section 2, we consider connected semi-simple Lie groups G with finite center. We construct a holomorphic family of representations $\pi_{(z)}$ of G on mixed L^p-spaces (see Section 2.2). Applying these representations to h_1, we obtain a holomorphic family of compact operators on these spaces (see Section 2.3). Using the Kunze-Stein phenomenon on semi-simple Lie groups (see Section 2.4), the eigenvectors of the operators $\pi_{(z)}(h_1)$ allow us to construct a holomorphic family of L^p-functions on G which are eigenvectors for $F(L)$, if $F \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})$. From the corresponding holomorphic family of eigenvalues we can read off that F admits a holomorphic extension in a neighborhood of some element in the spectrum of L (see Section 2.5). This gives us:

Theorem 1.1. *Let G be a non compact connected semi-simple Lie group with finite center. Then every sub-Laplacian on G is of holomorphic L^p-type, for $1 \leq p < \infty,\ p \neq 2$.*

Remark. Even if at the end of the proof, we consider only ordinary L^p-spaces, we need representations on mixed L^p-spaces. They are used to get some isometry property and then to apply the Kunze-Stein phenomenon.

In Sections 3.1 and 3.2, we discuss respectively p-induced representations and a generalization of the Coifman-Weiss transference principle [5]. We consider a separable locally compact group G, and an isometric representation ρ of a closed subgroup S of G on spaces of L^p-type, e.g. L^p-spaces $L^p(\Omega)$. Denote by $\pi_p := \mathrm{ind}_{p,S}^G \rho$ the p-induced representation of ρ. We prove, among other results, that, for any function $f \in L^1(G)$, the operator norm of $\pi_p(f)$ is bounded by the norm of the convolution operator $\lambda_G(f)$ on $L^p(G)$, provided the group S is amenable. Here, λ_G denotes the left-regular representation. It should be noted that we do not require the group G to be amenable. As an application we obtain the L^p-transference of a convolution operator on G to a convolution operator on the quotient group G/S, in the case where S is an amenable closed, normal subgroup.
When preparing this article, we were not aware of J.-Ph. Anker's article [1] which, to a large extent, contains these transference results, and which we also recommend for further references to this topic. We are indebted to N. Lohoué for informing us on Anker's work [1] as well as on the influence of C. Herz on the development of this field (compare [9]). For the convenience of the reader, we have nevertheless decided to include our approach to these transference results, since it differs from Anker's by the use of a suitable cross section for G/S, which we feel makes the arguments a bit easier.

Applying this transference principle, we obtain the following generalization of Theorem 1.1 in Section 4:

Theorem 1.2. *Let $G = \exp \mathfrak{g}$ be a connected Lie group, and denote by $S = \exp \mathfrak{s}$ its radical. If G/S is not compact, then every sub-Laplacian on G is of holomorphic L^p-type, for any $1 \leq p < \infty,\ p \neq 2$.*

It then suffices to study connected Lie groups for which G/S is compact. In Section 5, we shall consider groups G which are the semi-direct product of a compact group K with a non-symmetric exponential solvable group S from a certain class. The exponential solvable non-symmetric Lie groups have been completely classified by Poguntke [18] (with previous contributions by Leptin, Ludwig and Boidol) in terms of a purely Lie-algebraic condition (B). Let us describe this condition, which had been first introduced by Boidol in a different context [3].

Recall that the unitary dual of S is in one to one correspondence with the space of coadjoint orbits in the dual space $\mathfrak{s}^*$ of $\mathfrak{s}$ via the Kirillov map, which associates with a given point $\ell \in \mathfrak{s}^*$ an irreducible unitary representation π_ℓ (see, e.g., [8, Section 1]).

If ℓ is an element of $\mathfrak{s}^*$, denote by

$$\mathfrak{s}(\ell) := \{X \in \mathfrak{s} |\, \ell([X,Y]) = 0, \text{ for all } Y \in \mathfrak{s}\}$$

the *stabilizer* of ℓ under the coadjoint action ad^*. Moreover, if $\mathfrak{m}$ is any Lie algebra, denote by

$$\mathfrak{m} = \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \dots$$

the descending central series of $\mathfrak{m}$, i.e. $\mathfrak{m}^2 = [\mathfrak{m}, \mathfrak{m}]$, and $\mathfrak{m}^{k+1} = [\mathfrak{m}, \mathfrak{m}^k]$. Put

$$\mathfrak{m}^\infty = \bigcap_k \mathfrak{m}^k.$$

Then $\mathfrak{m}^\infty$ is the smallest ideal $\mathfrak{k}$ in $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{k}$ is nilpotent. Put

$$\mathfrak{m}(\ell) := \mathfrak{s}(\ell) + [\mathfrak{s}, \mathfrak{s}].$$

Then we say that ℓ respectively the associated coadjoint orbit $\Omega(\ell) := \mathrm{Ad}^*(\mathrm{G})\ell$ satisfies *Boidol's condition* (B), if

$$\text{(B)} \qquad\qquad \ell\,|_{\mathfrak{m}(\ell)^\infty} \neq 0.$$

According to [18], the group S is non-symmetric if and only if there exists a coadjoint orbit satisfying Boidol's condition.

If Ω is a coadjoint orbit, and if $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$, then

$$\Omega|_{\mathfrak{n}} := \{\ell|_{\mathfrak{n}} : \ell \in \Omega\} \subset \mathfrak{n}^*$$

will denote the restriction of Ω to $\mathfrak{n}$.

We show that the methods developped in [8] can also be applied to the case of a compact extension of an exponential solvable group and thus obtain

Theorem 1.3. *Let $G = K \ltimes S$ be a semi-direct product of a compact Lie group K with an exponential solvable Lie group S, and assume that there exists a coadjoint orbit $\Omega(\ell) \subset \mathfrak{s}^*$ satisfying Boidol's condition, whose restriction to the nilradical $\mathfrak{n}$ is closed in $\mathfrak{n}^*$. Then every sub-Laplacian on G is of holomorphic L^p-type, for $1 \leq p < \infty$, $p \neq 2$.*

Remarks.

(a) A sub-Laplacian L on G is of holomorphic L^p-type if and only if every continuous bounded multiplier $F \in \mathcal{M}_p(L)$ extends holomorphically to an open neighborhood of a non-isolated point in $\sigma_2(L)$.

(b) If the restriction of a coadjoint orbit to the nilradical is closed, then the orbit itself is closed (see [8, Thm. 2.2]).

(c) What we really use in the proof is the following property of the orbit Ω :

> Ω *is closed, and for every real character* ν *of* $\mathfrak{s}$ *which does not vanish on* $\mathfrak{s}(\ell)$*, there exists a sequence* $\{\tau_n\}_n$ *of real numbers such that* $\lim_{n\to\infty}(\Omega+\tau_n\nu)=\infty$ *in the orbit space.*

This property is a consequence of the closedness of $\Omega|_{\mathfrak{n}}$. There are, however, many examples where the condition above is satisfied, so that the conclusion of the theorem still holds, even though the restriction of Ω to the nilradical is not closed (see e.g. [8, Section 7]). We do not know whether the condition above automatically holds whenever the orbit Ω is closed.
Observe that, contrary to the semisimple case, we need to consider representations on mixed L^p-spaces till the end of the proof.

In all the sequel, if M is a topological space, $C_0(M)$ will mean the space of compactly supported continuous funcions on M.
As usual, if S is a Lie group, $\mathfrak{s}$ will denote its Lie algebra.

2 The semi-simple case

2.1 Preliminaries

If E is a vector space, denote by E^* its algebraic dual. If it is real, $E_{\mathbb{C}}$ denotes its complexification. Let F be a vector subspace of E. We identify in the sequel the restriction $\lambda_{|F}$ of $\lambda \in E^*$ or $E^*_{\mathbb{C}}$ to an element of respectively F^* or $F^*_{\mathbb{C}}$.
Let G be a connected semisimple real Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a Cartan involution θ of G and denote by K the fixed point group for θ. The Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to θ is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ the -1-eigenspace in $\mathfrak{g}$ for the differential of θ, denoted again by θ.
We fix a subspace $\mathfrak{a}$ of $\mathfrak{p}$ which is maximal with respect to the condition that it is an abelian subalgebra of $\mathfrak{g}$. It is endowed with the scalar product $(\cdot,\cdot)$ given by the Killing form B, which is positive definite on $\mathfrak{p}$. By duality, we endow $\mathfrak{a}^*$ with the corresponding, induced scalar product, which we also denote by $(\cdot,\cdot)$. Let $|\cdot|$ be the associated norm on $\mathfrak{p}$ and $\mathfrak{a}^*$.
For any root $\alpha \in \mathfrak{a}^*$, we denote by $\mathfrak{g}_\alpha$ the corresponding root space, i.e. $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | \, [H,X] = \alpha(X) \text{ for } H \in \mathfrak{a}\}$. We fix a set R^+ of positive roots of $\mathfrak{a}$ in $\mathfrak{g}$. Let P

denote the corresponding minimal parabolic subgroup of G, containing $A := \exp \mathfrak{a}$, and $P = MAN$ its Langlands decomposition.

Denote by ρ the linear form on $\mathfrak{a}$ given by

$$\rho(X) := \frac{1}{2} \operatorname{tr}(\operatorname{ad} X_{|\mathfrak{n}}) \text{ for all } X \in \mathfrak{a},$$

where $\mathfrak{n}$ is the Lie algebra of N.

Let $\| \cdot \|$ denote the "norm" on G defined in [2, §2]. Recall that, for $g \in G$, $\|g\|$ is the operator norm of $\operatorname{Ad} g$ considered as an operator on $\mathfrak{g}$, endowed with the real Hilbert structure, $(X, Y) \mapsto -B(X, \theta Y)$ as scalar product. This norm is K-biinvariant and, according to [2, Lemma 2.1], satisfies the following properties:

$$\begin{array}{ll} & \| \cdot \| \text{ is a continuous and proper function on } G, \\ & \|g\| = \|\theta(g)\| = \|g^{-1}\| \geq 1; \\ (2.1) & \|xy\| \leq \|x\| \, \|y\|; \\ & \text{there exists } c_1, c_2 > 0 \text{ such that, for } Y \in \mathfrak{p}, \text{ then } e^{c_1|Y|} \leq \| \exp Y\| \leq e^{c_2|Y|}; \\ & \text{for all } \quad a \in A, n \in N, \ \|a\| \leq \|an\|. \end{array}$$

We choose a basis for $\mathfrak{a}^*$, following, for example, [6, p. 220].

Let $\alpha_1, \ldots, \alpha_r$ denote the simple roots in R^+. By the Gram-Schmidt process, one constructs from the basis $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{a}^*$ an orthonormal basis $\{\beta_1, \ldots, \beta_r\}$ of $\mathfrak{a}^*$ in a such way that, for every $j = 1, \ldots, r$, the vector space $\operatorname{Vect}\{\beta_1, \ldots, \beta_j\}$ spanned by $\{\beta_1, \ldots, \beta_j\}$ agrees with $\operatorname{Vect}\{\alpha_1, \ldots, \alpha_j\}$, and, for every $1 \leq k < j \leq r$, $(\beta_j, \alpha_k) = 0$.

Define H_j $(j = 1, \ldots, r)$ as the element of $\mathfrak{a}$ given by $\beta_k(H_j) = \delta_{jk}$ $(k = 1, \ldots, r)$, and put

$$\begin{aligned} \mathfrak{a}_j &:= \mathbb{R} H_j \\ \mathfrak{a}^j &:= \textstyle\sum_{k=1}^{j} \mathfrak{a}_k \\ R^j &:= R^+ \cap \operatorname{Vect}\{\alpha_1, \ldots, \alpha_j\} \\ R_j &:= R^j \setminus R^{j-1}, \text{ with } R^0 := \emptyset \\ \mathfrak{n}^j &:= \textstyle\sum_{\alpha \in R^j} \mathfrak{g}_\alpha \\ \mathfrak{n}_j &:= \textstyle\sum_{\alpha \in R_j} \mathfrak{g}_\alpha. \end{aligned}$$

We define, for $j = 1, \ldots, r$, the reductive Lie subalgebra $\mathfrak{m}^j$ of $\mathfrak{g}$ by setting

$$\mathfrak{m}^j := \theta(\mathfrak{n}^j) + \mathfrak{m} + \mathfrak{a}^j + \mathfrak{n}^j.$$

In this way, we obtain a finite sequence of reductive Lie subalgebras of $\mathfrak{g}$,

$$\mathfrak{m} =: \mathfrak{m}^0 \subset \mathfrak{m}^1 \subset \cdots \subset \mathfrak{m}^r = \mathfrak{g},$$

such that

$$\mathfrak{m}^j = \theta(\mathfrak{n}_j) + \mathfrak{m}^{j-1} + \mathfrak{a}_j + \mathfrak{n}_j \quad (j = 1, \ldots, r).$$

Then $\mathfrak{m}^{j-1} \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$ is a parabolic subalgebra of $\mathfrak{m}^j$.

Observe that G is a real reductive Lie group in the Harish-Chandra class (see e.g. [7, p. 58], for the definition of this class of reductive Lie groups). We can then inductively define a decreasing sequence of reductive real Lie groups M^j in the Harish-Chandra class, starting from $M^r = G$, in the following way.

Let P_j denote the parabolic subgroup of M^j corresponding to the parabolic subalgebra $\mathfrak{m}^{j-1} \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$, and $P_j = M^{j-1}A_jN_j$ its Langlands decomposition. Here A_j (resp. N_j) is the analytic subgroup of M^j with Lie algebra $\mathfrak{a}_j$ (resp. $\mathfrak{n}_j$), $M^{j-1}A_j$ is the centralizer in M^j of A_j, and

$$M^{j-1} := \bigcap_{\chi \in \mathrm{Hom}(M^{j-1}A_j, \mathbb{R}_+^\times)} \mathrm{Ker}\, \chi$$

(see e.g. [7, Theorem 2.3.1]).

Moreover, $M^{j-1}A_j$ normalizes N_j and $\theta(N_j)$, and M^{j-1} is a reductive Lie subgroup of M^j, in the Harish-Chandra class, with Lie algebra $\mathfrak{m}^{j-1}$ (see [7, Proposition 2.1.5]).

Put $K^j := M^j \cap K$ $(j = 1, \ldots, r)$. Then K^j is the maximal compact subgroup of M^j related to the Cartan involution $\theta_{|M^j}$ of M^j (see e.g. [7, Theorem 2.3.2, p. 68]). Hence, M^j is the product

$$K^jP_j = K^jM^{j-1}A_jN_j.$$

Fix invariant measures dk, dm, da, dn, dm_j, dk_j, da_j, dn_j for respectively K, M, A, N, M^j, K^j, A_j, N_j.

Choose an invariant measure dx on G such that

$$\int_G \varphi(x)\, dx = \int_{K\times A\times N} a^{2\rho}\varphi(kan)\, dkdadn, \quad \text{for all } \varphi \in C_0(G) \tag{2.2}$$

(see, e.g., [7, Proposition 2.4.2]).

We shall next recall an integral formula. Let S be a reductive Lie group in the Harish-Chandra class, and let $S = K \exp \mathfrak{p}$ be its Cartan decomposition, where K is a maximal compact subgroup of S. Let Q be a parabolic subgroup of S related to the above Cartan decomposition, and let $Q = M_QA_QN_Q$ be its Langlands decomposition.

Let $K_Q := K \cap Q = K \cap M_Q$, and put, for $k \in K$, $[k] := kK_Q$ in K/K_Q. We extend this notation to S by putting, for $s = kman$, $(k, m, a, n) \in K \times M_Q \times A_Q \times N_Q$, $[s] := kK_Q$. This is still well defined even though the representation of s in $KM_QA_QN_Q$ is not unique. In fact,

$$s = kman = k'm'a'n'$$

if and only if $a' = a$, $n' = n$, and $k' = kk_Q$, $m' = k_Q^{-1}m$ for some $k_Q \in K_Q$ (see e.g. [7, Theorem 2.3.3]). From this we see that the decomposition above becomes unique, if we require m to be in $\exp(\mathfrak{m}_Q \cap \mathfrak{p})$.

Every $s \in S$ thus admits a unique decomposition $s = kman$, with $(k, m, a, n) \in K \times \exp(\mathfrak{m}_Q \cap \mathfrak{p}) \times A_Q \times N_Q$. We then write $k_Q(s) := k$, $m_Q(s) := m$, $a_Q(s) := a$ and $n_Q(s) := n$, i.e.

$$s = k_Q(s)m_Q(s)a_Q(s)n_Q(s).$$

In particular, $[s] = k_Q(s)K_Q$.

For $y \in S$ and $k \in K$, we define $y[k] \in K/K_Q$ as follows:

$$y[k] := [yk].$$

Moreover, for any $\gamma \in \mathfrak{a}^*_{\mathbb{C}}$ and $Y \in \mathfrak{a}$, we put $(\exp Y)^\gamma := e^{\gamma(Y)}$.

Lemma 2.1. *Fix an invariant measure dk on K and let $d[k]$ denote the corresponding left invariant measure on K/K_Q. For any $y \in S$, we then have*

$$d(y[k]) = a_Q(yk)^{-2\rho_Q} d[k],$$

where $\rho_Q \in \mathfrak{a}_Q^*$ is given by $\rho_Q(X) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}X_{|\mathfrak{n}_Q})$ $(X \in \mathfrak{a}_Q)$; that is, for any $f \in C(K/K_Q)$,

$$\int_{K/K_Q} f([k]) d[k] = \int_{K/K_Q} a_Q(yk)^{-2\rho_Q} f(y[k]) d[k].$$

Proof. We follow the proof of [7, Proposition 2.5.4]. Let $f \in C(K/K_Q)$. Consider f as a right K_Q-invariant function on K. Choose $\chi \in C_0(M_Q A_Q N_Q)$ such that

$$\int_{M_Q \times A_Q \times N_Q} a^{2\rho_Q} \chi(man) dm da dn = 1$$

and $\chi(k_Q q) = \chi(q)$, for all $k_Q \in K_Q$, $q \in Q$, and put, for $s \in S$ with $s = kman$, $(k, m, a, n) \in K \times M_Q \times A_Q \times N_Q$,

$$h(s) := f(k)\chi(man).$$

Notice that the function h is well defined, independently of the chosen decomposition $s = kman$ of s, since f (respectively χ) is right (respectively left) K_Q-invariant.
Let dm, da and dn be invariant measures on M_Q, A_Q and N_Q, respectively. Choose an invariant measure dx on S such that

$$\int_S \varphi(x)\, dx = \int_{K \times M_Q \times A_Q \times N_Q} a^{2\rho_Q} \varphi(kman)\, dk dm da dn, \quad \text{for all } \varphi \in C_0(S)$$

(see, e.g., [7, Proposition 2.4.3]). Then

$$\int_K f(k) dk = \int_S h(x)\, dx.$$

On the other hand, by left invariance,

$$\int_S h(x)\, dx = \int_S h(yx)\, dx = \int_{K \times M_Q \times A_Q \times N_Q} a^{2\rho_Q} h(ykman)\, dk dm da dn.$$

Write $yk = k_Q(yk) m_Q(yk) a_Q(yk) n_Q(yk)$. Since the elements of M_Q and A_Q commute, we get

$$ykman = k_Q(yk) m_Q(yk) m a_Q(yk) a n_Q(yk)^{(ma)^{-1}} n,$$

where $n_Q(yk)^{ma} = ma n_Q(yk)(ma)^{-1} \in N_Q$, since $M_Q A_Q$ normalizes N_Q. Therefore,

$$h(ykman) = f([yk]) \chi(m_Q(yk) m a_Q(yk) a n_Q(yk)^{(ma)^{-1}} n).$$

We thus obtain, by left invariance of dm, da, dn,

$$\begin{aligned} \textstyle\int_S h(yx)\, dx &= \textstyle\int_{K \times M_Q \times A_Q \times N_Q} a^{2\rho_Q} a_Q(yk)^{-2\rho_Q} f([yk]) \chi(man)\, dk dm da dn \\ &= \textstyle\int_K a_Q(yk)^{-2\rho_Q} f(y[k])\, dk. \end{aligned}$$

The lemma follows. □

Remark. Let $P = MAN$ be a minimal parabolic subgroup of G contained in Q. If we decompose $s \in S$ via the Iwasawa decomposition $S = KAN$ as

$$s = k(s)a(s)n(s),$$

where $k(s) \in K$, $a(s) \in A$ and $n(s) \in N$, we can check that $k(s) = k_Q(s)$ and $a(s) = a(m_Q(s))\, a_Q(s)$, where $a(m_Q(s))$ lies in fact in $\exp(\mathfrak{m}_Q \cap \mathfrak{a})$. Since this space is orthogonal to $\mathfrak{a}_Q$ with respect to the scalar product $(\cdot,\cdot)$ on $\mathfrak{p}$, for any $\lambda \in \mathfrak{a}_Q^*$ we have $\lambda|_{\mathfrak{m}_Q \cap \mathfrak{a}} = 0$, hence

$$a(s)^\lambda = a_Q(s)^\lambda.$$

With these considerations, the lemma above can also be deduced, for example, from [21, Lemma 2.4.1].

We return now to our semisimple Lie group G. In the sequel, we shall use another basis of $\mathfrak{a}^*$, given as follows. For $j = 1, \ldots, r$, let ρ_j denote the element of $\mathfrak{a}_j^*$ defined by

$$\rho_j(X) := \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}X_{|\mathfrak{n}_j}) \text{ for all } X \in \mathfrak{a}_j.$$

Notice that we can identify ρ_j with the restriction $\rho_{|\mathfrak{a}_j}$ of ρ to $\mathfrak{a}_j$.
By [6, Lemma 4.1], ρ_j and β_j are scalar multiples of each other. In particular, the family $\{\rho_j\}$ is an orthogonal basis of $\mathfrak{a}^*$, and therefore of $\mathfrak{a}_{\mathbb{C}}^*$. For every $\nu \in \mathfrak{a}^*$ (resp. $\nu \in \mathfrak{a}_{\mathbb{C}}^*$), define ν_j $(j = 1, \ldots, r)$ in $\mathbb{R}$ (resp. $\mathbb{C}$) by the following:

$$\nu = \sum_{j=1}^{r} \nu_j \rho_j.$$

Recall that, for $j = 1, \ldots, r$, P_j is a parabolic subgroup of the reductive real Lie group M^j, which lies in the Harish-Chandra class. Put therefore, by taking $(S, K, Q) := (M^j, K^j, P_j)$ in the discussion above, $k_j := k_{P_j}$, $m_{j-1} := m_{P_j}$, $a_j := a_{P_j}$ and $n_j := n_{P_j}$. Then, any $g \in M^j$ has a unique decomposition $g = k_j(g)m_{j-1}(g)a_j(g)n_j(g)$, with $k_j(g) \in K^j$, $m_{j-1}(g) \in \exp(\mathfrak{m}^{j-1} \cap \mathfrak{p})$, $a_j(g) \in A_j$ and $n_j(g) \in N_j$. Notice that $\mathfrak{m}_0 \cap \mathfrak{p} = \{0\}$, i.e. $m_0(g) = e$.

Lemma 2.2. *Denote by* r_y *the right multiplication with* $y \in G$. *Let* $j \in \{1, \ldots, r\}$, $g \in M^j$ *and* $k_l \in K^l$ $(l = 1, \ldots, j)$.
We define recursively the element g_l *of* M^l, $l = 1, \ldots, j$, *starting from* $l = j$, *by putting* $g_j := g$ *and* $g_{l-1} := m_{l-1}(g_l k_l)$, *i.e.*

$$g_l = m_l \circ (r_{k_{l+1}} \circ m_{l+1}) \circ \cdots \circ (r_{k_j} \circ m_j)(g), \quad 1 \le l \le j-1.$$

Then, the following estimate holds:

$$\|\Pi_{l=j}^{1} a_l(g_l k^l)\| \le \|g\|.$$

Proof. We first show that, for $1 \le p \le j$,

$$\|g\| = \|\Pi_{l=j}^{p} a_l(g_l k_l) \cdot m_{p-1}(g_p k_p) \cdot \Pi_{l=p}^{j} n_l(g_l k_l) k_l^{-1}\|. \tag{2.3}$$

(Here the products are non-commutative products, in which the order of the factors is indicated by the order of indices.) We use an induction, starting from $p = j$. If $p = j$ and $g \in M^j$, then

$$\|g\| = \|gk_jk_j^{-1}\| = \|k_j(gk_j)m_{j-1}(gk_j)a_j(gk_j)n_j(gk_j)k_j^{-1}\|.$$

Using the left K-invariance of the norm and the fact that $a_j(gk_j) \in A_j$ and $m_{j-1}(gk_j) \in M^{j-1}$ commute, we find that

$$\|g\| = \|a_j(gk_j)m_{j-1}(gk_j)n_j(gk_j)k_j^{-1}\|,$$

so that (2.3) holds for $p = j$. Assume now, by induction, that (2.3) is true for $p+1$ in place of p, i.e.

$$\|g\| = \|\Pi_{l=j}^{p+1} a_l(g_lk_l) \cdot m_p(g_{p+1}k_{p+1}) \cdot \Pi_{l=p+1}^{j} n_l(g_lk_l)k_l^{-1}\|.$$

We then decompose

$$m_p(g_{p+1}k_{p+1})k_p = g_pk_p = k_p(g_pk_p)m_{p-1}(g_pk_p)a_p(g_pk_p)n_p(g_pk_p).$$

Since $k_p(g_pk_p) \in K^p \subset M^l$, for $p \leq l \leq j$, it commutes with $a_l(g_lk_l)$, for $l = p+1, \ldots, j$, and therefore, because of the K-invariance of $\|\cdot\|$, we have

$$\|g\| = \|\Pi_{l=j}^{p+1} a_l(g_lk_l) \cdot m_{p-1}(g_pk_p)a_p(g_pk_p)n_p(g_pk_p)k_p^{-1} \cdot \Pi_{l=p+1}^{j} n_l(g_lk_l)k_l^{-1}\|.$$

Moreover, $a_p(g_pk_p)$ commutes with $m_{p-1}(g_pk_p)$, and so (2.3) follows.
Applying now (2.3) for $p = 1$, we obtain

$$\|\Pi_{l=j}^{1} a_l(g_lk_l)\Pi_{l=1}^{j} n_l(g_lk_l)k_l^{-1}\| = \|g\|. \tag{2.4}$$

By right K-invariance of the norm, the left hand side of this equation is equal to

$$\|\Pi_{l=j}^{1} a_l(g_lk_l)\Pi_{l=1}^{j} n_l(g_lk_l)k_l^{-1}\Pi_{l'=j}^{1} k_{l'}\|.$$

Notice that we can write $\Pi_{l=1}^{j} n_l(g_lk_l)k_l^{-1}\Pi_{l'=j}^{1} k_{l'}$ as follows:

$$n_1(g_1k_1)\big(k_1^{-1}n_2(g_2k_2)k_1\big)\big((k_2k_1)^{-1}n_3(g_3k_3)k_2k_1\big)\cdots\big((\Pi_{l=j-1}^{1} k_l)^{-1}n_j(g_jk_j)\Pi_{l'=j-1}^{1} k_{l'}\big).$$

Since $(\Pi_{l'=p-1}^{1} k_{l'})^{-1}$, $2 \leq p \leq j$, lies in $K^{p-1} \subset M^{p-1}$ and thus normalizes N_p, we get that

$$\Pi_{l=1}^{j} n_l(g_lk_l)k_l^{-1}\Pi_{l'=j}^{1} k_{l'} \in N.$$

Using the last property of the norm given in (2.1), the left hand side of (2.4) is then greater or equal to $\|\Pi_{l=j}^{1} a_l(g_lk_l)\|$, which proves the lemma. □

2.2 A holomorphic family of representations of G on mixed L^p- spaces

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $\mathcal{M}(G,P,\nu)$ denote the space of complex valued measurable functions f on G satisfying the following covariance property:

$$f(gman) = a^{-(\nu+\rho)}f(g) \text{ for all } g \in G,\ m \in M,\ a \in A,\ n \in N.$$

The space $\mathcal{M}(G,P,\nu)$ is endowed with the left regular action of G, denoted by $\tilde{\pi}_\nu$, i.e., $[\tilde{\pi}_\nu(g)f](g') = f(g^{-1}g')$. The representations $\tilde{\pi}_\nu$ form the class-one principal series.
Let $\mathcal{M}(K/M)$ be the space of right M-invariant measurable functions on K.
The restriction to K of functions on G gives us a linear isomorphism from $\mathcal{M}(G,P,\nu)$ onto $\mathcal{M}(K/M)$. Denote by $I_\nu : f \mapsto f_\nu$ the inverse mapping. Then $f_\nu \in \mathcal{M}(G,P,\nu)$ is given by

$$f_\nu(kan) := a^{-(\nu+\rho)}f(k) \text{ for all } k \in K,\ a \in A,\ n \in N,$$

if $G = KAN$ denotes the Iwasawa decomposition of G.
If we intertwine the representation $\tilde{\pi}_\nu$ with I_ν, we obtain a representation π_ν of G on $\mathcal{M}(K/M)$, given by

$$(\pi_\nu(g)f)_\nu = \tilde{\pi}_\nu(g)f_\nu, \text{ if } f \in \mathcal{M}(K/M),\ g \in G.$$

Denote by $d\dot{k}_j$, for $j = 1,\dots,r$, the quotient measure on K^j/K^{j-1} coming from dk_j. It is invariant by left translations. Notice that $K^{j-1} = K^j \cap M^{j-1}$.
We choose a left invariant measure $d\dot{k}$ on K/M such that, for any $f \in C(K/M)$,

$$\int_{K/M} f(k)d\dot{k} = \int_{K^r/K^{r-1}} \cdots \int_{K^1/M} f(k_r \cdots k_1)d\dot{k}_1 \cdots d\dot{k}_r. \tag{2.5}$$

Let $\underline{p} = (p_1,\dots,p_r) \in [1,+\infty[^r$.
One can easily see that, for every $f \in \mathcal{M}(K/M)$, $k' \in K$, the function on K^j given by

$$k \mapsto \Big(\int_{K^{j-1}/K^{j-2}} \cdots \big(\int_{K^1/M} \big|f(k'kk_{j-1}\cdots k_1)\big|^{p_1} d\dot{k}_1\big)^{p_2/p_1} \cdots d\dot{k}_{j-1}\Big)^{1/p_{j-1}},$$

is right K^{j-1}-invariant.
We can thus define the mixed L^p-space $L^{\underline{p}}(K/M)$, as the space of all (equivalent classes of) functions f in $\mathcal{M}(K/M)$ whose mixed L^p-norm

$$\|f\|_{\underline{p}} := \Big(\int_{K^r/K^{r-1}} \cdots \big(\int_{K^1/M} \big|f(k_r\cdots k_1)\big|^{p_1} d\dot{k}_1\big)^{p_2/p_1} \cdots d\dot{k}_r\Big)^{1/p_r}$$

is finite, endowed with this norm. This definition extends to the case where some of the p_j are infinite, by the usual modifications.
Let d denote the right G- and left K-invariant metric on G, given by

$$d(g,g') := \frac{1}{c_1}\log\|g'g^{-1}\| \quad (g,\ g' \in G),$$

where c_1 is the positive constant appearing in (2.1). Notice that $d(g,e)=0$ if and only if g lies in the center of G. In particular, d is not separating.
Then, for $a=\exp Y$, with $Y \in \mathfrak{a} \subset \mathfrak{p}$, and $\gamma \in \mathfrak{a}^*$, we have, in view of the fourth property of $\|\cdot\|$ in (2.1), that

$$a^\gamma = |e^{\gamma(Y)}| \leq e^{|\gamma|\,|Y|} \leq (e^{c_1|Y|})^{|\gamma|/c_1} \leq \|a\|^{\frac{|\gamma|}{c_1}} = e^{|\gamma|\,d(a,e)}. \tag{2.6}$$

Proposition 2.1. *For every $f \in L^{\underline{p}}(K/M)$ and $g \in G$, we have*

$$\|\pi_\nu(g)f\|_{\underline{p}} \leq e^{|\sum_j(\frac{2}{p_j}-\operatorname{Re}\nu_j-1)\rho_j|d(g,e)}\|f\|_{\underline{p}}.$$

Thus π_ν defines a representation $\pi_\nu^{\underline{p}}$ of G on $L^{\underline{p}}(K/M)$. Furthermore, this gives us an analytic family $\{\pi_\nu^{\underline{p}}\}_{\nu\in\mathfrak{a}^*_{\mathbb{C}}}$ of representations of G on $L^{\underline{p}}(K/M)$.

Before giving the proof, we show the following statement. We keep the same notations as in Lemma 2.2.

Lemma 2.3. *Let $g \in M^j, k \in K$ and $f_\nu \in \mathcal{M}(G,P,\nu)$. Then*

$$\begin{aligned}&\Big(\textstyle\int_{K^j/K^{j-1}}\cdots\big(\int_{K^1/M}\big|f_\nu(kgk_j\cdots k_1)\big|^{p_1}d\dot{k}_1\big)^{p_2/p_1}\cdots d\dot{k}_j\Big)^{1/p_j}\\ =&\Big(\textstyle\int_{K^j/K^{j-1}}\cdots\big(\int_{K^1/M}\big|\Pi_{l=j}^1 a_l(g_lk_l)^{-(\operatorname{Re}\nu_l+1)\rho_l}f_\nu(k\Pi_{l=j}^1k_l(g_lk_l))\big|^{p_1}d\dot{k}_1\big)^{p_2/p_1}\cdots d\dot{k}_j\Big)^{1/p_j}.\end{aligned}$$

Proof. We use induction on j. For $j=0$, one has, by right M-invariance of f and since $g\in M^0=M$, that

$$|f_\nu(kg)| = |f_\nu(k)|.$$

Assume that the statement is true for $j-1$. Observe that $a_j(gk_j)$ commutes with $k_{j-1}\cdots k_1 \in M^{j-1}$, and that $(k_{j-1}\cdots k_1)^{-1}n_j(gk_j)k_{j-1}\cdots k_1 \in N$. Therefore, the covariance property of f_ν applied to the integration over K^j/K^{j-1}, implies

$$\begin{aligned}&\Big(\textstyle\int_{K^j/K^{j-1}}\cdots\big(\int_{K^1/M}\big|f_\nu(kgk_j\cdots k_1)\big|^{p_1}d\dot{k}_1\big)^{p_2/p_1}\cdots d\dot{k}_j\Big)^{1/p_j}\\ =&\Big(\textstyle\int_{K^j/K^{j-1}}\cdots\big(\int_{K^1/M}\big|a_j(gk_j)^{-(\operatorname{Re}\nu_j+1)\rho_j}f_\nu(kk_j(gk_j)m_{j-1}(gk_j)k_{j-1}\cdots k_1)\big|^{p_1}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\dot{k}_1\big)^{p_2/p_1}\cdots d\dot{k}_j\Big)^{1/p_j}.\end{aligned}$$

The statement holds, using the induction hypothesis, since $g=g_j$ and $m_{j-1}(gk_j)=g_{j-1}\in M^{j-1}$. ☐

Proof of Proposition 2.1. If we apply (2.6) to $\gamma := \sum_{j=1}^r(\frac{2}{p_j}-\operatorname{Re}\nu_j-1)\rho_j$ and notice that the $\mathfrak{a}_l$'s are pairwise orthogonal with respect to $(\cdot,\cdot)$, we get, in view of Lemma 2.2:

$$\sup_{k_j\in K^j,\,j=1,\dots,r}\Pi_{j=1}^r a_j(g_jk_j)^{(\frac{2}{p_j}-\operatorname{Re}\nu_j-1)\rho_j} \leq \|g\|^{\frac{|\gamma|}{c_1}} = e^{|\gamma|d(g,e)}.$$

On the other hand, according to the above lemma and Lemma 2.1, applied successively to the integrations over K^j/K^{j-1}, $j=1,\dots,r$, we have

$$\|\pi_\nu(g^{-1})f\|_{\underline{p}} \leq \sup_{k_j\in K^j,\,j=1,\dots,r}(\Pi_{j=r}^1 a_j(g_jk_j)^{(\frac{2}{p_j}-\operatorname{Re}\nu_j-1)\rho_j})\|f\|_{\underline{p}}.$$

The first assertion of the proposition is now evident, since $d(g^{-1},e)=d(g,e)$.
In order to prove the analyticity of the family of representations $\pi_\nu^{\underline{p}}$, choose $\underline{p}=(p_1,\ldots,p_r)\in[1,\infty[^r$ and denote by $\underline{p}'=(p_1',\ldots,p_r')\in]1,\infty]^r$ the tuple of conjugate exponents, i.e., $1/p_j+1/p_j'=1$. Then, for $f\in L^{\underline{p}}(K/M)$, $u\in L^{\underline{p}'}(K/M)=(L^{\underline{p}}(K/M))'$ and $g\in G$, we have

$$\langle\pi_\nu^{\underline{p}}(g)f,u\rangle=\int_{K/M}(\pi_\nu^{\underline{p}}(g)f)(k)\overline{u(k)}d\dot{k}=\int_{K/M}a(g^{-1}k)^{-(\nu+\rho)}f(\kappa(g^{-1}k))\overline{u(k)}d\dot{k}.$$

Here, the functions $\kappa(\cdot),a(\cdot),n(\cdot)$ on G are given by the unique factorization $g=\kappa(g)a(g)n(g)$ of g, according to the Iwasawa decomposition $G=KAN$.
Obviously, the expression above is analytic in $\nu\in\mathfrak{a}_\mathbb{C}^*$, which finishes the proof. □

For $t=(t_1,\ldots,t_r)\in]0,+\infty[^r$, let $\Omega_t:=\{\nu\in\mathfrak{a}_\mathbb{C}^*\,|\,|\mathrm{Re}\,\nu_j|<t_j \text{ for all } j=1,\ldots,r\}$. Moreover, for $p\geq 0$, let $\overline{p}:=(p,\ldots,p)\in\mathbb{R}^r$.

Proposition 2.2.

(i) *For all* $\underline{p}\in[1,+\infty[^r$, $f\in L^{\underline{p}}(K/M)$, $\nu\in\Omega_t$, $g\in G$, *we have*

$$\|\pi_\nu^{\underline{p}}(g)f\|_{\underline{p}}\leq e^{\sum_j(t_j+1)|\rho_j|\,d(g,e)}\|f\|_{\underline{p}}.$$

(ii) *Let* $\nu\in\mathfrak{a}_\mathbb{C}^*$, *and let* $\underline{q}$ *be an element of* $[1,+\infty[^r$ *satisfying*

$$\mathrm{Re}\,\nu_j=\frac{2}{q_j}-1,\quad j=1,\ldots,r.$$

Then, for all $g\in G$, $f\in L^{\underline{q}}(K/M)$,

$$\|\pi_\nu^{\underline{q}}(g)f\|_{\underline{q}}=\|f\|_{\underline{q}}.$$

Furthermore, for $\nu\in i\mathfrak{a}^*$, $\pi_\nu^{\overline{2}}$ *is a unitary representation of* G.

Proof. (i) results immediately from the estimate given in Proposition 2.1 and (ii) from Lemmas 2.3 and 2.1, since, for such $\underline{q}$, we have $a^{-(\mathrm{Re}\,\nu_l+1)\rho_l}=a^{-2\rho_l/q_l}$, if $a\in A_l$. □

2.3 A holomorphic family of compact operators

Let $L=-\sum_1^k X_j^2$ be a fixed sub-Laplacian on G. The estimate (1.1), in combination with the estimate in Proposition 2.2 (i), easily implies that the operator

$$\pi_\nu^{\underline{p}}(h_1)f:=\int_G h_1(x)\pi_\nu^{\underline{p}}(x)f\,dx,\ \ f\in L^{\underline{p}}(K/M),$$

is well defined and bounded on $L^{\underline{p}}(K/M)$. In fact these operators are even compact. To see this, let us put, for $\nu\in\Omega_1$ and k_1, $k_2\in K$,

$$K_\nu(k_1,k_2):=c_G\int_{M\times A\times N}a^{-\nu+\rho}h_1(k_1(man)^{-1}k_2^{-1})dmdadn,\tag{2.7}$$

where c_G is the positive constant given by $d(x^{-1})=c_G dx$ (which exists, since G is unimodular).

Lemma 2.4. *The integral in* (2.7) *is absolutely convergent and defines a continuous, right M-invariant kernel function on $K \times K$, i.e. $K_\nu(k_1 m', k_2 m') = K_\nu(k_1, k_2)$ for every $m' \in M$.*

Proof. In order to prove that the integral in (2.7) is absolutely convergent, we put

$$I := \int_{M\times A\times N} |a^{-\nu+\rho} h_1(k_1(man)^{-1}k_2^{-1})| dm da dn.$$

Then, in view of (1.1), we have

$$I \leq C \int_{M\times A\times N} a^{-\mathrm{Re}\,\nu+\rho} e^{-cd(k_1(man)^{-1}k_2^{-1},e)^2} dm da dn.$$

Using the K-bi-invariance of the norm $\|\cdot\|$ on G and the inclusion $M \subset K$, we get that

$$d(k_1(man)^{-1}k_2^{-1}, e) = d(kan, e), \text{ for any } k \in K.$$

Moreover, by (2.6) and (2.1),

$$a^{-2\rho} a^{-\mathrm{Re}\,\nu+\rho} = a^{-\mathrm{Re}\,\nu-\rho} \leq e^{|\mathrm{Re}\,\nu+\rho| d(kan,e)}.$$

We thus get, since $|\mathrm{Re}\,\nu + \rho| \leq 2\sum_j |\rho_j|$ for $\nu \in \Omega_1$,

$$I \leq C \int_{K\times A\times N} a^{2\rho} e^{2\sum_j |\rho_j| d(kan,e)} e^{-cd(kan,e)^2} dk da dn,$$

for every k_1, $k_2 \in K$, which is in fact equal to

$$C \int_G e^{2\sum_j |\rho_j| d(x,e)} e^{-cd(x,e)^2}\, dx.$$

Since G is unimodular and has exponential volume growth, it is easy to see that this integral is finite. Moreover, since the integrand in (2.7) depends continuously on k_1 and k_2, we see that K_ν is continous.
In order to prove the right M-invariance of K_ν, let $m' \in M$. One has, for any $(m, a, n) \in M \times A \times N$,

$$(man)^{m'} = m^{m'} a n^{m'}.$$

According to the invariance of dm, we then have, for any k_1, $k_2 \in K$,

$$K_\nu(k_1 m', k_2 m') = c_G \int_{M\times A\times N} a^{-\nu+\rho} h_1(k_1(man^{m'})^{-1}k_2^{-1}) dm da dn.$$

Furthermore, it is easy to check that, for any $\varphi \in C_0(N)$,

$$\int_N \varphi(n^{m'})\, dn = \int_N \varphi(n)\, dn.$$

Indeed, since $G = KAN$, there exists $\phi \in C_0(G)$ such that

$$\varphi(n) = \int_{K\times A} a^{2\rho} \phi(kan)\, dk da.$$

According to our choice of the Haar measure dx on G (c.f. (2.2)), we have

$$\int_G \phi(x)\,dx = \int_{K\times A\times N} a^{2\rho}\phi(kan)\,dkdadn = \int_N \varphi(n)\,dn,$$

and using the invariance of dx and dk, in combination with the commutation and normalization properties of $m' \in M$, we see that

$$\int_N \varphi(n)\,dn = \int_G \phi(x^{m'})\,dx = \int_{K\times A\times N} a^{2\rho}\phi(kan^{m'})\,dkdadn = \int_N \varphi(n^{m'})\,dn.$$

We thus conclude that

$$K_\nu(k_1m', k_2m') = K_\nu(k_1, k_2).$$

□

Put, for $\nu \in \Omega_1$,

$$T(\nu) := \pi_\nu(h_1).$$

Proposition 2.3. *The operator $T(\nu)$ is represented by the integral kernel K_ν, i.e.*

$$(T(\nu)f)(k_1M) = \int_{K/M} K_\nu(k_1, k)f(k)\,d\dot{k},\ \ f \in L^1(K/M). \tag{2.8}$$

In particular, $T(\nu) = \pi_\nu^{\underline{p}}(h_1)$ is a compact operator on every mixed L^p-space $L^{\underline{p}}(K/M)$, $\underline{p} \in [1, +\infty[^r$, which we then shall also denote by $T_{\underline{p}}(\nu)$, in order to indicate the space on which it acts. Moreover, the family of compact operators $\nu \mapsto T_{\underline{p}}(\nu)$ is analytic (in the sense of Kato [10]) on Ω_1.
Furthermore, for $\nu \in i\mathfrak{a}^*$, the operator $T_{\overline{2}}(\nu)$ is self-adjoint on $L^{\overline{2}}(K/M)$.

Proof. We have, by definition, for any $k_1 \in K$, that

$$(T(\nu)f)(k_1) = \int_G h_1(x)(\pi_\nu(x)f)(k_1)\,dx.$$

By invariance of dx, this is equal to

$$c_G \int_G h_1(k_1x^{-1})f_\nu(x)\,dx.$$

Now, according to our choice of dx and using the covariance property of f_ν, we obtain

$$(T(\nu)f)(k_1) = c_G \int_{K\times A\times N} a^{2\rho}a^{-(\nu+\rho)}h_1(k_1(an)^{-1}k^{-1})f_\nu(k)\,dkdadn.$$

Since dk is invariant and $M \subset K$, this can be written, using also the right M-invariance of f_ν, as follows:

$$(T(\nu)f)(k_1) = c_G \int_{K\times M\times A\times N} a^{-\nu+\rho}h_1(k_1(man)^{-1}k^{-1})f_\nu(k)\,dkdmdadn.$$

But, $f_\nu = f$ on K, and thus (2.8) follows, by Fubini's theorem.
Since the kernel K_ν is continuous on the compact space $K \times K$, by Lemma 2.4, it follows that $T_{\underline{p}}(\nu)$ is a compact operator on $L^{\underline{p}}(K/M)$, and the analytic dependence of K_ν, which is evident from (2.8), implies that, for every $\underline{p} \in [1, +\infty[^r$, the family of operators $T_{\underline{p}}(\nu)$ is analytic on Ω_1.
Finally, if $\nu \in i\mathfrak{a}^*$, then $\pi_\nu^{\underline{2}}$ is unitary, and since $h_1(x) = \overline{h_1(x^{-1})}$, we see (from (1.2)) that the operator $\pi_\nu^{\underline{2}}(h_1)$ is self-adjoint. □

2.4 Some consequences of the Kunze-Stein phenomenon

Observe that, by Hölder's inequality, for any $\underline{p} \in [1, 2]^r$ and any $\underline{q} \in [\underline{p}, \underline{p}']$, we have

$$\|f\|_{\underline{q}} \leq \|f\|_{\underline{p}'}, \quad \text{for all } f \in L^{\underline{p}'}(K/M), \tag{2.9}$$

since the compact space K/M has normalized measure 1. Therefore, $L^{\underline{p}'}(K/M)$ is a subspace of $L^{\underline{q}}(K/M)$.
Notice also that $L^p(K/M) = L^{\overline{p}}(K/M)$ and $\|\cdot\|_{\overline{p}} = \|\cdot\|_p$, by our choice of measure on K/M (c.f. (2.5)).
As a consequence of the Kunze-Stein phenomenon (see [12] and [6]), we shall prove:

Proposition 2.4. *Let* $1 < p_0 < 2$ *and* $\nu_0 \in \mathfrak{a}^* \setminus \{0\}$. *There exist* $\varepsilon > 0$ *and* $C > 0$, *such that, for any* ξ, $\eta \in L^{p_0}(K/M)$ *and* $z \in \mathbb{C}$ *with* $|\mathrm{Re}\, z| < \varepsilon$,

$$\|\langle \pi_{z\nu_0}(\cdot)\xi, \eta\rangle\|_{L^{p_0'}(G)} \leq C\|\xi\|_{p_0'}\, \|\eta\|_{p_0'}. \tag{2.10}$$

Proof. Observe that, for every $\nu \in i\mathfrak{a}^*$, the representation π_ν is unitarily equivalent to $\tilde{\pi}_\nu$. Therefore, given $\delta > 0$, we obtain from [6], that there is a constant $C_\delta > 0$, such that, for any $2 + \delta \leq r' \leq \infty$ and ξ, $\eta \in L^2(K/M)$, we have:

$$\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\|_{L^{r'}(G)} \leq C_\delta \|\xi\|_2 \|\eta\|_2, \quad \text{provided } \mathrm{Re}\, \nu = 0. \tag{2.11}$$

Indeed, in [6], this is only stated for $\nu = 0$, but the proof easily extends to arbitrary $\nu \in i\mathfrak{a}^*$.
On the other hand, we have the estimate:

$$\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\|_{L^\infty(G)} \leq \|\xi\|_{\underline{q}} \|\eta\|_{\underline{q}'}, \quad \underline{q} \in [1, +\infty[^r, \tag{2.12}$$

for any $\xi \in L^{\underline{q}}(K/M)$, $\eta \in L^{\underline{q}'}(K/M)$, provided that:

$$\mathrm{Re}\, \nu_j = \frac{2}{q_j} - 1, \; j = 1, \ldots, r. \tag{2.13}$$

This is an immediate consequence of Proposition 2.2 (ii), since $\pi_\nu^{\underline{q}}$ is isometric, if (2.13) is satisfied.
Let $\theta_0 \in]0, 1[$ be given by $\frac{2}{p_0} = 1 + \theta_0$. Since $2 \in [p_0, p_0']$ and since $\underline{q} \in [\overline{p}_0, \overline{p}_0']$, if $\underline{q}$ satisfies (2.13) and $|\mathrm{Re}\, \nu_j| \leq \theta_0$, $j = 1, \ldots, r$, we can unify (2.11) and (2.12), using

(2.9), as follows.
Given $\delta > 0$, there is a constant $C_\delta \geq 1$ such that, for all $\xi \in L^{p_0}(K/M)$, $\eta \in L^{p_0'}(K/M)$:

$$\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\|_{L^{r'}(G)} \leq C_\delta \|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{for all } r' \in [2+\delta, +\infty], \text{ if } \operatorname{Re}\nu = 0,$$

and

$$\|\langle \pi_\nu(\cdot)\xi, \eta\rangle\|_{L^{\infty}(G)} \leq \|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{if } |\operatorname{Re}\nu_j| \leq \theta_0,\ j = 1, \ldots, r.$$

If we choose $\nu = z\nu_0$, and put $\Psi_z := \langle \pi_{z\nu_0}(\cdot)\xi, \eta\rangle$, for ξ, $\eta \in L^{p_0'}(K/M)$ fixed, this implies that

$$\|\Psi_{iy}\|_{L^{r'}(G)} \leq C_\delta \|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{for all } r' \in [2+\delta, +\infty] \text{ and } y \in \mathbb{R},$$

and

$$\|\Psi_{\pm\theta_1 + iy}\|_{L^{\infty}(G)} \leq C_\delta \|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{for all } y \in \mathbb{R},$$

if we put $\theta_1 := \theta_0 / \max_{j=1,\ldots,r} |\operatorname{Re}\nu_{0,j}|$.
Since Ψ_z depends analytically on z, we can apply Stein's interpolation theorem ([19, Theorem 4.1]), and obtain that, for every $r' \geq 2+\delta$,

$$\|\Psi_z\|_{L^{q'}(G)} \leq C_\delta \|\xi\|_{p_0'}\|\eta\|_{p_0'}, \quad \text{if } |\operatorname{Re} z| \leq \theta_1 \text{ and } q' := \tfrac{r'}{1 - |\operatorname{Re} z|/\theta_1}. \tag{2.14}$$

But, since $p_0' > 2$, we can choose $\delta > 0$ and $\varepsilon > 0$ so small that $(1 - \frac{\varepsilon}{\theta_1})p_0' \geq 2 + \delta$. Then, for $|\operatorname{Re} z| \leq \varepsilon$, if we choose $r' := p_0'(1 - \frac{|\operatorname{Re} z|}{\theta_1})$ in (2.14), we have $r' \geq 2+\delta$, and hence:

$$\|\Psi_z\|_{L^{p_0'}(G)} \leq C_\delta \|\xi\|_{p_0'}\|\eta\|_{p_0'}.$$

□

2.5 Proof of Theorem 1.1

Let $p \in [1, \infty[$, $p \neq 2$. The aim is to find a non-isolated point λ_0 in the L^2-spectrum $\sigma_2(L)$ of L and an open neighbourhood $\mathcal{U}$ of λ_0 in $\mathbb{C}$ such that, if $F_0 \in C_\infty(\mathbb{R})$ is an L^p-multiplier for L, then F_0 extends holomorphically to $\mathcal{U}$. Recall that $C_\infty(\mathbb{R})$ denotes the space of continuous functions on $\mathbb{R}$ vanishing at infinity.
Since the L^2-spectrum of L is contained in $[0, +\infty[$, we may assume that $F_0 \in C_\infty([0, +\infty[)$. Moreover, according to [8, Lemma 6.1], it suffices to consider the case where $2 < p' < \infty$. As in the introduction, we can replace F_0 by the function $F = F_0 e^{-\cdot}$, so that $F(L)$ acts on the spaces $L^q(G)$, $q \in [p, p']$ by convolution with the function $F(L)\delta \in \bigcap_{q=p}^{p'} L^q(G)$. The Kunze-Stein phenomenon implies now that every L^p function defines a bounded operator on $L^2(G)$ and also on every Hilbert space $\mathcal{H}$ of any unitary representation π of G, which is weakly contained in the left regular representation. Indeed, we know that for any coefficient $c^\pi_{\xi,\eta}(x) := \langle \pi(x)\xi, \eta\rangle$ of π, we have that

$$\|c^\pi_{\xi,\eta}\|_{p'} \leq C_p \|\xi\|\|\eta\|, \qquad \xi, \eta \in \mathcal{H},$$

for some constant $C_p > 0$. Hence for $f \in L^p(G)$,

$$|\int_G f(x) c^\pi_{\xi,\eta}(x)dx| \leq \|f\|_p \|c^\pi_{\xi,\eta}\|_{p'} \leq C_p \|f\|_p \|\xi\|\|\eta\|.$$

Hence there exists a unique bounded operator $\pi(f)$ on $\mathcal{H}$, such that $\|\pi(f)\|_{\mathrm{op}} \leq C_p\|f\|_p$ and

$$\langle \pi(f)\xi, \eta\rangle = \int_G f(x) c^\pi_{\xi,\eta}(x) dx, \quad \xi, \eta \in \mathcal{H}.$$

Choosing now a sequence $(f_\nu)_\nu$ of continuous functions with compact support, which converges in the L^p-norm to $F(L)\delta$, we see that the operators $\lambda(f_\nu)$ converge in the operator norm to $\lambda(F(L)) = F(\lambda(L))$, and so for every unitary representation $(\pi, \mathcal{H})$ of G which is weakly contained in the left regular representation λ, we have that:

$$\begin{aligned}&\int_G (F(L)\delta)(x) c^\pi_{\xi,\eta}(x)\, dx = \lim_{\nu\to\infty} \int_G f_\nu(x) c^\pi_{\xi,\eta}(x)\, dx \\ &= \lim_{\nu\to\infty} \langle \pi(f_\nu)\xi, \eta\rangle = \langle \pi(F(L))\xi, \eta\rangle = \langle F(\pi(L))\xi, \eta\rangle, \qquad \xi, \eta \in \mathcal{H}.\end{aligned}$$

In particular,

$$\begin{aligned}(F(L)\delta) * (c^\pi_{\xi,\eta})\check{}\,(x) \quad &= \int_G (F(L)\delta)(y) \langle \pi(y)\xi, \pi(x)\eta\rangle\, dy \\ &= \langle F(\pi(L))\xi, \pi(x)\eta\rangle, \qquad x \in G,\ \xi, \eta \in \mathcal{H}.\end{aligned} \tag{2.15}$$

In a first step in order to find $\lambda_0 \in \mathbb{R}$ and its neighborhood $\mathcal{U}$, we choose a suitable direction ν_0 in $\mathfrak{a}^*$. To this end, let ω be the Casimir operator of G, and let $\nu \in i\mathfrak{a}^*$. Then π_ν is a unitary representation, and we can define the operator $d\pi_\nu(\omega)$ on the space of smooth vectors in $L^2(K/M)$ with respect to π_ν. Moreover, π_ν is irreducible (see [11, Theorem 1]), and therefore

$$d\pi_\nu(\omega) = \chi(\nu)\mathrm{Id},$$

where χ is a polynomial function on $\mathfrak{a}^*$, given by the Harish-Chandra isomorphism. Thus, p is in fact a quadratic form.

Choose $\nu_0 \in \mathfrak{a}^*$, $\nu_0 \neq 0$, such that $p(\nu_0) \neq 0$. Then, clearly,

$$|\chi(iy\nu_0)| \to +\infty \text{ as } y \to +\infty \text{ in } \mathbb{R}. \tag{2.16}$$

Put $p_0 := p'$. According to Proposition 2.4, there is an $\varepsilon > 0$ and a constant $C > 0$, such that (2.10) holds, for every $z \in U_1 := \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < \varepsilon\}$. Put

$$\pi_{(z)} := \pi_{z\nu_0} \text{ and } \widetilde{T}(z) := T(z\nu_0).$$

Then $(\widetilde{T}(z))_{z\in U_1}$ is an analytic family of compact operators on $L^{p_0}(K/M)$ (see Proposition 2.3).

And, by an obvious analogue to [8, Proposition 5.4], there exists an open connected neighbourhood U_{y_0} of some point iy_0 in U_1, with $y_0 \in \mathbb{R}$, and two holomorphic mappings

$$\lambda : U_{y_0} \to \mathbb{C} \text{ and } \xi : U_{y_0} \to L^{p_0'}(K/M)$$

such that, for all $z \in U_{y_0}$ and some constant $C > 0$,

$$\begin{aligned}&\widetilde{T}(z)\xi(z) = \lambda(z)\xi(z); \\ &\xi(z) \neq 0 \text{ and } \|\xi(z)\|_{p_0'} \leq C.\end{aligned} \tag{2.17}$$

Since $\pi_{(iy)}$ is unitary for every $y \in \mathbb{R}$, λ is real-valued on $U_{y_0} \cap i\mathbb{R}$.
Fix a non-trivial function η in $C^\infty(K/M)$.
Let Φ_z, $z \in U_{y_0}$, denote the function on G given by

$$\Phi_z(g) := \langle \pi_{(z)}(g^{-1})\xi(z), \eta\rangle.$$

Then $\Phi_z(g)$ depends continuously on z and g. Moreover, by (2.10) and (2.17), there exists a constant $C_0 > 0$, such that:

$$\|\Phi_z\|_{L^{p_0'}(G)} \leq C_0, \quad \text{for all } z \in U_{y_0}. \tag{2.18}$$

Thus, for any $z \in U_{y_0}$, $\Phi_z \in L^{p_0'}(G)$, and consequently $F(L)\Phi_z \in L^{p_0'}(G)$ is well-defined, since F is an $L^{p_0'}$-multiplier for L.
Put $\mu(z) := -\log \lambda(z)$ $(z \in U_{y_0})$, where log denotes the principal branch of the logarithm. Since, for $z \in U_{y_0}$, $\xi(z)$ is an eigenvector of $\widetilde{T}(z) = \pi_{(z)}(h_1)$ associated to the eigenvalue $\lambda(z)$, where h_1 is the convolution kernel of e^{-L}, one has by (2.15), for all $z \in U_{y_0} \cap i\mathbb{R}$, $g \in G$:

$$\begin{aligned} (F(L)\Phi_z)(g) &= \langle F(\pi_{(z)}(L))\xi(z), \pi_{(z)}\eta\rangle \\ &= F(\mu(z))\langle \pi_{(z)}(g^{-1})\xi(z), \eta\rangle. \end{aligned} \tag{2.19}$$

Let ψ be a fixed element of $C_0(G)$ such that:

$$\int_G \Phi_{iy_0}(x)\psi(x)\,dx \neq 0.$$

By shrinking U_{y_0}, if necessary, we may assume that $\int_G \Phi_z(x)\psi(x)\,dx \neq 0$ for all $z \in U_{y_0}$. Then, (2.19) implies that:

$$(F \circ \mu)(z) = \frac{\int_G (F(L)\Phi_z)(x)\psi(x)\,dx}{\int_G \Phi_z(x)\psi(x)\,dx}, \quad \text{for } z \in U_{y_0} \cap i\mathbb{R}. \tag{2.20}$$

Observe that the enumerator and the denominator in the right-hand side of (2.20) are holomorphic functions of $z \in U_{y_0}$. Indeed, $\langle F(L)\Phi_z, \overline{\psi}\rangle = \langle \Phi_z, F(L)^*\overline{\psi}\rangle)$, where $F(L)^*\overline{\psi} \in L^{p_0}$ and $||\Phi_z||_{L^{p_0'}} \leq C$, by (2.18). This implies that the mapping $z \mapsto \langle F(L)\Phi_z, \overline{\psi}\rangle$ is continuous, and the holomorphy of this mapping then follows easily from Fubini's and Morera's theorems.
Therefore, $F \circ \mu$ has a holomorphic extension to U_{y_0}.
Moreover, since $\omega h_1 \in L^1(G)$, in view of Proposition 2.2, the norm

$$\|\pi_{(iy)}(\omega h_1)\|_{op} \leq \|\omega h_1\|_{L^1(G)}$$

is uniformly bounded, for $y \in \mathbb{R}$. On the other hand, $\pi_{(iy)}(\omega h_1) = d\pi_{(iy)}(\omega)\pi_{(iy)}(h_1) = \chi(iy\nu_0)\pi_{(iy)}(h_1)$, and so (2.16) implies that

$$\lim_{y\to+\infty} \|\widetilde{T}(iy)\| = \lim_{y\to+\infty} \|\pi_{(iy)}(h_1)\| = 0.$$

This shows that λ is not constant, and hence, varying y_0 slightly, if necessary, we may assume that $\mu'(iy_0) \neq 0$. Then μ is a local bi-holomorphism near iy_0, which implies, in combination with (2.20), that F has a holomorphic extension to a complex neighbourhood of $\lambda_0 := \mu(iy_0) \in \mathbb{R}$.

3 Transference for p-induced representations

3.1 p-induced representations

Let G be a separable locally compact group and $S < G$ a closed subgroup.
By [15], there exists a Borel measurable cross-section $\sigma : G/S \to G$ for the homogeneous space $H := G/S$ (i.e. $\sigma(x) \in x$ for every $x \in G/S$) such that $\sigma(K)$ is relatively compact for any compact subset K of H. Then, every $g \in G$ can be uniquely decomposed as

$$g = \sigma(x)s, \quad \text{with } x \in H, s \in S.$$

We put $\Phi : H \times S \to G,\ \Phi(x, s) := \sigma(x)s$.
Then Φ is a Borel isomorphism, and we write

$$\Phi^{-1}(g) =: (\eta(g), \tau(g)).$$

Then

$$g = \sigma \circ \eta(g)\tau(g), \quad g \in G.$$

For later use, we also define

$$\begin{aligned} \tau(g, x) &:= \tau(g^{-1}\sigma(x)), \\ \eta(g, x) &:= \eta(g^{-1}\sigma(x)), \quad g \in G, x \in H\,. \end{aligned}$$

Let dg denote the left-invariant Haar measure on G, and Δ_G the modular function on G, i.e.

$$\int_G f(gh)dg = \Delta_G(h)^{-1} \int_G f(g)dg,\ h \in G\,.$$

Similarly, ds denotes the left-invariant Haar measure on S, and Δ_S its modular function.
On a locally compact measure space Z, we denote by $\mathcal{M}_b(Z)$ the space of all essentially bounded measurable functions from M to $\mathbb{C}$, and by $\mathcal{M}_0(Z)$ the subspace of all functions which have compact support, in the sense that they vanish a.e. outside a compact subset of Z. For $f \in \mathcal{M}_0(G)$, let $\tilde{f}$ be the function on G given by

$$\tilde{f}(g) := \int_S f(gs)\Delta_{G,S}(s)ds, \quad g \in G\,,$$

where we have put $\Delta_{G,S}(s) = \Delta_G(s)/\Delta_S(s)$, $s \in S$. Then $\tilde{f}$ lies in the space

$$\begin{aligned} \mathcal{E}(G, S) := \{\tilde{h} \in \mathcal{M}_b(G) : \tilde{h} &\text{ has compact support modulo } S, \text{ and} \\ \tilde{h}(gs) &= (\Delta_{G,S}(s))^{-1}\tilde{h}(g) \text{ for all } g \in G, s \in S\} \end{aligned}$$

In fact, one can show that $\mathcal{E}(G, S) = \{\tilde{f} : f \in \mathcal{M}_0(G)\}$. Moreover, one checks easily, by means of the use of a Bruhat function, that $\tilde{f} = 0$ implies $\int_G f(g)dg = 0$.
From here it follows that there exists a unique positive linear functional, denoted by $\int_{G/S} d\dot{g}$, on the space $\mathcal{E}(G, S)$, which is left-invariant under G, such that

$$\begin{aligned} \int_G f(g)dg &= \int_{G/S} \tilde{f}(g)d\dot{g} \\ &= \int_{G/S}\int_S f(gs)\Delta_{G,S}(s)ds\ d\dot{g}\,. \end{aligned} \tag{3.1}$$

By means of the cross-section σ, we can next identify the function $\tilde{h} \in \mathcal{E}(G,S)$ with the measurable function $h \in \mathcal{M}_0(H)$, given by

$$h(x) := R\tilde{h}(x) := \tilde{h}(\sigma(x)), \quad x \in H\,.$$

Notice that, given $h \in \mathcal{M}_0(H)$, the corresponding function $\tilde{h} =: R^{-1}h \in \mathcal{E}(G,S)$ is given by

$$\tilde{h}(\sigma(x)s) = h(x)\Delta_{G,S}(s)^{-1}\,.$$

The mapping $h \mapsto \int_{G/S} \tilde{h}(g)d\dot{g}$ is then a positive Radon measure on $C_0(H)$, so that there exists a unique regular Borel measure dx on $H = G/S$, such that

$$\int_{G/S} \tilde{h}(g)d\dot{g} = \int_H h(x)dx, \quad h \in C_0(H)\,. \tag{3.2}$$

Formula (3.1) can then be re-written as

$$\int_G f(g)dg = \int_H \int_S f(\sigma(x)s)\Delta_{G,S}(s)ds\; dx\,. \tag{3.3}$$

Notice that the left-invariance of $\int_{G/S} d\dot{g}$ then translates into the following quasi-invariance property of the measure dx on H:

$$\int_H h(\eta(g,x))\Delta_{G,S}(\tau(g,x))^{-1}dx = \int_H h(x)dx \;\text{ for every } g \in G\,. \tag{3.4}$$

Formula (3.3) remains valid for all $f \in L^1(G)$.
Next, let ρ be a strongly continuous isometric representation of S on a complex Banach space $(X, ||\cdot||_X)$, so that in particular

$$||\rho(s)v||_X = ||v||_X \;\text{ for every } s \in S, v \in X.$$

Fix $1 \le p < \infty$, and let $L^p(G,X;\,\rho)$ denote the Banach space of all Borel measurable functions $\tilde{\xi} : G \to X$, which satisfy the covariance condition

$$\tilde{\xi}(gs) = \Delta_{G,S}(s)^{-1/p}\rho(s^{-1})[\tilde{\xi}(g)], \quad \text{for all } g \in G, s \in S\,,$$

and have finite L^p-norm $||\tilde{\xi}||_p := \left(\int_{G/S} ||\tilde{\xi}(g)||_X^p d\dot{g}\right)^{1/p}$.
Notice that the function $g \mapsto ||\tilde{\xi}(g)||_X^p$ satisfies the covariance property of functions in $\mathcal{E}(G,S)$, so that the integral $\int_{G/S} ||\tilde{\xi}(g)||_X^p d\dot{g}$ is well-defined.

The *p-induced representation* $\pi_p = \mathrm{ind}_{p,S}^G \rho$ is then the left-regular representation $\lambda_G = \lambda$ of G acting on $L^p(G,X;\rho)$, i.e.

$$\left[\pi_p(g)\tilde{\xi}\right](g') := \tilde{\xi}(g^{-1}g'), \quad g, g' \in G,\, \tilde{\xi} \in L^p(G,X;\,\rho).$$

By means of the cross-section σ, one can realize π_p on the L^p-space $L^p(H,X)$.
To this end, given $\tilde{\xi} \in L^p(G,X;\rho)$, we define $\xi \in L^p(H,X)$ by

$$\xi(x) := \mathcal{T}\tilde{\xi}(x) := \tilde{\xi}(\sigma(x)), \quad x \in H\,.$$

Because of (3.2), $\mathcal{T} : L^p(G, X;\, \rho) \to L^p(H, X)$ is a linear isometry, with inverse

$$\mathcal{T}^{-1}\xi(\sigma(x)s) := \tilde{\xi}(\sigma(x)s) = \Delta_{G,S}(s)^{-1/p}\rho(s^{-1})[\xi(x)]\,.$$

Since, for $g \in G$, $y \in H$ and $\tilde{\xi} \in L^p(G, X;\, \rho)$,

$$\begin{aligned}
\tilde{\xi}(g^{-1}\sigma(y)) &= \tilde{\xi}(\sigma \circ \eta(g^{-1}\sigma(y))\tau(g^{-1}\sigma(y))) \\
&= \tilde{\xi}(\sigma(\eta(g, y))\tau(g, y)) \\
&= \Delta_{G,S}(\tau(g, y))^{-1/p}\rho(\tau(g, y)^{-1})\left[\tilde{\xi}(\sigma(\eta(g, y)))\right],
\end{aligned}$$

we see that the induced representation π_p can also be realized on $L^p(H, X)$, by

$$[\pi_p(g)\xi](y) = \Delta_{G,S}(\tau(g, y))^{-1/p}\rho(\tau(g, y)^{-1})\,[\xi(\eta(g, y))]\,, \tag{3.5}$$

for $g \in G, y \in H,\ \xi \in L^p(H, X)$.
Observe that $\pi_p(g)$ acts isometrically on $L^p(H, X)$, for every $g \in G$. This is immediate from the original realization of π_p on $L^p(G, X;\, \rho)$, but follows also from (3.4), in the second realization given by (3.5).

Examples 3.1.

(a) If $S \lhd G$ is a closed, normal subgroup, then $H = G/S$ is again a group, and one finds that, for a suitable normalization of the left-invariant Haar measure dx on H, we have

$$\int_G f(g)dg = \int_H \int_S f(\sigma(x)s)ds\, dx, \quad f \in L^1(G)\,.$$

In particular, $\Delta_G|_S = \Delta_S$, so that $\Delta_{G,S} = 1$ and dx in (3.3) agrees with the left-invariant Haar measure on H.
Furthermore, there exists a measurable mapping $q : H \times H \to S$,

such that

$$\sigma(x)^{-1}\sigma(y) = \sigma(x^{-1}y)q(x, y), \quad x, y \in H\,,$$

since $\sigma(x)^{-1}\sigma(y) \equiv \sigma(x^{-1}y)$ modulo S. Thus, if $g = \sigma(x)s$, then

$$\begin{aligned}
g^{-1}\sigma(y) &= s^{-1}\sigma(x)^{-1}\sigma(y) = s^{-1}\sigma(x^{-1}y)q(x, y) \\
&= \sigma(x^{-1}y)((s^{-1})^{\sigma(x^{-1}y)^{-1}}q(x, y)).
\end{aligned}$$

(Here we use the notation $s^g := gsg^{-1},\ s \in S, g \in G$.)
This shows that $\tau(g, y) = (s^{-1})^{\sigma(x^{-1}y)^{-1}}q(x, y)$ and $\eta(g, y) = x^{-1}y$. Hence π_p is given as follows:

$$[\pi_p(\sigma(x)s)\xi](y) = \rho(q(x, y)^{-1}s^{\sigma(x^{-1}y)^{-1}})\left[\xi(x^{-1}y)\right], \tag{3.6}$$

for $(x, s) \in H \times S,\ y \in H,\ \xi \in L^p(H, X)$.
We remark that it is easy to check that:

$$q(x, y)^{-1}s^{\sigma(x^{-1}y)^{-1}} = s^{\sigma(y)^{-1}\sigma(x)}q(x, y)^{-1}.$$

Notice that (3.6) does not depend on p.

(b) In the special case where $\rho = 1$ and S is normal, the induced representation $\iota = \mathrm{ind}_S^G 1$ is given by

$$[\iota(\sigma(x)s)\xi]\,(y) = \xi(x^{-1}y)\,.$$

For the integrated representation, we then have

$$\begin{aligned}
[\iota(f)\xi]\,(y) &= \int_H \int_S f(\sigma(x)s)\xi(x^{-1}y)ds\,dx \\
&= \int_H \tilde{f}(x)\xi(x^{-1}y)dx \\
&= \left[\lambda_H(\tilde{f})\xi\right](y)\,,
\end{aligned}$$

i.e.

$$\iota(f) = \lambda_H(\tilde{f})\,,$$

where

$$\tilde{f}(x) := \int_S f(\sigma(x)s)ds\,,$$

i.e. $\tilde{f}$ is the image of f under the quotient map from G onto G/S.

3.2 A transference principle

If $\xi \in L^p(H, X)$, and if $\phi : S \to \mathbb{C}$, we define the *ρ-twisted tensor product*

$$\xi \otimes_\rho^p \phi : G \to X \quad \text{by}$$
$$\left[\xi \otimes_\rho^p \phi\right](\sigma(x)s) := \phi(s)\Delta_{G,S}(s)^{-1/p}\rho(s^{-1})[\xi(x)], \quad (x,s) \in H \times S\,.$$

Let us denote by X^* the dual space of X. For any complex vector space Y, we denote by $\overline{Y}$ its complex conjugate, which, as an additive group, is the space Y, but with scalar multiplication given by $\overline{\lambda}y$, for $\lambda \in \mathbb{C}$ and $y \in Y$. In the sequel, we assume that X contains a dense, ρ- invariant subspace X_0, which embeds via an anti-linear mapping $i : X_0 \hookrightarrow \overline{X^*}$ into the complex conjugate of the dual space of X, in such a way that

$$||x|| = \sup_{\{v \in X_0 :\, ||i(v)||_{X^*}=1\}} |\langle x, v\rangle| \quad \text{for every } x \in X. \tag{3.7}$$

Here, we have put

$$\langle x, v\rangle := i(v)(x), \quad v \in X_0, x \in X.$$

Moreover, we assume that

$$||i(\rho(s)v)||_{X^*} = ||i(v)||_{X^*} \quad \text{for every } v \in X_0, s \in S, \tag{3.8}$$

and

$$\langle \rho(s)x, \rho(s)v\rangle = \langle x, v\rangle \quad \text{for every } x \in X, v \in X_0, s \in S. \tag{3.9}$$

The most important example for us will be an L^p-space $X = L^p(\Omega)$, $1 \le p < \infty$, on a measure space $(\Omega, d\omega)$, and a representation ρ of G which acts isometrically on $L^p(\Omega)$

as well as on its dual space $L^{p'}(\Omega)$ (i.e. $\frac{1}{p}+\frac{1}{p'}=1$). In this case, by interpolation, we have $||\rho(g)\xi||_r \leq ||\xi||_r$, for $|\frac{1}{r}-\frac{1}{2}| \leq |\frac{1}{p}-\frac{1}{2}|$, $g \in G$, which implies that indeed $\rho(g)$ acts isometrically on $L^r(\Omega)$, for $|\frac{1}{r}-\frac{1}{2}| \leq |\frac{1}{p}-\frac{1}{2}|$. In particular, ρ is a unitary representation on $L^2(\Omega)$. We can then choose $X_0 := L^{p'}(\Omega) \cap L^p(\Omega) \subset L^2(\Omega)$, and put

$$i(\eta)(\xi) := \int_\Omega \xi(\omega)\overline{\eta(\omega)}\, d\omega, \quad \eta \in L^{p'}(\Omega) \cap L^p(\Omega), \xi \in L^p(\Omega).$$

Notice that (3.8) and (3.9) are always satisfied, if ρ is a unitary character.

Lemma 3.1. *Let* $\phi \in L^p(S), \psi \in L^{p'}(S), \xi \in L^p(H, X_0)$ *and* $\eta \in L^{p'}(H, X_0)$, *where* $\frac{1}{p}+\frac{1}{p'}=1$. *Then, for every* $g \in G$,

$$\langle \lambda_G(g) \left(\xi \otimes^p_\rho \phi\right), \eta \otimes^{p'}_\rho \psi\rangle = \int_H \phi * \overset{\vee}{\psi}\, (\tau(g,x))\langle [\pi_p(g)\xi](x),\, \eta(x)\rangle\, dx. \tag{3.10}$$

Proof. By (3.3), we have

$$\begin{aligned}
&\langle \lambda_G(g) \left(\xi \otimes^p_\rho \phi\right), \eta \otimes^{p'}_\rho \psi\rangle \\
= \quad & \int_H \int_S \langle \xi \otimes^p_\rho \phi(g^{-1}\sigma(x)s),\, \eta \otimes^{p'}_\rho \psi(\sigma(x)s)\rangle\, \Delta_{G,S}(s) ds\, dx \\
= \quad & \int_H \int_S \langle \xi \otimes^p_\rho \phi\left(\sigma(\eta(g,x))\tau(g,x)s\right),\, \eta \otimes^{p'}_\rho \psi(\sigma(x)s)\rangle\, \Delta_{G,S}(s) ds\, dx \\
= \quad & \int_H \int_S \Delta_{G,S}(\tau(g,x)s)^{-\frac{1}{p}} \Delta_{G,S}(s)^{-\frac{1}{p'}} \phi(\tau(g,x)s)\overline{\psi}(s) \\
& \qquad\qquad \langle \rho(s^{-1}\tau(g,x)^{-1})\, [\xi(\eta(g,x))],\, \rho(s^{-1})[\eta(x)]\rangle\, \Delta_{G,S}(s)\, ds\, dx \\
= \quad & \int_H \int_S \Delta_{G,S}(\tau(g,x))^{-\frac{1}{p}} \phi(\tau(g,x)s)\overline{\psi}(s)\, ds \\
& \qquad\qquad \langle \rho(\tau(g,x)^{-1})[\xi(\eta(g,x))],\, \eta(x)\rangle\, dx\,.
\end{aligned}$$

Here, we have used that, by (3.9), $\langle \rho(s^{-1})v_1, \rho(s^{-1})v_2\rangle = \langle v_1, v_2\rangle$ for all $v_1, v_2 \in X_0$. But,

$$\int_S \phi(\tau(g,x)s)\overline{\psi}(s) ds = \int_S \phi(s)\psi(\tau(g,x)^{-1}s) ds = \phi * \overset{\vee}{\psi}\, (\tau(g,x)),$$

and

$$\Delta_{G,H}(\tau(g,x))^{-\frac{1}{p}} \langle \rho(\tau(g,x)^{-1})[\xi(\eta(g,x))], \eta(x)\rangle = \langle [\pi_p(g)\xi](x), \eta(x)\rangle\,,$$

and thus (3.10) follows. □

From now on, we shall assume that the group S is *amenable*.

Since G is separable, we can then choose an increasing sequence $\{A_j\}_j$ of compacta in S such that $A_j^{-1} = A_j$ and $S = \bigcup_j A_j$, and put

$$\phi_j = \phi_j^p := \frac{\chi_{A_j}}{|\, A_j\,|^{1/p}}, \quad \psi_j = \psi_j^{p'} := \frac{\chi_{A_j}}{|\, A_j\,|^{1/p'}},$$

where χ_A denotes the characteristic function of the subset A. Then $\overset{\vee}{\psi_j}= \psi_j$, $||\phi_j||_p = ||\psi_j||_{p'} = 1$, and, because of the amenability of S (see [16]), we have

(3.11) $\qquad \chi_j := \phi_j * \psi_j$ tends to 1, uniformly on compacta in S.

Proposition 3.1. *Let $\pi_p = \mathrm{ind}_{p,S}^G \rho$ be as before, where S is amenable, and let $\xi, \eta \in C_0(H, X_0)$.*
Then

$$\langle \pi_p(g)\xi, \eta\rangle = \lim_{j\to\infty} \langle \lambda_G(g)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle\,,$$

uniformly on compacta in G.

Proof. By Lemma 3.1,

$$\langle \lambda_G(g)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle = \int_H \chi_j(\tau(g,x))\langle [\pi_p(g)\xi](x), \eta(x)\rangle\, dx\,.$$

Fix a compact set $K = K^{-1} \subset H$ containing the supports of ξ and η, and let $Q \subset G$ be any compact set. We want to prove that $\{\tau(g,x) |\, g \in Q, x \in K\}$ is relatively compact, for then, by (3.11), we immediately see that

$$\lim_{j\to\infty} \langle \lambda_G(g)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle = \int_H \langle [\pi_p(g)\xi](x), \eta(x)\rangle dx = \langle \pi_p(g)\xi, \eta\rangle\,,$$

uniformly for $g \in Q$.
Recall that $\tau(g,x) = \tau(g^{-1}\sigma(x))$. Therefore, since $\sigma(K)$ is relatively compact, it suffices to prove that τ maps compact subsets of G into relatively compact sets in S. So, let again Q denote a compact subset of G, and put $M := Q \bmod S \subset H = G/S$. Then M is compact, so that $\overline{\sigma(M)}$ is compact in S. And, since $\tau(\sigma(x)s) = s$ for every $x \in H$, $s \in S$, we have

$$\begin{aligned} \tau(Q) &= \{s \in S |\, \sigma(x)s \in Q \text{ for some } x \in M\} \\ &= \sigma(M)^{-1}Q\,, \end{aligned}$$

which shows that $\tau(Q)$ is indeed relatively compact. □

Theorem 3.1. *For every bounded measure $\mu \in M^1(G)$, we have*

$$||\, \pi_p(\mu)\, ||_{L^p(H,X)\to L^p(H,X)} \le ||\, \lambda_G(\mu)\, ||_{L^p(G,X)\to L^p(G,X)}\,.$$

Proof. Let $\xi, \eta \in C_0(H, X_0)$. Observe first that, for $g \in G$,

$$\begin{aligned} & |\, \langle \lambda_G(g)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle\, | \\ \le\ & ||\, \lambda_G(g)(\xi \otimes_\rho^p \phi_j)\, ||_{L^p(G,X)}\ ||\, i \circ (\eta \otimes_\rho^{p'} \psi_j)\, ||_{L^{p'}(G,X^*)} \\ =\ & ||\, \xi \otimes_\rho^p \phi_j\, ||_{L^p(G,X)}\ ||\, i \circ (\eta \otimes_\rho^{p'} \psi_j)\, ||_{L^{p'}(G,X^*)}\,, \end{aligned}$$

where

$$\begin{aligned}
&||\, \xi \otimes_\rho^p \phi_j \,||^p_{L^p(G,X)} \\
= \;& \int_H \int_S |\, \phi_j(s)\,|^p \Delta_{G,S}(s)^{-1} ||\, \rho(s^{-1})[\xi(x)]\,||^p_X \Delta_{G,S}(s) ds\, dx \\
= \;& \int_S |\, \phi_j(s)\,|^p ds \quad \int_H ||\, \xi(x)\,||^p_X dx \\
= \;& ||\, \xi \,||^p_{L^p(H,X)}\,,
\end{aligned}$$

since $\rho(s^{-1})$ is isometric on X, so that

$$||\, \xi \otimes_\rho^p \phi_j \,||_{L^p(G,X)} = ||\xi||_{L^p(H,X)}, \tag{3.12}$$

and similarly, because of (3.8),

$$||\, i \circ (\eta \otimes_\rho^{p'} \psi_j)\,||_{L^{p'}(G,X^*)} = ||\, i \circ \eta \,||_{L^{p'}(H,X^*)}\,. \tag{3.13}$$

This implies

$$|\,\langle \lambda_G(g)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle| \;\leq ||\xi||_{L^p(H,X)}\; ||\, i \circ \eta \,||_{L^{p'}(H,X^*)}.$$

Therefore, if $\mu \in M^1(G)$, Proposition 3.1 implies, by the dominated convergence theorem, that

$$\langle \pi_p(\mu)\xi, \eta\rangle = \lim_{j\to\infty} \langle \lambda_G(\mu)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle\,. \tag{3.14}$$

Moreover, by (3.12) and (3.13),

$$\begin{aligned}
&|\,\langle \lambda_G(\mu)(\xi \otimes_\rho^p \phi_j), \eta \otimes_\rho^{p'} \psi_j\rangle\,| \\
&= |\int_G \langle \lambda_G(\mu)(\xi \otimes_\rho^p \phi_j),\, \eta \otimes_\rho^{p'} \psi_j\rangle\, dg| \\
&\leq ||\, \lambda_G(\mu)\,||_{L^p(G,X)\to L^p(G,X)}\, ||\xi||_{L^p(H,X)}\; ||\, i \circ \eta \,||_{L^{p'}(H,X^*)}.
\end{aligned}$$

By (3.14), we therefore obtain

$$|\,\langle \pi_p(\mu)\xi, \eta\rangle\,| \leq ||\, \lambda_G(\mu)\,||_{L^p(G,X)\to L^p(G,X)}\, ||\xi||_{L^p(H,X)}\; ||\, i \circ \eta \,||_{L^{p'}(H,X^*)}. \tag{3.15}$$

In view of (3.7), this implies the theorem, since $C_0(H, X_0)$ lies dense in $L^p(H, X)$. □

Corollary of Theorem 3.1 (Transference). *Let $X = L^p(\Omega)$. Then, for every $\mu \in M^1(G)$, we have*

$$||\, \pi_p(\mu)\,||_{L^p(H,L^p(\Omega))\to L^p(H,L^p(\Omega)))} \leq ||\, \lambda_G(\mu)\,||_{L^p(G)\to L^p(G)}\,.$$

Proof. If $X = L^p(\Omega)$ and $h \in L^p(G, X)$, then, by Fubini's theorem,

$$||\lambda_G(\mu)h||^p_{L^p(G,X)} = \int_\Omega ||\mu * h(\cdot,\omega)||^p_{L^p(G)}\, d\omega \leq ||\, \lambda_G(\mu)\,||^p_{L^p(G)\to L^p(G)} ||h||_{L^p(G,X)},$$

hence

$$||\lambda_G(\mu)||_{L^p(G,X)\to L^p(G,X)} \leq ||\, \lambda_G(\mu)\,||_{L^p(G)\to L^p(G)}.$$

In combination with (3.15), we obtain the desired estimate. □

Remark. We call a Banach space X to be of **L^p-type**, $1 \leq p < \infty$, if there exists an embedding $\iota : X \hookrightarrow L^p(\Omega)$ into an L^p-space such that

$$\frac{1}{C}||x||_X \leq ||\iota(x)||_{L^p(\Omega)} \leq C||x||_X \quad \text{for every } x \in X,$$

for some constant $C \geq 1$.
For instance, any separable Hilbert space $\mathcal{H}$ is of L^p-type, for $1 \leq p < \infty$, or, more generally, any space $L^p(Y, \mathcal{H})$. This follows easily from Khintchin's inequality. Corollary 3.2 remains valid for spaces X of L^p-type, by an obvious modification of the proof.

Denote by $C^*_r(G)$ the reduced C^*-algebra of G. If $p = 2$, we can extend (3.14) to $C^*_r(G)$.

Proposition 3.2. *If $p = 2$ and $X = L^2(\Omega)$, then the unitary representation π_2 is weakly contained in the left-regular representation λ_G. In particular, for any $K \in C^*_r(G)$, the operator $\pi_2(K) \in \mathcal{B}(L^2(H, L^2(\Omega)))$ is well-defined.*
Moreover, for all $\xi, \eta \in C_0(H, L^2(\Omega))$, we have

$$\text{(3.16)} \qquad \langle \pi_2(K)\xi, \eta\rangle \quad = \quad \lim_{j\to\infty} \langle \lambda_G(K)(\xi \otimes^2_\rho \phi_j), \eta \otimes^2_\rho \phi_j, \rangle\,.$$

Proof. If $K \in C^*_r(G)$, then we can find a sequence $\{f_k\}_k$ in $L^1(G)$, such that $\lambda_G(K) = \lim_{k\to\infty} \lambda_G(f_k)$ in the operator norm $||\cdot||$ on $L^2(G)$. But, (3.15) implies that

$$\text{(3.17)} \qquad ||\,\pi_2(f)\,|| \leq ||\,\lambda_G(f)\,||\,, \ \text{ for all } f \in L^1(G),$$

where $||\cdot||$ denotes the operator norm on $\mathcal{B}(L^2(H, L^2(\Omega)))$ and $\mathcal{B}(L^2(G))$, respectively. Therefore, the $\{\pi_2(f_k)\}_k$ form a Cauchy sequence in $\mathcal{B}(L^2(H, L^2(\Omega)))$, whose limit we denote by $\pi_2(K)$.
It does not depend on the approximating sequence $\{f_k\}_k$. Moreover, from (3.17) we then deduce that

$$\text{(3.18)} \qquad ||\,\pi_2(K)\,|| \;\leq\; ||\,\lambda_G(K)\,|| \;=\; ||\,K\,||_{C^*_r(G)}\,, \ \text{ for all } K \in C^*_r(G)\,.$$

In particular, we see that π_2 is weakly contained in λ_G. It remains to show (3.16).
Given $\varepsilon > 0$, we choose $f \in C_0(G)$ such that $||\,K - f\,||_{C^*_r(G)} < \varepsilon/4$. Next, by (3.15), we can find j_0 such that

$$|\langle \pi_2(f)\xi, \eta\rangle \;-\; \langle \lambda_G(f)(\xi \otimes^2_\rho \phi_j),\, \eta \otimes^2_\rho \phi_j\rangle\,| < \varepsilon/4 \quad \text{for all } \ j \geq j_0\,.$$

Assume without loss of generality that $||\,\xi\,||_2 = ||\,\eta\,||_2 = 1$. Then, by (3.18),

$$|\,\langle \pi_2(K)\xi, \eta\rangle \;-\; \langle \pi_2(f)\xi, \eta\rangle\,| \leq ||\,K - f\,||_{C^*_r(G)}\, ||\,\xi\,||_2\, ||\,\eta||_2 \quad < \varepsilon/4,$$

and furthermore

$$\begin{aligned} & |\langle \lambda_G(K)(\xi \otimes^2_\rho \phi_j), \eta \otimes^2_\rho \phi_j\rangle - \langle \lambda_G(f)(\xi \otimes^2_\rho \phi_j), \eta \otimes^2_\rho \phi_j\rangle\,| \\ \leq\; & ||\,K - f\,||_{C^*_r(G)}\, ||\,\xi \otimes^2_\rho \phi_j\,||_2\, ||\,\eta \otimes^2_\rho \phi_j\,||_2 \\ <\; & \frac{\varepsilon}{4}\, ||\,\xi\,||_2\, ||\,\eta\,||_2 = \varepsilon/4\,. \end{aligned}$$

Combining these estimates, we find that

$$|\,\langle \pi_2(K)\xi, \eta\rangle - \langle \lambda_G(K)(\xi \otimes^2_\rho \phi_j), \eta \otimes^2_\rho \phi_j\rangle\,| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon \quad \text{for all } \ j \geq j_0\,.$$

□

Corollary of Proposition 3.2. *Assume that* ρ *is a unitary representation on a separable Hilbert space* X*, for instance a unitary character of* S*, and that* $\Delta_{G,S} = 1$*. Let* $K \in C_r^*(G)$*, and assume that* $\lambda_G(K)$ *extends from* $L^2(G) \cap L^p(G)$ *to a bounded linear operator on* $L^p(G)$*, where* $1 \leq p < \infty$*.*
Then $\pi_2(K)$ *extends from* $L^2(G/S) \cap L^p(G/S)$ *to a bounded linear operator on* $L^p(G/S)$*, and*

$$(3.19) \qquad ||\pi_2(K)||_{L^p(G/S)\to L^p(G/S)} \leq ||\lambda_G(K)||_{L^p(G)\to L^p(G)}.$$

Moreover, for $f \in L^1(G)$*, we have* $\pi_p(f) = \pi_2(f)$ *on* $C_0(G/S)$*.*

Proof. If $\xi, \eta \in C_o(H)$, then, since $\Delta_{G,S} = 1$,

$$\langle \pi_2(K)\xi, \eta\rangle = \lim_{j\to\infty} \langle \lambda_G(K)(\xi \otimes_\rho^2 \phi_j^2), \eta \otimes_\rho^2 \psi_j^2\rangle = \lim_{j\to\infty} \langle \lambda_G(K)(\xi \otimes_\rho^p \phi_j^p), \eta \otimes_\rho^{p'} \psi_j^{p'}\rangle.$$

And,

$$\begin{aligned} & |\langle \lambda_G(K)(\xi \otimes_\rho^p \phi_j^p), \eta \otimes_\rho^{p'} \psi_j^{p'}\rangle| \\ \leq\ & ||\lambda_G(K)||_{L^p(G)\to L^p(G)}\, ||\xi \otimes_\rho^p \phi_j^p||_{L^p(G)}\, ||\eta \otimes_\rho^{p'} \psi_j^{p'}||_{L^{p'}(G)} \\ \leq\ & ||\lambda_G(K)||_{L^p(G)\to L^p(G)}\, ||\xi||_{L^p(H)}\, ||\eta||_{L^{p'}(H)}. \end{aligned}$$

Estimate (3.19) follows.
That $\pi_p(f) = \pi_2(f)$ on $C_0(H)$, if $f \in L^1(G)$, is evident, since $\Delta_{G,S} = 1$. □

4 The case of a non-compact semi-simple factor

In this section, we shall give our proof of Theorem 1.2.
Let us first notice the following consequence of Corollary 3.2.

Assume that S is a closed, normal and amenable subgroup of G, and let $L = -\sum_j X_j^2$ be a sub-Laplacian on G. Denote by $\iota_2 := \mathrm{ind}_S^G 1$ the representation of G induced by the trivial character of S (compare Example 3.1), and let $\tilde{L} = -\sum_j (X_j \bmod \mathfrak{s})^2 = d\iota_2(L)$ be the corresponding sub-Laplacian on the quotient group $H := G/S$. Then

$$(4.1) \qquad \mathcal{M}_p(L) \cap C_\infty(\mathbb{R}) \subset \mathcal{M}_p(\tilde{L}) \cap C_\infty(\mathbb{R}).$$

In particular, if $\tilde{L}$ is of holomorphic L^p-type, then so is L.
In order to prove (4.1), assume that F is an L^p-multiplier for L contained in $C_\infty(\mathbb{R})$. Then $F(L)$ lies in $C_r^*(G)$, and by Corollary 3.2 the operator $\iota_2(F(L)) = F(d\iota_2(L)) = F(\tilde{L})$ extends from $L^2(H) \cap L^p(H)$ to a bounded operator on $L^p(H)$, so that $F \in \mathcal{M}_p(\tilde{L}) \cap C_\infty(\mathbb{R})$.

Let now G be a connected Lie group, with radical $S = \exp \mathfrak{s}$. Then there exists a connected, simply connected semi-simple Lie group H such that G is the semi-direct product of H and S, and this Levi factor H has a discrete center Z (see [4]). Let L be a sub-Laplacian on G, and denote by $\tilde{L}$ the corresponding sub-Laplacian on $G/S \simeq H$

and by $\tilde{\tilde{L}}$ the sub-Laplacian on H/Z corresponding to $\tilde{L}$ on H. We have that Z and S are amenable groups, and H/Z has finite center. From Theorem 1.1, we thus find that $\tilde{\tilde{L}}$ is of holomorphic L^p-type for every $p \neq 2$, if we assume that H is non-compact, and (4.1) then allows us to conclude that the same is true of $\tilde{L}$, and then also of L.

5 Compact extensions of exponential solvable Lie groups

5.1 Compact operators arizing in induced representations

Let now $K = \exp \mathfrak{k}$ be a connected compact Lie group acting continuously on an exponential solvable Lie group $S = \exp \mathfrak{s}$ by automorphisms $\sigma(k) \in$ Aut (S), $k \in K$. We form the semi-direct product $G = K \ltimes S$ with the multiplication given by:

$$(k, s) \cdot (k', s') = (kk', \sigma({k'}^{-1})ss'), \quad k, k' \in K, s, s' \in S.$$

The left Haar measure dg is the product of the Haar measure of K and the left Haar measure of S. Let us choose a K-invariant scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{s}$ of S. Denote by $\mathfrak{n}$ the nil-radical of $\mathfrak{s}$. Since every derivation d of $\mathfrak{s}$ maps the vector space $\mathfrak{s}$ into the nil-radical, it follows that the orthogonal complement $\mathfrak{b}$ of $\mathfrak{n}$ in $\mathfrak{s}$ is in the kernel of $d\sigma(X)$ for every $X \in \mathfrak{k}$. The following decomposition of the solvable Lie algebra $\mathfrak{s}$ has been given in [3]. Choose an element $X \in \mathfrak{b}$, which is in general position for the roots of $\mathfrak{s}$, i.e., for which $\lambda(X) \neq \mu(X)$ for all roots $\mu \neq \lambda$ of $\mathfrak{s}$. Let $\mathfrak{s}_0 = \{Y \in \mathfrak{s};\ \mathrm{ad}^l(X)Y = 0 \text{ for some } l \in \mathbb{N}^*\}$. Then $\mathfrak{s}_0$ is a nilpotent subalgebra of $\mathfrak{s}$, which is K-invariant (since $[X, \mathfrak{k}] = \{0\}$) and $\mathfrak{s} = \mathfrak{s}_0 + \mathfrak{n}$. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n} \cap \mathfrak{s}_0$ in $\mathfrak{s}_0$. Then $\mathfrak{a}$ is also a K-invariant subspace of $\mathfrak{s}$ (but not in general a subalgebra) and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$. Let $N = \exp \mathfrak{n} \subset S$ be the nil-radical of the group S. Then S is the topological product of $A = \exp \mathfrak{a}$ and N. Finally our group G is the topological product of K, A and N. Hence every element g of G has the unique decomposition:

$$g = k_g \cdot a_g \cdot n_g, \text{ where } k_g \in K,\ a_g \in A \text{ and } n_g \in N.$$

We shall use the notations and constructions of [8] in the following but we have to replace there the symbol G with the letter S.

Let $h : G \to \mathbb{C}$ be a function. For every $x \in G$, we denote by $\tilde{h}(x)$ the function on S defined by:

$$\tilde{h}(x)(s) = h(xs),\ s \in S.$$

Also for a function $r : S \to \mathbb{C}$ and for $x \in G$, we let ${}^x r : S \to \mathbb{C}$ be defined by:

$${}^x r(s) := r(xsx^{-1}).$$

We say that a Borel measurable function $\omega : G \to \mathbb{R}_+$ is a weight, if $1 \leq \omega(x) = \omega(x^{-1})$ and $\omega(xy) \leq \omega(x)\omega(y)$ for every $x, y \in G$. Then the space

$$L^p(G, \omega) = \{f \in L^p(G) |\, \|f\|_{\omega,p} := \int_G |f(g)|^p \omega(g)\, dg < \infty\},$$

for $1 \leq p \leq \infty$, is a subspace of $L^p(G)$ and for $p = 1$ it is even a Banach algebra for the norm $\| \cdot \|_{\omega,1}$.

Proposition 5.1. *Let G be a locally compact group and let S be a closed normal subgroup of G. Let ω be a continuous weight on G such that the inverse of its restriction to S is integrable with respect to the Haar measure on S. Let $f, g : G \to \mathbb{C}$ be two continuous functions on G, such that $\omega \cdot g$ is uniformly bounded and such that $f \in L^1(G, \omega)$. Let $h := f * g \in L^1(G, \omega)$. Then for every $t \in G$, the function $\tilde{h}(t)$ is in $L^1(S)$ and the mapping $G \times G \to L^1(S); (t, u) \mapsto {}^u\tilde{h}(t)$ is continuous.*

Proof. Since ω is a weight, we have that $\omega(s) \leq \omega(u)\omega(u^{-1}s)$, i.e. $\frac{1}{\omega(u^{-1}s)} \leq \frac{\omega(u)}{\omega(s)}, \ \ s, u \in G$. Hence, for $t \in G, \ s \in S$,

$$\begin{aligned}
|\tilde{h}(t)(s)| &= |\int_G f(u)g(u^{-1}ts)\, du| = |\int_G f(tu)g(u^{-1}s)\, du| \\
&\leq \int_G |f(tu)||g(u^{-1}s)|\frac{\omega(u^{-1}s)}{\omega(u^{-1}s)}\, du \leq \int_G |f(tu)|\omega(u)|g(u^{-1}s)|\frac{\omega(u^{-1}s)}{\omega(s)}\, du
\end{aligned}$$

and so

$$\begin{aligned}
\|\tilde{h}(t)\|_1 &\leq \int_S \int_G |f(tu)|\omega(u)|g(u^{-1}s)|\frac{\omega(u^{-1}s)}{\omega(s)}\, duds \\
&\leq \int_S \int_G |f(tu)|\omega(u)\frac{\|g\|_{\omega,\infty}}{\omega(s)}\, duds \\
&\leq \int_S \int_G \omega(t^{-1})|f(tu)|\omega(tu)\frac{\|g\|_{\omega,\infty}}{\omega(s)}\, duds = \omega(t)\|f\|_{\omega,1}\|g\|_{\omega,\infty}\|(\frac{1}{\omega})_{|S}\|_1 \qquad (5.1)
\end{aligned}$$

So for every $t \in G$, the function $\tilde{h}(t)$ is in $L^1(S)$. Furthermore, for $t, t' \in G$, by (5.1),

$$\begin{aligned}
\|h(t) - h(t')\|_1 &\leq \int_S \int_G |f(tu) - f(t'u)|\omega(u)\frac{\|g\|_{\omega,\infty}}{\omega(s)}\, duds \\
&\leq \|(\lambda(t^{-1})f - \lambda(t'^{-1})f)\|_{\omega,1}\|g\|_{\omega,\infty}\|(\frac{1}{\omega})_{|S}\|_1,
\end{aligned}$$

where λ denotes left translation by elements of G. Since left translation in $L^1(G, \omega)$ and conjugation in $L^1(S)$ are continuous, it follows that the mapping $(t, u) \mapsto {}^u\tilde{h}(t)$ from $G \times G$ to $L^1(S)$ is continuous too. $\square$

Let as in (1.1) δ denote the Carathéodory distance associated to our sub-Laplacian L on G and $(h_t)_{t>0}$ its heat kernel. Then the function $\omega_d(g) := e^{d\delta(x,e)}, \ g \in G, \ d \in \mathbb{R}_+$, defines a weight on G. Since we have the Gaussian estimate

$$|h_t(g)| \leq C_t e^{-C_t \delta(g,e)^2}, \text{ for all } g \in G, t > 0,$$

it follows that:

$$h_t \in L^1(G, \omega_d) \cap L^\infty(G, \omega_d) \text{ for every } t > 0 \text{ and } d > 0. \qquad (5.2)$$

Proposition 5.2. *Let G be the semidirect product of a connected compact Lie group K acting on an exponential solvable Lie group S. Then there exists a constant $d > 0$, such that $\frac{1}{\omega_d}|_S$ is in $L^1(S)$.*

Proof. Let U be a compact symmetric neighborhood of e in G containing K. Since S is connected, we know that $G = \cup_{k\in\mathbb{N}} U^k$. This allows us to define $\tau_U = \tau : G \to \mathbb{N}$ by:

$$\tau(x) = \min\{k \in \mathbb{N} |\, x \in U^k\}.$$

Then τ is sub-additive and defines thus a distance on G, which is bounded on compact sets. Since τ is clearly connected in the sense of [20], it follows that τ and the Carathéodory distance δ are equivalent at infinity, i.e.

$$1 + \tau(x) \leq D(1 + \delta(x)) \leq D'(1 + \tau(x)),\ x \in G.$$

We choose now a special compact neighborhood of e in the following way. We take our K-invariant scalar-product on $\mathfrak{s}$, the unit-ball $B_{\mathfrak{a}}$ in $\mathfrak{a}$ and the unit-ball $B_{\mathfrak{n}} \in \mathfrak{n}$. Both balls are K-invariant. Let $U_{\mathfrak{a}} = \exp B_{\mathfrak{a}}$ and $U_{\mathfrak{n}} = \exp B_{\mathfrak{n}}$. Then $U = KU_{\mathfrak{a}}U_{\mathfrak{n}} \cap U_{\mathfrak{n}}U_{\mathfrak{a}}K$ is a compact symmetric neighborhood of e. Let us give a rough estimate of the radia of the "balls" U^l, $l \in \mathbb{N}$. For simplicity of notation, we shall denote all the positive constants which will appear in the following arguments (and which will be assumed to be integers, if necessary) by C.
Let $k_i a_i n_i \in KU_{\mathfrak{a}}U_{\mathfrak{n}}$, $i = 1, \cdots, l$ and $g := \Pi_{i=1}^l k_i a_i n_i$. We have

$$g = \Pi_{i=1}^l k_i a_i n_i = (\Pi_{i=1}^l k_i a_i)\left((k_2 a_2 \cdots k_l a_l)^{-1} n_1 (k_2 a_2 \cdots k_l a_l) \cdots (k_l a_l) n_{l-1} (k_l a_l) n_l\right).$$

Since $U_{\mathfrak{a}}$ is K-invariant, it follows that:

$$g = \Pi_{i=1}^l k_i a_i s_i = k' a' \Pi_{i=1}^l (a_i'' k_i'') n_i (a_i'' k_i'')^{-1},$$

where $k', k_1'', \cdots k_l'' \in K$, $a' \in U_{\mathfrak{a}}^l$, $a_1'' \in U_{\mathfrak{a}}^{l-1}, \cdots, a_{l-1}'' \in U_a$. Hence there exists $X_1, \cdots, X_l \in B_{\mathfrak{a}}$, such that

$$a' = \exp X_1 \cdots \exp X_l = \exp(X_1 + \cdots + X_l) \exp q_l(X_1, \cdots, X_l)$$

for some element $q_l(X_1, \cdots, X_l) \in \mathfrak{n} \cap \mathfrak{s}_0$. Since $\mathfrak{s}_0$ is a nilpotent Lie algebra we have that $\|q_l(X_1, \cdots, X_l)\| \leq C(1 + l)^C$, $l \in \mathbb{N}$. Hence

$$a' \in \exp(lB_a) \exp[C(1 + l)^C B_{\mathfrak{n}}] \subset \exp(lB_{\mathfrak{a}}) U_{\mathfrak{n}}^{C(1+l)^C}.$$

Furthermore, because $U_{\mathfrak{a}}$ is compact, $\sup_{a\in U_{\mathfrak{a}}} \| \operatorname{Ad}(a)\|_{op} \leq C < \infty$ and so $(a_i'' k_i'') n_i (a_i'' k_i'')^{-1} \in \exp C^{(l-i)} B_{\mathfrak{n}} \subset U_{\mathfrak{n}}^{C^{l-i}}$ $(i = 1, \cdots, l)$. Finally for some integer constants C,

$$\begin{aligned} g = k' a' \Pi_{i=1}^l (a_i'' k_i'') n_i (a_i'' k_i'')^{-1} \quad &\in \quad K \exp lU_a U_{\mathfrak{n}}^{C(1+l)^C} \left(\Pi_{i=1}^{l-1} U_{\mathfrak{n}}^{C^{l-i}}\right) U_{\mathfrak{n}} \\ &\subset \quad K \exp lU_a U_{\mathfrak{n}}^{C(1+l)^C + \sum_{i=1}^{l-1} C^{l-i} + 1} \\ &\subset \quad K \exp lU_a U_{\mathfrak{n}}^{C^l} \\ &\subset \quad K \exp lU_a \exp C^l B_{\mathfrak{n}} \end{aligned} \tag{5.3}$$

Hence for any $g \in G$, for $\tau_U(g) = l$, we have that $g \in (KU_{\mathfrak{a}}U_{\mathfrak{n}})^l$ and so, denoting by $\mathrm{Log} : S \to \mathfrak{s}$ the inverse map of $\exp : \mathfrak{s} \to S$, $g = k_g a_g n_g$, with $k_g \in K$, $a_g \in \exp \mathfrak{a}$, $\|\mathrm{Log}(a_g)\| \leq l = \tau_U(g)$ and $n_g \in N$ with $\|\mathrm{Log}(n_g)\| \leq C^l$, i.e. $\log(1 + \|\mathrm{Log}(n_g)\|) \leq Cl = C\tau_G(g)$. Whence for our weight ω_d, ($d \in \mathbb{R}_+$), we have that:

$$\begin{aligned}\omega_d(g) = e^{d\delta(g)} &\geq Ce^{dC\tau_U(g)} \geq Ce^{dC(\|\mathrm{Log}(a_g)\| + \log(1+\|\mathrm{Log}(n_g)\|))}\\ &= Ce^{dC\|\mathrm{Log}(a_g)\|}(1 + \|\mathrm{Log}(n_g)\|)^{dC}.\end{aligned}$$

Therefore, for d big enough,

$$\begin{aligned}\int_S \frac{1}{\omega_d(s)} ds &= \int_{\mathfrak{a}} \int_{\mathfrak{s}} \frac{1}{\omega_d(\exp X \exp Y)} \, dY dX\\ &\leq C \int_{\mathfrak{a}} \int_{\mathfrak{s}} e^{-dC\|X\|} \frac{1}{(1 + \|Y\|)^{dC}} \, dY dX < \infty.\end{aligned}$$

□

Proposition 5.3. *Let T be a compact topological space and let $k : T \times T \to \mathcal{K}(\mathcal{H})$ be a continuous mapping into the space of compact operators on a Hilbert space $\mathcal{H}$. Let μ be a Borel probability measure on T. Then the linear mapping*

$$\begin{aligned}K &: L^2(T, \mathcal{H}) \to L^2(T, \mathcal{H}),\\ K\xi(t) &:= \int_T k(t, u)\xi(u) \, du, \ t \in T, \ \xi \in L^2(T, \mathcal{H}),\end{aligned}$$

is compact too.

Proof. We show that K is the norm-limit of a sequence of operators of finite rank. Let $\varepsilon > 0$. Since T is compact and k is continuous, there exists a finite partition of unity of $T \times T$ consisting of continuous non-negative functions $(\varphi_i)_{i=1}^N$, such that $\|k(t, t') - k(u, u')\|_{op} < \frac{\varepsilon}{2}$ for every $(t, t'), (u, u')$ contained in the support φ_i. Choose for $i = 1, \cdots, N$ an element (t_i, t'_i) in supp φ_i. Since $k(t_i, t'_i)$ is a compact operator, we can find a bounded endomorphism F_i of $\mathcal{H}$ of finite rank, such that $\|k(t_i, t'_i) - F_i\|_{op} < \frac{\varepsilon}{2}$, hence $\|k(t, t') - F_i\|_{op} < \varepsilon$ for every $(t, t') \in$ supp φ_i, $i = 1, \cdots, N$. The finite rank operator F_i has the expression $F_i = \sum_{k=1}^{N_i} P_{\eta_{i,k}, \eta'_{i,k}}$, where for $\eta, \eta' \in \mathcal{H}$, $P_{\eta,\eta'}$ denotes the rank one operator $P_{\eta,\eta'}(\eta'') = \langle \eta'', \eta' \rangle \eta$, $\eta'' \in \mathcal{H}$.

We approximate the continuous functions φ_i uniformly on $T \times T$ up to an error of at most $\frac{\varepsilon}{R}$ by tensors $\psi_i = \sum_{j=1}^{M_i} \varphi_{i,j} \otimes \varphi'_{i,j} \in C(T, \mathbb{R}_+) \otimes C(T, \mathbb{R}_+)$ for some $R > 0$ to be determined later on. Let K_ε be the finite rank operator

$$K_\varepsilon = \sum_{i=1}^N \sum_{j=1}^{M_i} \sum_{k=1}^{N_i} P_{\varphi_{i,j} \otimes \eta_{i,k}, \ \varphi'_{i,j} \otimes \eta'_{i,k}}.$$

In order to estimate the difference $K - K_\varepsilon$, we let first $K_{\varepsilon,1}$ be the kernel operator with kernel $k_{\varepsilon,1}(s, t) = \sum_{i=1}^N \varphi_i(s, t) F_i$. Then for $\xi \in L^2(T, \mathcal{H})$

$$\|K_{\varepsilon,1}\xi - K\xi\|_2^2 = \int_T \| \sum_{i=1}^N \int_T \varphi_i(s, t)(k(s, t) - F_i)\xi(t) dt\|^2 \, ds$$

$$\leq \int_T (\sum_{i=1}^N \int_T \varphi_i(s,t)\varepsilon\|\xi(t)\|dt)^2\, ds = \int_T(\int_T \varepsilon\|\xi(t)\|dt)^2\, ds \leq \varepsilon^2\|\xi\|^2,$$

hence $\|K - K_{\varepsilon,1}\|_{\text{op}} \leq \varepsilon$. Moreover

$$\|(K_{\varepsilon,1} - K_\varepsilon)\xi\|^2 = \int_T \| \int_T \sum_{i=1}^N (\varphi_i(s,t) - \sum_{j=1}^{M_i} \varphi_{i,j}(s)\varphi'_{i,k}(t))F_i\xi(t)\, dt\|^2\, ds$$

$$\leq \int_T(\int_T \sum_{i=1}^N \frac{\varepsilon}{R}\|F_i\|_{\text{op}}\|\xi(t)\|dt)^2\, ds \leq \frac{\varepsilon^2}{R^2}(\sum_{i=1}^N \|F_i\|_{\text{op}})^2\|\xi\|^2.$$

So, if we let $R = \frac{1}{1+\sum_{i=1}^N \|F_i\|_{\text{op}}}$, then

$$\|K - K_\varepsilon\|_{op} \leq \|K - K_{\varepsilon,1}\|_{op} + \|K_{\varepsilon,1} - K_\varepsilon\|_{op} \leq 2\varepsilon.$$

□

Let now π be an isometric representation of the group S on a Banach space X and denote by $\rho^p := ind_S^G\pi$ be the corresponding induced representation of G on $L^p(G, X; \pi)$. Here we follow notation of Section 3.1.
Let h be in $L^1(G)$ and assume furthermore that $\tilde{h}(g) \in L^1(S)$ for all $g \in G$ and that the mapping $\tilde{h} : G \to L^1(S)$ is continuous. Then the operator $\rho^p(h)$ is a kernel operator, whose kernel $k(t,u)$, $t, u \in G$, is given by:

$$k(t,u) = \Delta_G(u^{-1})\pi({}^u\tilde{h}(tu^{-1})) \tag{5.4}$$

(in the notations of Proposition 5.1). Indeed, for $\xi \in L^p(G, X; \pi)$, $t \in G$,

$$\begin{aligned}
[\rho^p(h)\xi](t) &= \int_G h(g)\xi(g^{-1}t)\, dg = \int_G \Delta_G(g^{-1})h(tg^{-1})\xi(g)\, dg \\
&= \int_{G/S}\int_S \Delta_G(s^{-1}g^{-1})h(ts^{-1}g^{-1})\xi(gs)\, dsdg \\
&= \int_{G/S}\int_S \Delta(g^{-1})\Delta_S(s^{-1})h(ts^{-1}g^{-1})\xi(gs)\, dsdg \\
&= \int_{G/S}\int_S \Delta(g^{-1})h(tg^{-1}(gsg^{-1}))\pi(s)\xi(g)\, dsdg \\
&= \int_{G/S} \Delta(g^{-1})\pi({}^g\tilde{h}(tg^{-1}))\xi(g)\, dg.
\end{aligned}$$

Moreover the kernel k satisfies the following covariance property under S:

$$k(ts, us') = \pi(s^{-1})k(t,u)\pi(s'), \qquad t,\, u \in G,\, s,\, s' \in S. \tag{5.5}$$

Proposition 5.4. *Let G be the semidirect product of a connected compact Lie group K acting on an exponential solvable Lie group S. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of the normal closed subgroup S of G whose Kirillov-orbit $\Omega_\pi = \Omega \subset \mathfrak{s}^*$ is closed. Let $\rho = ind_S^G\pi$. Then the operator $\rho(h_t)$ is compact for every $t > 0$.*

Proof. By the relation (5.2) the function h_t is in $L^1(G,\omega_d)\cap L^\infty(G,\omega_d)$ for t and $d>0$. Furthermore we have that $h_t = h_{t/2} * h_{t/2}$. Hence by the Propositions 5.2 and 5.1 the mapping $G\times G\to L^1(S)$, $(s,u)\mapsto {}^u\tilde{h}_t(su^{-1})$, is continuous and so the operator valued kernel function $k(s,u) := \Delta_G(u^{-1})\pi({}^u\tilde{h}_t(su^{-1}))$ is continuous too. It follows from the preceeding discussion that the k is just the integral kernel of the operator $\rho(h_t)$. The fact that the Kirillov orbit of $\pi\in\widehat{S}$ is closed in $\mathfrak{s}^*$ implies that for every $\varphi\in L^1(S)$, the operator $\pi(\varphi)=\int_S f(s)\pi(s)ds$ is compact (see [13] and [8]). Hence $k(s,u)$ is compact for every $(s,u)\in G\times G$ and in particular for every $(s,u)\in K\times K$. We apply Proposition 5.3 to the restriction of k to $K\times K$. The related kernel operator on $L^2(K,\mathcal{H})$ is then compact. Now, since π is unitary, the restriction map to K is an isometric isomorphism from $L^2(G,\mathcal{H};\pi)$ onto $L^2(K,\mathcal{H})$, and we thus see that $\rho(h_t)$ is compact too. □

5.2 Proof of Theorem 1.3

We now turn to the proof of Theorem 1.3, which follows closely the notation and argumentation in [8]. In the sequel, we always make the following

> **Assumption.** $\ell\in\mathfrak{s}^*$ satisfies Boidol's condition (B), and $\Omega(\ell)|_{\mathfrak{n}}$ is closed. Moreover, we assume that $p\in[1,\infty[$, $p\neq 2$, is fixed.

Since ℓ satisfies (B), the stabilizer $\mathfrak{s}(l)$ is not contained in $\mathfrak{n}$. Let ν be the real character of $\mathfrak{s}$, which has been defined in [8, Section 5], trivial on $\mathfrak{n}$ and different from 0 on $\mathfrak{s}(\ell)$. We denote by $\pi_\ell=\mathrm{ind}_P^S\chi_\ell$ the irreducible unitary representation of S associated to ℓ by the Krillov map; here $P=P(\ell)$ denotes a suitable polarizing subgroup for ℓ, and χ_ℓ the character $\chi_\ell(p):=e^{i\ell(\log p)}$ of P.

For any complex number z in the strip

$$\Sigma := \{\zeta\in\mathbb{C} : |\operatorname{Im}\zeta|<1/2\},$$

let Δ_z be the complex character of S given by

$$\Delta_z(\exp X) := e^{-iz\nu(X)},\quad X\in\mathfrak{s},$$

and χ_z the unitary character

$$\chi_z(\exp X) := e^{-i\operatorname{Re} z\nu(X)},\quad X\in\mathfrak{s}.$$

If we define $p(z)\in]1,\infty[$ by the equation

$$\operatorname{Im} z = 1/2 - 1/p(z), \tag{5.1}$$

it is shown in [8] that the representation π_ℓ^z, given by

$$\pi_\ell^z(x) := \Delta_z(x)\pi_\ell(x) = \chi_z(x)\pi_\ell^{\overline{p(z)}}(x),\quad x\in G, \tag{5.2}$$

is an isometric representation on the mixed L^p-space $L^{\overline{p(z)}}(S/P,\ell)$. Here, $\pi_\ell^{\overline{p(z)}}$ denotes the $\overline{p(z)}$-induced representation of S on $L^{\overline{p(z)}}(S/P,\ell)$ defined in [8], where $\overline{p(z)}$ is a

multi-index of the form $(p(z),\dots,p(z),2,\dots,2)$.
Observe that for $\tau \in \mathbb{R}$, we have $p(\tau) = 2$, and $\pi_\ell^\tau = \chi_\tau \otimes \pi_\ell$ is a unitary representation on $L^{\overline{2}}(S/P,\ell)$. Moreover,

$$\pi_\ell^\tau \simeq \pi_{\ell-\tau\nu}, \tag{5.3}$$

since the mapping $f \mapsto \overline{\chi}_\tau f$ intertwines the representations $\chi_\tau \otimes \pi_\ell$ and $\pi_{\ell-\tau\nu}$.

We take now for $z \in \Sigma$ the $p(z)$-induced representation $\rho_\ell^z := \mathrm{ind}_{p(z),S}^G \pi_\ell^z$ of G which acts on the space

$$L^{\overline{p(z)}}(G/P,\ell) := L^{p(z)}(G, L^{\overline{p(z)}}(S/P,\ell); \pi_\ell^z).$$

Let us shortly write

$$L^{\overline{p}} := L^{\overline{p(z)}}(G/P,\ell), \quad 1 \le p < \infty,$$

for the space of ρ_ℓ^z.
We can extend the character Δ_z, $z \in \Sigma$, of S to a function on G by letting

$$\Delta_z(kan) := \Delta_z(an) = e^{-iz\nu(\mathrm{Log}(a))},\ k \in K, a \in A, n \in N.$$

Since ν is trivial on $\mathfrak{n}$ and since $kak^{-1} \in aN$ for all $k \in K, a \in A$, we have that

$$\Delta_z(kank') = \Delta_z(an),\ k, k' \in K, a \in A, n \in N,$$

and in particular Δ_z is a character of G.
Define the operator $T(z)$, $z \in \Sigma$, by:

$$T(z) := \rho_\ell^z(h_1).$$

Then by the relations (5.4) and (5.2), for $z \in \Sigma$ and $\xi \in L^{\overline{p}}$, (since Δ_z is K-invariant)

$$\begin{aligned} T(z)\xi(k) &= \int_K \pi_\ell^z({}^{k'}\tilde{h}_1(kk'^{-1}))\xi(k')dk' \\ &= \int_K \pi_\ell((\Delta_{z|S})^{k'}\tilde{h}_1(kk'^{-1}))\xi(k')dk' \\ &= \int_K \pi_\ell({}^{k'}\widetilde{(\Delta_z h_1)}(kk'^{-1}))\xi(k')dk' \\ &= [\rho_\ell(\Delta_z h_1)](\xi(k)). \end{aligned}$$

Hence

$$T(z) = \rho_\ell^z(h_1) = \rho_\ell(\Delta_z h_1),\ z \in \Sigma. \tag{5.4}$$

Since by (5.2), for every continuous character χ of G which is trivial on N the function χh_1 is in $L^1(G)$, it follows from [8, Corollary 5.2 and Proposition 3.1] that the operator $T(z)$ leaves $L^{\overline{q}}$ invariant for every $1 \le q < \infty$, and is bounded on all these spaces. Moreover, by Proposition 5.4, $T(\tau)$ is compact for $\tau \in \mathbb{R}$. From here on we can proceed exactly as in the proof of [8, Theorem 1], provided that we can prove a "Riemann-Lebesgue" type lemma like [8, Theorem 2.2] in our present setting, since $G = K \ltimes S$ is amenable.

We must show that $T(\tau)$ tends to 0 in the operator norm if τ tends to ∞ in $\mathbb{R}$. The condition we have imposed on the coadjoint orbit Ω of ℓ, namely that the restriction of Ω to $\mathfrak{n}$ is closed, tells us that $\lim_{\tau\to\infty} \Omega + \tau\nu = \infty$ in the orbit space, which means that $\lim_{\tau\to\infty} \|\pi_{\ell+\tau\nu}(f)\|_{op} = 0$ for every $f \in L^1(S)$. Now, by (5.4) the operator $T(\tau) = \rho_\ell^\tau(h_1)$ is a kernel operator whose kernel K_τ has values in the bounded operators on $\mathcal{H}_\ell$. the kernel K_τ is given by:

$$K_\tau(k, k') = \int_S \Delta_\tau(s) h_1(k^{-1} s k'^{-1}) \pi_\ell(s) ds = \pi_\ell^\tau(h_1(k, k')),$$

where $h_1(k, k')$ is the function on S defined by $h_1(k, k')(s) := h_1(ksk'^{-1})$. Hence

$$\lim_{\tau\to\infty} \|\pi_\ell^\tau(h_1(k, k'))\|_{op} = 0$$

for every $k, k' \in K$. Moreover for $k, k' \in K$,

$$\|\pi_\ell^\tau(h_1(k, k'))\|_{op} \leq \|h_1(k, k')\|_1 \leq \sup_{k''\in K} \|\tilde{h}_1(k'')\|_1.$$

We know from Proposition 5.1 that, for every $k'' \in K$,

$$\|h_1(k'')\|_1 \leq \|\omega_d|_K\|_\infty \|h_{1/2}\|_{\omega_d,1} \|h_{1/2}\|_{\omega_d,\infty} \|(\frac{1}{\omega_d})|_S\|_1,$$

which is finite by Proposition 5.2 and relation (5.2) (if d is big enough) . Hence, by Lebesgue's dominated convergence theorem, we see that:

$$\lim_{\tau\to\infty} \int_K \int_K \|\pi_\ell^\tau(h_1(k, k')\|_{op}^2 \, dkdk' = 0.$$

This shows that:

$$\lim_{\tau\to\infty} \|\rho_\ell^\tau(h_1)\|_{op} = 0.$$

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