Dual topology of the motion groups $SO(n) \ltimes \mathbb{R}^n$.

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Abstract

Let $n \in \mathbb{N}^*$ and let $M_n = SO(n) \ltimes \mathbb{R}^n$ be the corresponding motion group. In this paper, we describe the topology of the dual space $\hat{M}_n$ and identifying $\hat{M}_n$ with the subspace of admissible coadjoint orbits $\mathfrak{m}_n^\ddagger / M_n$, we show that this identification is a homeomorphism.

1 Introduction.

It is wellknown that for a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = \exp\mathfrak{g}$, its dual space $\widehat{G}$ is homeomorphic to the space of co-adjoint orbits $\mathfrak{g}^*/G$ through the Kirillov mapping (see [Lep-Lud]). If we consider semi-direct products $G = K \ltimes N$ of compact connected Lie groups K acting on simply connected nilpotent Lie groups N, then again we have an orbit picture of the dual space of G (see [Lipsman]) and one can imagine that the topology of $\widehat{G}$ is linked to the topology of the admissible co-adjoint orbits.

In this paper we consider the motion groups $M_n := SO(n) \ltimes \mathbb{R}^n$ and we show that in this case the topology of their unitary dual spaces $\hat{M}_n$ is determined by the topology of the space of admissible co-adjoint orbits. For every admissible linear functional ℓ of the Lie algebra $\mathfrak{m}_n$ of M_n, we can construct an irreducible unitary representation π_ℓ by holomorphic induction and every irreducible representation of M_n arises in this manner. We obtain in this fashion a map from the set $\mathfrak{m}_n^\ddagger$ of the admissible linear functionals onto the dual space $\hat{M}_n$ of M_n. Since π_ℓ is equivalent to $\pi_{\ell'}$ if and only if ℓ and ℓ' are on the same M_n-orbit, we obtain finally a homeomorphism between the space of admissible co-adjoint orbits $\mathfrak{m}_n^\ddagger/M_n$ and the dual space $\hat{M}_n$ of M_n in Theorem 5.5.

Some Mathematicians had seen the dual space topology of the motion group M_n. In this case Kaniuth had showed in [Kan-Ta] that for all compact subset K of M_n and all $\pi \in \hat{M}_n$, the mapping defined by

$$\psi_K(\pi) = \inf_{\xi \in \mathcal{H}_\pi^1} (\max_{x \in K} \|\pi(x)\xi - \xi\|)$$

is continuous on $\hat{M}_n \backslash \widehat{SO(n)}$, that is, on the set of infinite dimensional representations of M_n, where $\mathcal{H}_\pi^1$ is the unit sphere in $\mathcal{H}_\pi$, the Hilbert space of π.

Here is a brief section-by-section description of the contents of the paper. In paragraph 2 we recall the description of the topology of the unitary dual space $\widehat{G}$ of a locally compact group G. In paragraph 3, we determine the dual space of our motion groups M_n, the representations attached to an admissible linear functional are obtained via Mackey's little-group method and the dual space of M_n is given by the parameter space $\mathcal{P}_n := \{(r, \rho), r > 0, \rho \in \widehat{SO(n-1)}\} \bigcup \widehat{SO(n)}$. In paragraph 4, we link the convergence of sequences of elements of $\hat{M}_n$ to the convergence in $\mathcal{P}_n$. In paragraph 5, we use the convergence in the parameter space to show that the orbit space $\mathfrak{m}_n^{\ddagger}/M_n$ and $\hat{M}_n$ are homeomorphic.

1.1 Some notations and basic facts.

We recall first some basic notions and facts.

- Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$, i.e.,

$$\mathrm{ad}_{\mathfrak{g}}(X)Y = \mathrm{ad}(X)Y = [X, Y], \text{ for all } X, Y \in \mathfrak{g}$$

  and the group G acts on $\mathfrak{g}$ by the adjoint representation Ad, and on $\mathfrak{g}^*$ by the co-adjoint representation Ad^*, i.e.,

$$\langle \mathrm{Ad}^*(g)\ell, X\rangle = \langle g.\ell, X\rangle = \langle \ell, \mathrm{Ad}(g^{-1})X\rangle, g \in G, \ell \in \mathfrak{g}^*, X \in \mathfrak{g}.$$

- The set $G.\ell = \{g.\ell,\ g \in G\} =: \mathcal{O}_\ell$ is called the G-orbit of ℓ. We denote by $\mathfrak{g}^*/G$ the space of coadjoint orbits and by $p_G : \mathfrak{g}^* \to \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e, a subset U in $\mathfrak{g}^*/G$ is open if and only $p_G^{-1}(U)$ is open in $\mathfrak{g}^*$. The following Lemma can also be found in [Lep-Lud].

  Lemma 1.1. *Let $(\mathcal{O}_k)_{k\in\mathbb{N}}$ be a sequence of elements in $\mathfrak{g}^*/G$. Then $(\mathcal{O}_k)_k$ converges to the orbit $\mathcal{O}$ in $\mathfrak{g}^*/G$ if and only if for any $\ell \in \mathcal{O}$, there exist $\ell_k \in \mathcal{O}_k, k \in \mathbb{N}$, such that $\ell = \lim\limits_{k+\infty} \ell_k$.*

  Proof. If for every $k \in \mathbb{N}$, we have an $\ell_k \in \mathcal{O}_k$ such that $\lim\limits_{k\to+\infty} \ell_k = \ell \in \mathcal{O}$, then for every G-invariant neighborhood U of $\mathcal{O}$ in $\mathfrak{g}^*$, we have that $\ell_k \in U$ for k large enough. Hence $\mathcal{O}_k \subset U$ for all these k's.

  Conversely suppose that the sequence $(\mathcal{O}_k)$ converges to $\mathcal{O}$ in the orbit space. Then for every $\ell \in \mathcal{O}$, choose a decreasing family (V_n) of relatively compact open neighborhoods of ℓ, such that $\overline{V_{n+1}} \subset V_n$ for every n and such that $\bigcap V_n = \{\ell\}$. Then the open sets $U_n := \mathrm{A}d(G)V_n$ are G-invariant neighborhoods of $\mathcal{O}$ and so there exists $R_n > 0$, such that $\mathcal{O}_k \subset U_n$ for every $k \geq R_n$. We can assume that (R_n) is an increasing sequence with $\lim\limits_{n\to+\infty} R_n = +\infty$. Choose for $R_n \leq k < R_{n+1}$ an element $\ell_k \in V_n \cap \mathcal{O}_k$. If V is a neighborhood of ℓ, then V contains V_n for some n and so $\ell_k \in V$ for every $k \geq R_n$. This shows that $\lim\limits_{k\to+\infty} \ell_k = \ell$. $\square$

- Let dg be a left Haar measure on the unimodular locally compact group G and let $C_c(G)$ denote the space of continuous functions on G with compact support. Then $C_c(G)$ is an involutive $*$-algebra with the convolution product

$$f_1 * f_2(x) = \int_G f_1(g) f_2(g^{-1}x) dg$$

and involution

$$f^*(x) = \overline{f(x^{-1})}.$$

- Let π be a unitary representation of G on the Hilbert space $\mathcal{H}_\pi$. For all $\xi, \eta \in \mathcal{H}_\pi$, we define the coefficient $C^\pi_{\xi,\eta}$ of the representation π by

  $$C^\pi_{\xi,\eta}(g) = \langle \pi(g)\xi, \eta \rangle,\ g \in G.$$

  If G is compact, then for two irreducible unitary representations $(\pi, \mathcal{H}_\pi)$ and $(\pi', \mathcal{H}_{\pi'})$, the orthogonality relation says that for all $\xi, \eta \in \mathcal{H}_\pi,\ \xi', \eta' \in \mathcal{H}_{\pi'}$,

  $$\langle C^\pi_{\xi,\eta}, C^{\pi'}_{\xi',\eta'} \rangle := \int_G C^\pi_{\xi,\eta}(g)\overline{C^{\pi'}_{\xi',\eta'}(g)}dg = \begin{cases} 0 & \text{if } \pi \text{ and } \pi' \text{ are not equivalent,} \\ \frac{\langle \xi,\xi' \rangle \langle \eta',\eta \rangle}{d_\pi} & \text{if } \pi \text{ and } \pi' \text{ are equivalent,} \end{cases} \tag{1}$$

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  where d_π denotes the dimension of the representation π.

If G is a Lie group, then we denote by $C_c^\infty(G)$ the infinitely differentiable functions with compact support.
$\mathbb{R}$ denotes the real, $\mathbb{C}$ the complex, $\mathbb{Z}$ the integers and $\mathbb{N}$ the entire numbers. We define on $\mathbb{R}^n$ the norm $\|.\|$ by $\|X\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ and we denote by $B(X, r)$ the set of all $Y \in \mathbb{R}^n$ such that $\|X - Y\| < r$, for $X \in \mathbb{R}^n$ and r positive real.

1.2 Induced representations.

Let H be a closed subgroup of a unimodular Lie group G such that the quotient space G/H admits a left invariant Borel measure $d\dot{g}$. Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of H. Define the space of mappings

$$\begin{aligned} \mathcal{E}(G/H, \rho) = \{\xi : G \longrightarrow \mathcal{H}_\rho,\ &\text{continuous and with compact support modulo } H \\ &\text{such that } \xi(gh) = \rho(h)^*(\xi(g)),\ \text{for all } g \in G, h \in H\}. \end{aligned}$$

We remark that the space $\mathcal{E}(G/H, \rho)$ is left translation invariant and that for each ξ, η in $\mathcal{E}(G/H, \rho)$, the mapping $g \longmapsto \langle \xi(g), \eta(g) \rangle_{\mathcal{H}_\rho}$ belongs to the space $C_c(G/H)$. This relation allows us to define a scalar product on $\mathcal{E}(G/H, \rho)$ in the following way

$$\langle \xi, \eta \rangle := \int_{G/H} \langle \xi(g), \eta(g) \rangle_{\mathcal{H}_\rho} d\dot{g},$$

and an L^2-norm by

$$\|\xi\|_2 = \Big(\int_{G/H} \|\xi(g)\|^2_{\mathcal{H}_\rho} d\dot{g} \Big)^{\frac{1}{2}}.$$

We define the induced representation $\mathrm{Ind}_H^G \rho$ of G as the left regular representation of G on the completion $L^2(G/H, \rho)$ of $\mathcal{E}(G/H, \rho)$ with respect the norm $\|.\|_2$ defined above, i.e.,

$$(\mathrm{Ind}_H^G \rho)(x)(\xi)(y) = \xi(x^{-1}y), \forall x, y \in G, \xi \in L^2(G/H, \rho).$$

2 The topology for the dual space of a locally compact group.

Let G be a second countable locally compact group, and $\widehat{G}$ be the space of the equivalence classes of irreducible unitary representations of G. Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation on the Hilbert space $\mathcal{H}_\pi$.

Definition 2.1. *A continuous function* $\varphi : G \longrightarrow \mathbb{C}$ *is said to be of positive type if the kernel defined on* $G \times G$ *by* $(g_1, g_2) \mapsto \varphi(g_j^{-1} g_i)$ *is of positive type, i.e. for all* $g_1, g_2, ..., g_n \in G$ *and all* $c_1, c_2, ..., c_n \in \mathbb{C}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \varphi(g_j^{-1} g_i) \geq 0.$$

Proposition 2.2. *Let* ξ *be a vector in* $\mathcal{H}_\pi$. *Then the function*

$$C_\xi^\pi : G \longrightarrow \mathbb{C},\ g \longmapsto \langle \pi(g)\xi, \xi \rangle$$

is of positive type.

Remark 2.3. $\mathcal{H}_\pi$ is the completion of the space $V :=$ span $\{\lambda(g) C_\xi^\pi, g \in G\}$, (where λ is the left regular representation) with respect to the following scalar inner product

$$\langle \sum_i c_i \lambda(g_i) C_\xi^\pi, \sum_j c_j \lambda(g_j) C_\xi^\pi \rangle := \sum_{i,j} c_i \overline{c_j} C_\xi^\pi (g_j^{-1} g_i).$$

Theorem 2.4. *([Dixmier]) Let* $(\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}$ *be a family of irreducible unitary representations of* G. *Then* $(\pi_k)_k$ *converges to* π *if and only if for some non-zero (res. for every) vector* ξ *in* $\mathcal{H}_\pi$, *there exist* $\xi_k \in \mathcal{H}_{\pi_k}, k \in \mathbb{N}$, *such that the sequence* $(C_{\xi_k}^{\pi_k})_k$ *of functions converges uniformly on compacta to* C_ξ^π.

The topology of $\widehat{G}$ can also be expressed by the weak convergence of the coefficient functions.

`t1` **Theorem 2.5.** *([Dixmier]) Let* $(\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}$ *be a sequence of irreducible unitary representations of* G. *Then* $(\pi_k)_k$ *converges to* π *if and only if for some non-zero (resp. for every) vector* ξ *in* $\mathcal{H}_\pi$, *there are* $\xi_k \in \mathcal{H}_{\pi_k}$ *such that the sequence of linear functionals* $(C_{\xi_k}^{\pi_k})_k \subset C^*(G)'$ *converges weakly on some dense subspace of the* C^**-algebra* $C^*(G)$ *of* G *to the linear functional* C_ξ^π.

If G is a Lie group, then we denote respectively by $\mathfrak{g}$ the Lie algebra of G and by $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. For a unitary representation $(\pi, \mathcal{H}_\pi)$ of G, let $\mathcal{H}_\pi^\infty$ be the subspace of $\mathcal{H}_\pi$ consisting of the C^∞-vectors for π.

`cinfcon` **Corollary 2.6.** *Let* $(\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}$ *be a sequence of irreducible unitary representations of the Lie group* G. *If* $(\pi_k)_k$ *converges to* π *then for every unit vector* ξ *in* $\mathcal{H}_\pi^\infty$, *there exist* $\xi_k \in \mathcal{H}_{\pi_k}^\infty, k \in \mathbb{N}$, *such that the numerical sequence* $(\langle d\pi_k(D)\xi_k, \xi_k \rangle)_k$ *converges to* $\langle d\pi(D)\xi, \xi \rangle$, *for each* $D \in \mathcal{U}(\mathfrak{g})$.

Proof. Let $\xi \in \mathcal{H}_\pi^\infty$ be a unit vector . It follows from [Dix-Mal], that there exist $f_1, \cdots, f_s \in C_c^\infty(G)$ and linearely independent vectors $\xi_1, \cdots, \xi_s \in \mathcal{H}_\pi$, such that $\xi = \pi(f_1)\xi_1 + \cdots + \pi(f_s)\xi_s$. Since π is irreducible, we can find for any non-zero vector $\eta \in \mathcal{H}_\pi$, elements q_j in the C^*-algebra of G, such that $\xi_j = \pi(q_j)\eta, j = 1 \cdots, s$. Hence $\xi = \sum_{j=1}^{s} \pi(f_j)\pi(q_j)\eta$.

Choose now for $k \in \mathbb{N}$ vectors $\eta_k \in \mathcal{H}_{\pi_k}$, such that the coefficients $C^{\pi_k}_{\eta_k}$ converge weakly to the coefficient C^{π}_{η}. Let $\xi_k := \sum_{j=1}^{s} \pi_k(f_j)\pi_k(q_j)\eta_k, k \in \mathbb{N}$. Then, for $D \in \mathcal{U}(\mathfrak{g})$ it follows that

$$\begin{aligned}
\lim_{k+\infty} \langle d\pi_k(D)\xi_k, \xi_k\rangle &= \lim_{k+\infty} \langle \sum_{j=1}^{s} \pi_k(D * f_j)\pi_k(q_j)\eta_k, \sum_{i=1}^{s} \pi_k(f_i)\pi_k(q_i)\eta_k\rangle \\
&= \sum_{i,j=1}^{s} \lim_{k+\infty} \langle \pi_k(q_i^* * f_i^* * D * f_j * q_j)\eta_k, \eta_k\rangle \\
&= \sum_{i,j=1}^{s} \langle \pi(q_i^* * f_i^* * D * f_j * q_j)\eta, \eta\rangle \\
&= \langle d\pi(D)\xi, \xi\rangle.
\end{aligned}$$

□

Example 2.7. Let $G = \mathbb{R}^n$. Then $\widehat{G}$ is the set of all characters χ_ℓ defined by

$$\chi_\ell(x) = e^{-i\langle \ell, x\rangle}$$

for $x, \ell \in \mathbb{R}^n$.
Let $\mathcal{E} := \{e_1, \cdots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. We denote by $\partial_j, j = 1, \cdots, n$ the partial derivative in the direction e_j.
Let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of linear forms on $\mathbb{R}^n$. It is easily seen that the sequence $(\chi_{\ell_k})_k$ converges uniformly on compacta to χ_ℓ if and only if $\lim\limits_{k+\infty} \ell_k = \ell$. Indeed, let if $(\ell_k)_k$ converges to ℓ, then for $\varepsilon, r > 0$ there exists $k_0 = k(\varepsilon, r) \in \mathbb{N}$ such that for any $k \geq k_0$, we have that $\|\ell_k - \ell\| < \frac{\varepsilon}{r}$. Then for each $x \in B(0, r)$

$$|\chi_{\ell_k}(x) - \chi_\ell(x)| = |e^{-i\langle \ell_k - \ell, x\rangle} - 1| \leq \|\ell_k - \ell\|\|x\| < \varepsilon.$$

Conversely, it is enough to prove that

$$\lim_{k+\infty} \langle \ell_k, e_j\rangle = \langle \ell, e_j\rangle \qquad (2) \quad \boxed{\texttt{f1}}$$

for each $j = 1, ..., n$. We choose a function φ in $C_c^\infty(G)$ such that

$$\widehat{\varphi}(\ell) = \int_{\mathbb{R}^n} e^{-i\langle \ell, x\rangle}\varphi(x)dx = 1. \qquad (3) \quad \boxed{\texttt{f2}}$$

¿From Theorem (2.5), it follows that

$$\lim_{k+\infty} \langle \chi_{\ell_k}, \partial_j\varphi\rangle = \langle \chi_\ell, \partial_j\varphi\rangle. \qquad (4) \quad \boxed{\texttt{f3}}$$

In addition, we have

$$\langle \chi_{\ell_k}, \partial_j\varphi\rangle = -\int_{B(0,r)} \partial_j(e^{-i\langle \ell_k, x\rangle})\varphi(x)dx = i\langle \ell_k, e_j\rangle\widehat{\varphi}(\ell_k). \qquad (5) \quad \boxed{\texttt{f4}}$$

Thus, by (3), (4) and (5) we obtain (2).
Hence, we get the (wellknown) result that the dual space of $\mathbb{R}^n$ is homeomorphic with $\mathbb{R}^n$.

2.1 The Motion groups.

We consider now the rotation group $SO(n)$ acting on the abelian group $\mathbb{R}^n$ by automorphisms. In this text, $\mathbb{R}^n$ is identified with the space of $n \times 1$ real matrices. Hence, we can form the semi-direct product $M_n = SO(n) \ltimes \mathbb{R}^n$, equipped with the group law

$$(A, x) \centerdot (B, y) := (AB, x + Ay). \tag{6}$$

Throughout this paper, we identify $(1, x)$ with the vector x, $(A, 0)$ with the matrix A, and we write also $x \centerdot A$ for the element $(1, x)(A, 0) = (A, x)$ of M_n.
We denote by $\mathfrak{m}_n = \mathfrak{so}(n) \oplus \mathbb{R}^n$ the Lie algebra of M_n, and $\mathfrak{m}_n^*$ the vector dual space of $\mathfrak{m}_n$. Then, for all $(A, a) \in M_n$ et $(B, b) \in \mathfrak{m}_n$ we get

$$\begin{aligned}
Ad((A,a)^{-1})(B,b) &:= \frac{d}{ds}\Big|_{s=0} Ad((A,a)^{-1})(e^{sB}, sb) \\
&= \frac{d}{ds}\Big|_{s=0} (A,a)^{-1} \centerdot (e^{sB}, sb) \centerdot (A,a) \\
&= \frac{d}{ds}\Big|_{s=0} (A^t, -A^t a) \centerdot (e^{sB}A, e^{sB}a + sb) \\
&= \frac{d}{ds}\Big|_{s=0} (A^t e^{sB} A, A^t e^{sB} a + sA^t b - A^t a) \\
&= (A^t BA, A^t Ba + A^t b).
\end{aligned}$$

¿From this identity we deduce the Lie bracket

$$[(A, x), (B, y)] = (AB - BA, Ay - Bx) \ (A, B \in \mathfrak{so}(n), x, y \in \mathbb{R}^n).$$

On the Lie algebra $\mathfrak{m}_n$, we have the natural scalar product:

$$\langle (A, x), (B, y) \rangle := \frac{1}{2}\mathrm{tr}(AB^t) + x^t y \ (A, B \in \mathfrak{so}(n), x, y \in \mathbb{R}^n).$$

This scalar product can now be used to identify $\mathfrak{m}_n^*$ with $\mathfrak{m}_n$ and $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. Every linear functional F on $\mathfrak{m}_n$ corresponds to a unique element $\xi_F \in \mathfrak{m}_n$, such that

$$F(\xi_F) = \langle \xi_F, \eta \rangle, \eta \in \mathfrak{m}_n.$$

It follows that for all $(A, a) \in M_n$, $(B, b) \in \mathfrak{m}_\mathfrak{n}$ and $(U, u) \in \mathfrak{m}_n^*$

$$\begin{aligned}
\langle Ad^*((A,a))(U,u), (B,b) \rangle &:= \langle (U,u), Ad((A,a)^{-1})(B,b) \rangle \\
&= \langle (U,u), (A^t BA, A^t Ba + A^t b) \rangle \\
&= \frac{1}{2}\mathrm{tr}(UA^t B^t A) + u^t(A^t Ba) + u^t(A^t b) \\
&= \frac{1}{2}\mathrm{tr}((AUA^t)B^t) + (Au)^t(Ba) + (Au)^t b.
\end{aligned}$$

On the other hand, the fact that $B = (B_{ij})_{1 \le i,j \le n}$ is a skew-symmetric matrix, implies that

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}((va^t - av^t)B^t) &= \frac{1}{2} \sum_{1 \le i,j \le n} (v_i a_j - a_i v_j) B_{ij} \\
&= \sum_{1 \le i,j \le n} v_i B_{ij} a_j \\
&= v^t Ba, \quad \text{for all } v \in \mathbb{R}^n.
\end{aligned}$$

Hence, we obtain

$$\langle Ad^*((A,a))(U,u),(B,b)\rangle = \langle (AUA^t + ((Au)a^t - a(Au)^t), Au), (B,b)\rangle \qquad (7)$$

i.e.

$$Ad^*((A,a))(U,u) \;=\; (AUA^t + [(Au)a^t - a(Au)^t], Au). \qquad (8)$$

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Therefore, for $u \neq 0$, the co-adjoint orbit $\mathcal{O}_{U,u}$ is given by

$$\begin{aligned} \mathcal{O}_{U,u} &= \mathrm{Ad}^*(M_n)(U,u) = \{(AUA^t + [(Au)a^t - a(Au)^t], Au), A \in SO(n), a \in \mathbb{R}^n\} \qquad (9) \\ &= \{(AUA^t, Au), A \in SO(n)\} + (AW_uA^t, 0), \end{aligned}$$

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where $W_u = \{ua^t - au^t, a \in \mathbb{R}^n\}$ is a subspace of dimension $n-1$ of $\mathfrak{so}(n)$. We deduce from this expression that the orbit $\mathcal{O}_{U,u}$ is closed and that the M_n-invariant measure $d\beta_{U,u}$ of the orbit $\mathcal{O}_{U,u}$ can be written as

$$\int_{\mathcal{O}_{U,u}} \varphi(q)\,d\beta_{U,u}(q) = \int_{SO(n)}\int_{W_u} \varphi((AUA^t, Au) + (ABA^t, 0))\,dB\,dA, \varphi \in C_c(\mathcal{O}_{U,u}). \qquad (10)$$

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2.2 The dual space of $SO(n)$.

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We need a precise description of the irreducible representations of $SO(n)$ (see [Knapp] for details). A Cartan subalgebra of $\mathfrak{so}(n)$ can be taken to consist of the two-by-two diagonal blocks $\begin{pmatrix} 0 & \theta_j \\ -\theta_j & 0 \end{pmatrix}, j = 1, \cdots, [n/2]$ starting from the upper left (here $[m], m \in \mathbb{R}$, denotes the largest integer smaller then m). For an integer $j \in [1, [n/2]]$ denote by e_j the associated evaluation functional on the complexification of the Cartan subalgebra. When n is even, say $n = 2d$, the roots are the functionals $\pm e_i \pm e_j$ with $1 \leq i < j \leq d$. When n is odd, say $n = 2d+1$, the roots are the functionals $\pm e_i \pm e_j$ with $1 \leq i < j \leq d$ and also the $\pm e_j$ with $1 \leq j \leq d$. We take the positive roots to be the $e_i \pm e_j$ with $i < j$ and, when n is odd, the e_j. The dominant integral forms λ for $SO(n)$ are given by expressions

$$\lambda_1 e_1 + ... + \lambda_d e_d \longleftrightarrow \lambda = (\lambda_1, ..., \lambda_d) \qquad (11)$$

$$\text{with} \begin{cases} \lambda_1 \geq ... \geq \lambda_{d-1} \geq |\lambda_d| & \text{when } n = 2d \\ \lambda_1 \geq ... \geq \lambda_d \geq 0 & \text{when } n = 2d+1, \end{cases}$$

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with all the λ_j's understood to be integers. Hence the dual space of $SO(n)$ is determined by the representations τ_λ, given by its highest weight λ.

If ξ_λ is the highest weight vector in the space $\mathcal{H}_\lambda$ of τ_λ, then $\tau_\lambda(T)\xi_\lambda = \chi_\lambda(T)\xi_\lambda$ for every T in the maximal torus $\mathbb{T}_n$ of $SO(n)$, where χ_λ is the character of $\mathbb{T}_n$ determined by λ.
The branching theorem for $SO(2d+1)$ with respect to $SO(2d)$ states that the representation of $SO(2d+1)$ with highest weight $(\lambda_1, ..., \lambda_d)$ restricted to $SO(2d)$ decomposes with multiplicity one and the representations of $SO(2d)$ that appear are exactly those with highest weights $\mu = (\mu_1, ..., \mu_d)$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{d-1} \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|. \qquad (12)$$

Similarly the branching theorem for $SO(2d)$ with respect to $SO(2d-1)$ says that the representation of $SO(2d)$ with highest weight $(\lambda_1, ..., \lambda_d)$ decomposes on $S(2d-1)$ with multiplicity 1 and the representations of $SO(2d-1)$ that appear are exactly those with highest weights $\mu = (\mu_1, ..., \mu_{d-1})$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{d-1} \geq \mu_{d-1} \geq |\lambda_d|. \tag{13}$$

3 The dual space of M_n.

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3.1 Description of $\hat{M}_n$.

The dual space of M_n has been described by G. Mackey (for details, see [Mackey1] and [Mackey2]).

For each linear form ℓ on $\mathbb{R}^n$ and any irreducible unitary representation ρ of the stabilizer S_ℓ of ℓ in $SO(n)$, we have that

$$\sigma_{(\rho,\ell)} := \rho \otimes \chi_\ell \tag{14}$$

is an irreducible unitary representation of $H_\ell = S_\ell \ltimes \mathbb{R}^n$ whose restriction to $\mathbb{R}^n$ is a multiple of χ_ℓ, and the induced representation $\pi_{(\rho,\ell)} := \mathrm{ind}_{S_\ell}^{G} \sigma_{(\rho,\ell)}$ is an irreducible representation of M_n. If ℓ and ℓ' are in the same sphere centered at 0, then $\ell' = A \cdot \ell$ for some $A \in SO(n)$ and $S_{\ell'} = A S_\ell A^t$. The representations $\pi_{(\rho,\ell)}$ and $\pi_{(\rho',\ell')}$ (where $\rho'(B) := \rho(A^t B A)$, $B \in S_{\ell'}$) are equivalent (cf. [Mackey1] paragraph 3.9). If $r > 0$ is the radius of the sphere, one notes χ_r the character associated with the linear form ℓ_r which is identified with the vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix}$.

The stabilizer S_{ℓ_r} of ℓ_r is the subgroup $SO(n-1)$. Let us write ρ_μ instead of ρ for the representation of $SO(n-1)$ with highest weight μ and $\pi_{(\mu,r)}$ instead of $\pi_{(\rho_\mu,\ell_r)}$.

In this way we obtain all the irreducible representations of M_n, which are not trivial on its normal subgroup $\mathbb{R}^n$.

On the other hand, every irreducible unitary representation τ_λ of $SO(n)$ extends to an irreducible representation (also denoted by τ_λ) of the entire group M_n, defined by

$$\tau_\lambda(A, x) := \tau_\lambda(A), A \in SO(n), x \in \mathbb{R}^n.$$

Now Mackey's theory tells us that

Proposition 3.1. *$\widehat{SO(n)} \ltimes \mathbb{R}^n$ is in bijection with the set of parameters $\mathcal{P}_n := \widehat{SO(n-1)} \times \mathbb{R}_+^* \bigcup \widehat{SO(n)}$.*

orbatt

3.2 Co-adjoint orbits attached to irreducible representations.

We associate to the representation $\pi_{(\mu,r)}$ the linear functional (J_μ, ℓ_r) in $\mathfrak{m}_n^*$ where

$$J_\mu = \begin{pmatrix} \mu_1 J & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_d J & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

if $n = 2d + 1$ is odd and if $n = 2d$ is even, then

$$J_\mu = \begin{pmatrix} \mu_1 J & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \mu_{d-1} J & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

We see that the stabilizer $M_n(\ell)$ of $\ell = (J_\mu, \ell_r)$ in M_n is equal to $M_n(\ell) = SO(n)(\ell) \ltimes \mathbb{R}^n(\ell)$. Indeed, by (8), we have that

$$\begin{aligned} M_n(\ell) &= \{(A, a) \in M_n; (AJ_\mu A^t + (A\ell_r a^t - a(A\ell_r)^t), A\ell_r) = (J_\mu, \ell_r)\} \\ &= \{(A, a) \in M_n; A \in SO(n-1), AJ_\mu A^t + (\ell_r a^t - a(\ell_r)^t) = J_\mu\} \\ &= \{(A, a) \in M_n; a \in \mathbb{R}\ell_r, A \in SO(n-1), AJ_\mu A^t = J_\mu\}, \end{aligned}$$

since $AJ_\mu A^t \in \mathfrak{so}(n-1)$ and

$$\ell_r a^t - a(\ell_r)^t = \begin{pmatrix} 0 & \dots & 0 & -ra_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & ra_{n-1} \\ ra_1 & \dots & ra_{d-1} & 0 \end{pmatrix}.$$

Therefore $a \in \mathbb{R}\ell_r = \mathbb{R}^n(\ell)$ and $A \in SO(n)(\ell)$. Hence, ℓ is aligned (see [Lipsman] Lemma 4.2). A linear functional $\ell \in \mathfrak{m}_n^*$ is called admissible, if there exists a unitary character χ of the connected component of $M_n(\ell)$, such that $d\chi = i\ell_{|\mathfrak{m}_n(\ell)}$. It is clear now that the linear functionals (J_μ, ℓ_r) are all admissible and so, according to [Lipsman], the representation of M_n obtained by holomorphic induction from the linear functional (J_μ, ℓ_r) is equivalent to the representation $\pi_{(\mu,r)}$ (see [Lipsman]).

For τ_λ we take the linear functional $(J_\lambda, 0)$ of $\mathfrak{m}_n^*$ defined in the following way:
Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We identify the linear form λ with the element J_λ in $\mathfrak{so}(n)$ where

$$J_\lambda = \begin{pmatrix} \lambda_1 J & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_d J \end{pmatrix},$$

if $n = 2d$ is even. If $n = 2d + 1$ is odd, then we put

$$J_\lambda = \begin{pmatrix} \lambda_1 J & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda_d J & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then the representation of M_n obtained by holomorphic induction from $(J_\lambda, 0)$ is equivalent to τ_λ.
We denote by $\mathcal{O}_\lambda$ the co-adjoint orbit of $(J_\lambda, 0)$ and by $\mathcal{O}_{(\mu,r)}$ the co-adjoint orbit of (J_μ, ℓ_r). Let $\mathfrak{m}_n^\ddagger \subset \mathfrak{m}_n^*$ be the union of all the $\mathcal{O}_{(\mu,r)}$ and of all the $\mathcal{O}_\lambda$ and by $\mathfrak{m}_n^\ddagger / M_n$ the corresponding set in the orbit space. It follows now from [Lipsman], that $\mathfrak{m}_n^\ddagger$ is just the set of all admissible linear functionals of $\mathfrak{m}_n$.

4 The topology of the dual space of the motion group M_n

In this section, we characterize the topology of the dual space of the semidirect product $M_n = SO(n) \ltimes \mathbb{R}^n$ in terms of the data $(r > 0, \rho_\mu \in \widehat{SO(n-1)}, \tau_\lambda \in \widehat{SO(n)})$.

4.1

Let us first write down explicitly the representation $\pi_{(\mu,r)} = \mathrm{ind}_{S_{\ell_r}}^{M_n} \sigma_{(\rho_\mu,\ell_r)}$. Its Hilbert space $\mathcal{H}_{\pi_{(\mu,r)}}$ can be identified with the space

$$L^2(M_n/SO(n-1) \ltimes \mathbb{R}^n, \sigma_{(\rho_\mu,\ell_r)}) \simeq L^2(SO(n)/SO(n-1), \rho_\mu).$$

Let ξ be a unit vector in $\mathcal{H}_{\pi_{(\mu,r)}}$. For all $x \in \mathbb{R}^n$, and all $A, B \in SO(n)$ we have

$$\begin{aligned} (\pi_{(\mu,r)}(A,x)\xi)(B) &= (\pi_{(\mu,r)}(x \cdot A)\xi)(B) \\ &= \xi(A^{-1}B(B^{-1} \cdot x^{-1} \cdot B) \\ &= \overline{\chi_r(B^{-1} \cdot x^{-1} \cdot B)}\xi(A^{-1}B). \end{aligned}$$

However

$$B^{-1} \cdot x^{-1} \cdot B = (1, -B^{-1}x) =: -B^{-1}x.$$

Therefore

$$(\pi_{(\mu,r)}(A,x)\xi)(B) = e^{-i\langle B\ell_r, x\rangle}\xi(A^{-1}B). \qquad (15)$$

`infmur`

It follows that

$$C_\xi^{\pi_{(\mu,r)}}(A,x) = \int_{SO(n)} e^{-i\langle B\ell_r, x\rangle}\langle \xi(A^{-1}B), \xi(B)\rangle_{\mathcal{H}_{\rho_\mu}} dB. \qquad (16)$$

`coeffmur`

`l2mu`

4.2

Let us use the notations of the subsection (2.2). By the theorems of Murnaghan and Frobenius (see subsection (2.2)), we have

$$\pi_\mu := \pi_{(\mu,r)|SO(n)} \simeq \mathrm{ind}_{SO(n-1)}^{SO(n)} \rho_\mu = \begin{cases} \sum\limits_{\substack{\tau_\lambda \in \widehat{SO(2d+1)} \\ \lambda_1 \geq \mu_1 \geq \ldots \geq \lambda_{d-1} \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|}} \tau_\lambda & (\text{ if } n = 2d+1) \\ \sum\limits_{\substack{\tau_\lambda \in \widehat{SO(2d)} \\ \lambda_1 \geq \mu_1 \geq \ldots \geq \lambda_{d-1} \geq \mu_{d-1} \geq |\lambda_d|}} \tau_\lambda & (\text{if } n = 2d) \end{cases} \qquad (17)$$

`resdesin`

Since ρ_μ is a subrepresentation of $\mathrm{ind}_{\mathbb{T}_{n-1}}^{SO(n-1)} \chi_\mu$, we can identify the Hilbert space $\mathcal{H}_{\pi_{(\mu,r)}}$ of the representation $\pi_{(\mu,r)}$ with a closed subspace L^2_μ of the space $L^2(SO(n)/\mathbb{T}_{n-1}, \chi_\mu)$. Here $\mathbb{T}_{n-1} \subset \mathbb{T}_n$ denotes the maximal torus of $SO(n-1)$. Now every irreducible representation τ_λ of $SO(n)$ can be realized as a subrepresentation of $L^2(SO(n))$ via the intertwining operator

$$U_\lambda : \mathcal{H}_\lambda \to L^2(SO(n)); U_\lambda(\xi)(k) := \langle \xi, \pi_\lambda(k)\xi_\lambda\rangle, k \in SO(n).$$

For $\tau_\lambda \in \widehat{SO(n)}$ we take an orthonormal basis $\mathcal{B}^\lambda = \{\phi_j^\lambda; j = 1, \cdots, d_\lambda\}$ of $\mathcal{H}_{\tau_\lambda}$ consisting of eigenvectors for $\mathbb{T}_n$ of $\mathcal{H}_{\tau_\lambda}$ (d_λ denotes the dimension of τ_λ), and for every eigenvalue

χ_ν of $\mathbb{T}_{n-1}$ appearing in τ_λ we denote by $I(\lambda,\nu)$ the set of indices i for which $\tau_\lambda(A)\phi_i^\lambda = \chi_\nu(A)\phi_i^\lambda, A \in \mathbb{T}_{n-1}$. It follows then from the theorem of Peter-Weyl, that

$$L^2_\mu \subset \sum_{\substack{\tau_\lambda \in \widehat{SO(2d+1)} \\ \tau_\lambda \in \pi_\mu}} \sum_{1\leq j\leq d_\lambda} \sum_{i\in I(\lambda,\mu)} \mathbb{C}\overline{C^\lambda_{i,j}}, \tag{18}$$

`phijbasis`

where for simplicity of notations, we have written $C^\lambda_{i,j} := C^{\tau_\lambda}_{\phi_i^\lambda,\phi_j^\lambda}, 1 \leq i,j \leq d_\lambda$.

4.3

In the proofs of the following theorems, we shall often make us of the fact that the Laplacian $\Delta := \sum_{j=1}^n \partial_j^2$ on $\mathbb{R}^n$ is central in the enveloping algebra of $\mathfrak{g}$.

`laplaceirr` **Lemma 4.1.** *For every irreducible representation $\pi_{(\mu,r)}(r > 0, \rho_\mu \in SO(n-1))$ of M_n, we have that*

$$d\pi_{(\mu,r)}(\Delta) = -r^2\mathbb{I}.$$

Proof. It follows from (15), that for all $B \in SO(n)$ and all $i = 1,...,n$,

$$d\pi_{(\mu,r)}(e_i)\xi(B) = \left.\frac{d}{dt}\right|_{t=0} \pi_{(\mu,r)}(te_i)\xi(B) = -i\langle B\ell_r, e_i\rangle\xi(B). \tag{19}$$

Hence

$$d\pi_{(\mu,r)}(e_i^2)\xi(B) = -\langle B\ell_r, e_i\rangle^2\xi(B). \tag{20}$$

Therefore

$$\begin{aligned}
\langle d\pi_{(\mu,r)}(\Delta)\xi,\xi\rangle &= -\sum_{1\leq i\leq n} \langle d\pi_{(\mu,r)}(e_i^2)\xi,\xi\rangle \\
&= -\int_{SO(n)/SO(n-1)} \Big(\sum_{1\leq i\leq n} \langle B\ell_r, e_i\rangle^2\Big)|\xi(B)|^2 dB \\
&= -\int_{SO(n)/SO(n-1)} \|\ell_r\|^2 |\xi(B)|^2 dB = -r^2.
\end{aligned}$$

□

`rnenull` **Theorem 4.2.** *Let $r > 0$ and $\rho_\mu \in \widehat{SO(n-1)}$. Then a sequence $(\pi_{(\mu^k,r_k)})_k$ of irreducible representations of M_n converges in $\hat{M}_n$ to $\pi_{(\mu,r)}$ if and only if $(r_k)_k$ tends to r as $k \longrightarrow +\infty$ and $\mu^k = \mu$ for k large enough.*

Proof. Suppose at first that $\lim\limits_{k+\infty} r_k = r$ and $\mu^k = \mu$ for k large enough.
We choose $\xi_k = \xi$ for all $k \in \mathbb{N}$. Thus for $f \in C_c^\infty(SO(n) \ltimes \mathbb{R}^n)$ and for every $k \in \mathbb{N}$ we have

$$\langle C_{\xi_k}^{\pi_{(\mu^k,r_k)}}, f\rangle = \int_{R^n}\int_{SO(n)}\int_{SO(n)} e^{-i\langle B\ell_{r_k},x\rangle} f(A,x)\langle\xi(A^{-1}B),\xi(B)\rangle_{\mathcal{H}_{\rho_\mu}} dBdAdx. \tag{21}$$

Then, by Lebesgue's theorem $(\langle C_{\xi_k}^{\pi_{(\mu^k,r_k)}}, f\rangle)_k$ converges to $\langle C_\xi^{\pi_{(\mu,r)}}, f\rangle$.
Conversely, suppose that $(\pi_{(\mu^k,r_k)})_k$ converges to $\pi_{(\mu,r)}$. It follows from Corollary 2.6 that

for a unit vector $\xi \in \mathcal{H}^{\infty}_{\pi_{(\mu^k,r_k)}}$, there exist $\xi_k \in \mathcal{H}_{\pi_{(\mu^k,r_k)}}$ such that $\|\xi_k\|_{\mathcal{H}_{\pi_{(\mu^k,r_k)}}} = 1$ and $(\langle d\pi_{(\mu^k,r_k)}(\Delta)\xi_k, \xi_k\rangle)_k$ converges to $\langle d\pi_{(\mu,r)}(\Delta)\xi, \xi\rangle$. By Lemma 4.1 we have that

$$-r_k^2 = \langle d\pi_{(\mu^k,r_k)}(\Delta)\xi_k, \xi_k\rangle \to \langle d\pi_{(\mu,r)}(\Delta)\xi, \xi\rangle = -r^2.$$

Thus, r_k tends to r as $k \longrightarrow +\infty$. It remains for us to show that $\mu^k = \mu$ for k large enough.

Let ξ be any unit vector in $\mathcal{H}_{\pi_{(\mu,r)}}$. So by theorem (2.5) there are $\xi_k \in \mathcal{H}_{\pi_{(\mu^k,r_k)}}$ such that $\|\xi_k\|_{\mathcal{H}_{\pi_{(\mu^k,r_k)}}} = 1$ and $(C^{\pi_{(\mu^k,r_k)}}_{\xi_k})_k$ converges uniformly on compacta to $C^{\pi_{(\mu,r)}}_{\xi}$. In particular, we have that

$$\begin{aligned}
\lim_{k\to\infty} C^{\pi_{(\mu^k,r_k)}}_{\xi_k}(A,0) &= \lim_{k\to\infty} \langle \pi_{(\mu^k,r_k)}(A,0)\xi_k, \xi_k\rangle \\
&= \lim_{k\to\infty} \int_{SO(n)} \xi_k(A^{-1}B)\overline{\xi_k(B)}dB \\
&= \lim_{k\to\infty} \int_{SO(n)} \xi(A^{-1}B)\overline{\xi(B)}dB \\
&= C^{\pi_{(\mu,r)}}_{\xi}(A,0).
\end{aligned} \tag{22} \quad \boxed{\texttt{unifconva}}$$

uniformly in $A \in SO(n)$. However, by (18) we can write

$$\xi_k = \sum_{\substack{\tau_\lambda \in \widehat{SO(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1\le j\le d_\lambda} \sum_{i \in I(\lambda,\mu^k)} a^{(\lambda,k)}_{i,j}\overline{C^{\lambda}_{i,j}}. \tag{23}$$

and

$$1 = \|\xi_k\|^2_{\mathcal{H}_{\pi_{(\mu^k,r_k)}}} = \sum_{\substack{\tau_\lambda \in \widehat{SO(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1\le j\le d_\lambda} \sum_{i \in I(\lambda,\mu^k)} \frac{|a^{(\lambda,k)}_{i,j}|^2}{d_\lambda}. \tag{24} \quad \boxed{\texttt{e7}}$$

In addition, for all $A, B \in SO(n)$

$$C^{\lambda}_{i,j}(A^{-1}B) = \langle \tau_\lambda(A^{-1}B)\phi^\lambda_i, \phi^\lambda_j\rangle = C^{\tau_\lambda}_{\phi^\lambda_i, \tau_\lambda(A)\phi^\lambda_j}(B). \tag{25}$$

Consequently, by using the orthogonality relation (1), we have

$$\begin{aligned}
C^{\pi_{(\mu^k,r_k)}}_{\xi_k}(A,0) &= \\
& \sum_{\substack{\tau_\lambda \in \widehat{SO(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1\le j,j'\le d_\lambda} \sum_{i,i' \in I(\lambda,\mu^k)} a^{(\lambda,k)}_{i,j}\overline{a^{(\lambda,k)}_{i',j'}}\,\overline{\langle C^{\tau_\lambda}_{\phi^\lambda_i,\tau_\lambda(A)\phi^\lambda_j}, C^{\tau_\lambda}_{\phi^\lambda_{i'},\phi^\lambda_{j'}}\rangle} \qquad (26) \quad \boxed{\texttt{decomcij}} \\
&= \sum_{\substack{\tau_\lambda \in \widehat{SO(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1\le j,j'\le d_\lambda} \sum_{i,i' \in I(\lambda,\mu^k)} \frac{a^{(\lambda,k)}_{i,j}\overline{a^{(\lambda,k)}_{i',j'}}}{d_\lambda}\overline{\langle \phi^\lambda_i, \phi^\lambda_{i'}\rangle\langle \phi^\lambda_{j'}, \tau_\lambda(A)\phi^\lambda_j\rangle} \\
&= \sum_{\substack{\tau_\lambda \in \widehat{SO(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1\le j,j'\le d_\lambda} \sum_{i \in I(\lambda,\mu^k)} \frac{a^{(\lambda,k)}_{i,j}\overline{a^{(\lambda,k)}_{i,j'}}}{d_\lambda} C^{\lambda}_{j,j'}(A).
\end{aligned}$$

Let $\tilde{\mu} = (\mu_1, ..., \mu_{d-1}, |\mu_d|)$ if $n = 2d+1$ is odd and $\tilde{\mu} = (\mu_1, ..., \mu_{d-1}, 0)$ if $n = 2d$ is even. Then $I(\tilde{\mu}, \mu)$ consists of one point, since $\tilde{\mu}$ is dominant integral and we can take $I(\tilde{\mu}, \mu) = \{1\}$. We choose now $\overline{\xi_\mu} := \overline{\xi} := \sqrt{d_{\tilde{\mu}}} C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}}} \in L^2_\mu$ and we obtain

$$\begin{array}{rcl} C^{\pi(\mu,r)}_{\xi}(A,0) & = & \displaystyle\int_{SO(n)} \xi(A^{-1}B)\overline{\xi(B)}dB \\ & = & d_{\tilde{\mu}} \overline{\langle C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \tau_{\tilde{\mu}}(A)\phi_1^{\tilde{\mu}}}, C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}}} \rangle} \\ & = & \overline{\langle \phi_1^{\tilde{\mu}}, \tau_{\tilde{\mu}}(A)\phi_1^{\tilde{\mu}} \rangle} = C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}}}(A), A \in SO(n). \end{array}$$

It follows from (22) and (26), that the numerical series defined by

$$\begin{array}{rcl} S_k & := & \langle C^{\pi(\mu^k, r_k)}_{\xi^k}(.,0), C^{\pi(\mu,r)}_{\xi}(.,0) \rangle \\ & = & \displaystyle\sum_{\substack{\tau_\lambda \in \widehat{SO(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \le j,j' \le d_\lambda} \sum_{i \in I(\lambda, \mu^k)} \frac{a^{(\lambda,k)}_{i,j} \overline{a^{(\lambda,k)}_{i,j'}}}{d_\lambda} \langle C^\lambda_{j,j'}, C^{\tilde{\mu}}_{1,1} \rangle \\ & = & \displaystyle\sum_{i \in I(\lambda, \mu^k)} \frac{|a^{(\tilde{\mu},k)}_{i,1}|^2}{d^2_{\tilde{\mu}}} (k \in \mathbb{N}) \end{array}$$

converges to the number $\langle C^{\pi(\mu,r)}_{\xi}(.,0), C^{\pi(\mu,r)}_{\xi}(.,0) \rangle = \frac{1}{d_{\tilde{\mu}}} \neq 0$. Hence by the orthogonality relation (1) and (11), we must have that $\tau_{\tilde{\mu}} \in \pi_{\mu^k}$ for k large enough, since otherwise the right hand side of S_k is 0 for an infinity of $k's$. Therefore we deduce from (17) that

$$\mu_1 \ge \mu_1^k \ge \cdots \ge \mu_{d-1} \ge \mu^k_{d-1} \ge |\mu_d| \ge |\mu_d^k| (\text{ resp. } \mu_1 \ge \mu_1^k \ge \cdots \ge \mu_{d-2} \ge \mu^k_{d-2} \ge |\mu_{d-1}| \ge |\mu^k_{d-1}|)$$

and also that

$$\lim_{k \to \infty} \sum_{i \in I(\lambda, \mu^k)} \frac{|a^{(\tilde{\mu},k)}_{i,1}|^2}{d_{\tilde{\mu}}} = 1. \qquad (27) \quad \boxed{\texttt{a1kto1}}$$

Whence, by (24)

$$\lim_{k \to \infty} \left[\sum_{\substack{\tau_\lambda \neq \tau_{\tilde{\mu}} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \le j \le d_\lambda} \sum_{i \in I(\lambda, \mu^k)} \frac{|a^{(\lambda,k)}_{i,j}|^2}{d_\lambda} + \sum_{2 \le j \le d_{\tilde{\mu}}} \sum_{i \in I(\lambda, \mu^k)} \frac{|a^{(\tilde{\mu},k)}_{i,j}|^2}{d_{\tilde{\mu}}} \right] = 0.$$

Thus, we have $\xi_k = \sum_{i \in I(\lambda, \mu^k)} a^{(\tilde{\mu},k)}_{1,i} \overline{C^{\tilde{\mu}}_{1,i}} + E_k$ where $E_k \in L^2_{\mu^k}$ with $\lim\limits_{k \to \infty} \|E_k\|_2 = 0$ $(k \in \mathbb{N})$. Let $\eta_k := \sum_{i \in I(\lambda, \mu^k)} a^{(\tilde{\mu},k)}_{1,i} \overline{C^{\tilde{\mu}}_{1,i}}, k \in \mathbb{N}$. Since the sequence $(\mu^k)_k$ is seen to be bounded, we can decompose it (apart from a finite number of indices) in a finite union of constant subsequences. Let us show that all these constant subsequences are equal to μ. Take such a constant subsequence $(\mu^{k_j})_j$, i.e, we have that $\mu^{k_j} = \mu', j \in \mathbb{N}$, with $\mu_1 \ge \mu_1' \ge ... \ge \mu_{d-2} \ge \mu'_{d-2} \cdots$.

Then, we obtain for $x \in \mathbb{R}^n$ that

$$\begin{aligned} C_{\xi_{k_j}}^{\pi_{(\mu^{k_j}, r_{k_j})}}(1,x) = C_{\xi_{k_j}}^{\pi_{(\mu', r_{k_j})}}(1,x) \\ &= \int_{SO(n)} e^{-i\langle B\ell_{r_{k_j}}, x\rangle} |\xi_{k_j}(B)|^2 dB \\ &= \int_{SO(n)} e^{-i\langle B\ell_r, x\rangle} |\eta_k(B)|^2 dB + \varepsilon_{k_j}(x), \end{aligned}$$

where $\varepsilon_{k_j}(x)$ tends uniformly to zero as k tends to infinity. Since by (27) (for another subsequence) $\eta_{k_j} = \sum_{i \in I(\lambda,\mu')} a_{1,i}^{(\tilde{\mu},k_j)} \overline{C_{1,i}^{\tilde{\mu}}}$ tends to an element $\xi_{\mu'} = \sum_{i \in I(\lambda,\mu')} a_{1,i}^{(\tilde{\mu})} \overline{C_{1,i}^{\tilde{\mu}}} \in L^2_{\mu'}$, we have

$$\lim_{k\to\infty} C_{\xi_{k_j}}^{\pi_{(\mu^{k_j}, r_{k_j})}}(1,x) = \int_{SO(n)} e^{-i\langle B\ell_r, x\rangle} |\xi_{\mu'}(B)|^2 dB. \tag{28}$$

Consequently, we find

$$\int_{SO(n)} e^{-i\langle B\ell_r, x\rangle} |\xi_{\mu'}(B)|^2 dB = \int_{SO(n)} e^{-i\langle B\ell_r, x\rangle} |\xi_{\mu}(B)|^2 dB. \tag{29}$$

e8

We define two measures δ_μ and $\delta_{\mu'}$ on $\mathbb{R}^n$ by

$$\delta_\mu(f) = \int_{SO(n)} f(B\ell_r) |\xi_\mu(B)|^2 dB$$

and

$$\delta_{\mu'}(f) = \int_{SO(n)} f(B\ell_r) |\xi_{\mu'}(B)|^2 dB,$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. From (29), it follows that $\widehat{\delta_\mu} = \widehat{\delta_{\mu'}}$ i.e. $\delta_\mu = \delta_{\mu'}$ and $|\xi_\mu| = |\xi_{\mu'}|$. Hence

$$0 \neq \langle \phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}} \rangle = |\xi_\mu(\mathbb{I}_n)| = |\xi_{\mu'}(\mathbb{I}_n)| = |\sum_{i \in I(\lambda,\mu')} a_{1,i}^{(\tilde{\mu})} \langle \phi_i^{\tilde{\mu}}, \phi_1^{\tilde{\mu}} \rangle|.$$

This implies that $\langle \phi_{1,i}^{\tilde{\mu}}, \phi_{1,1}^{\tilde{\mu}} \rangle \neq 0$ for at least one $i \in I(\tilde{\mu}, \mu')$. This proves that $\mu = \mu'$. $\square$

Let us use again the notations of the subsection (2.2).

Theorem 4.3. *Let $(\pi_{(\mu^k, r_k)})_k$ be sequence of irreducible representations of M_n. Then $(\pi_{(\mu^k, r_k)})_k$ converges to τ_λ in $\hat{M}_n$ if and only if $\lim\limits_{k\to\infty} r_k = 0$ and $\tau_\lambda \in \pi_{\mu^k}$ for k large enough.*

Proof. Suppose first that $n = 2d+1$ is odd. Then the condition $\tau_\lambda \in \pi_{\mu^k}$ means that $\lambda_1 \geq \mu_1^k \geq ... \geq \lambda_{d-1} \geq \mu_{d-1}^k \geq \lambda_d \geq |\mu_d^k|$. Suppose now that this condition is fulfilled or k large enough and that $\lim\limits_{k\to\infty} r_k = 0$. Hence the sequence $(\mu^k)_k$ is bounded and we can again write $(\mu^k)_k$ as a finite union of eventually constant sequences. Take such an infinite subset $I \subset \mathbb{N}$, such that $\mu^{k_j} = \mu = \mu(I)$ for all $j \in I$. We choose a unit vector $\xi \in \mathcal{H}_{\tau_\lambda} \subset \mathcal{H}_{\pi_{(\mu^k, r_k)}}$. Hence we have that

$$\begin{aligned} \langle \tau_\lambda(A)\xi, \xi \rangle_{\mathcal{H}_{\tau_\lambda}} &= \langle (\mathrm{ind}_{SO(n-1)}^{SO(n)} \rho_\mu)(A)\xi, \xi \rangle_{L^2} \\ &= \int_{SO(n)} \langle \xi(A^{-1}B), \xi(B) \rangle_{\mathcal{H}_{\rho_\mu}} dB, \end{aligned}$$

for all $A \in SO(n)$. Thus, we can choose $\xi_{k_j} = \xi$ for all $j \in I$. We obtain, for all f in $C_c^\infty(SO(n) \ltimes \mathbb{R}^n)$

$$\begin{aligned} &\langle C_{\xi_{k_j}}^{\pi_{(\mu^{k_j}, r_{k_j})}}, f\rangle = \langle C_{\xi}^{\pi_{(\mu, r_{k_j})}}, f\rangle \\ = &\int_{R^n}\int_{SO(n)}\int_{SO(n)} \chi_{r_k}(B^{-1}x) f(A,x) \langle \xi(A^{-1}B), \xi(B)\rangle_{\mathcal{H}_{\rho_\mu}} dBdAdx. \end{aligned}$$

This integral converges to

$$\begin{aligned} &\int_{SO(n)}\int_{\mathbb{R}^n} f(A,x) \int_{SO(n)} \langle \xi(A^{-1}B), \xi(B)\rangle_{\mathcal{H}_{\rho_\mu}} dBdAdx \\ = &\int_{SO(n)}\int_{R^n} f(A,x) \langle \tau_\lambda(A,x)\xi, \xi\rangle_{\mathcal{H}_{\tau_\lambda}} dAdx \\ = &\langle C_\xi^{\tau_\lambda}, f\rangle. \end{aligned}$$

By considering all possible subsets I of this kind, we see that $\pi_{(\mu^k, r_k)}$ has as limit point the representation τ_λ.

Conversely, it is clear from Lemma (4.1), and Corollary (2.6) that $\lim\limits_{k+\infty} r_k = 0$, since τ_λ is trivial on $\mathbb{R}^n$. It remains for us to show that $\lambda_1 \geq \mu_1^k \geq ... \geq \lambda_{d-1} \geq \mu_{d-1}^k \geq \lambda_d \geq |\mu_d^k|$ for k large enough. We use the notations and proceedings of the proof of Theorem 4.2.
Let $\xi \in \mathcal{H}_{\tau_\lambda} = \phi_1^\lambda$ be a unit vector associated to the heighest weight λ. Then there exist $\xi_k \in \mathcal{H}_{\pi_{(\mu^k, r_k)}}$ of length 1 such that for all $A \in SO(n)$ we have $\lim\limits_{k\to\infty} C_{\xi_k}^{\pi_{(\mu^k, r_k)}}(A, 0) = C_\xi^{\tau_\lambda}(A)$. Then by (18) we can write

$$\xi_k = \sum_{\substack{\tau_{\lambda'} \in \widehat{SO(n)} \\ \tau_{\lambda'} \in \pi_{\mu^k}}} \sum_{1 \leq j \leq d_\lambda} \sum_{i \in I(\lambda', \mu^k)} a_{i,j}^{(\lambda', k)} \overline{C_{i,j}^{\lambda'}}.$$

and

$$1 = \|\xi_k\|^2_{\mathcal{H}_{\pi_{(\mu^k, r_k)}}} = \sum_{\substack{\tau_{\lambda'} \in \widehat{SO(n)} \\ \tau_{\lambda'} \in \pi_{\mu^k}}} \sum_{1 \leq j \leq d_{\lambda'}} \sum_{i \in I(\lambda', \mu^k)} \frac{|a_{i,j}^{(\lambda', k)}|^2}{d_{\lambda'}}.$$

The numerical series S_k defined by

$$\begin{aligned} S_k &:= \langle C_{\xi^k}^{\pi_{(\mu^k, r_k)}}(.,0), C\tau_\lambda)_\xi(.,0)\rangle \\ &= \sum_{\substack{\tau_{\lambda'} \in \widehat{SO(n)} \\ \tau_{\lambda'} \in \pi_{\mu^k}}} \sum_{1 \leq j, j' \leq d_{\lambda'}} \sum_{i \in I(\lambda', \mu^k)} \frac{a_{i,j}^{(\lambda', k)} \overline{a_{i,j'}^{(\lambda', k)}}}{d_\lambda} \langle C_{j,j'}^{\lambda'}, C_{1,1}^\lambda\rangle \end{aligned}$$

converges to $\langle C_\xi^{\tau_\lambda}, C_\xi^{\tau_\lambda}\rangle = \frac{1}{d_\lambda} \neq 0$. By the orthogonality relation (1), it follows that $\tau_\lambda \in \pi_{\mu^k}$ for k large enough, i.e. that $\lambda_1 \geq \mu_1^k \geq ... \geq \lambda_{d-1} \geq \mu_{d-1}^k \geq \lambda_d \geq |\mu_d^k|$ for k large enough. The same reasoning applies if n is even. $\square$

Remark 4.4. It follows from the preceding theorems that a sequence $(\pi_{(\mu^k,r_k)})_k$ can only have a limit point if the sequences $(\mu^k)_k$ and $(r_k)_k$ are bounded. Furthermore we see that subspace $\widehat{SO(n-1)}\times\mathbb{R}_+^*$ of $\hat{M}_n$ has a Hausdorff topology, but that sequences in $\widehat{SO(n-1)}\times\mathbb{R}_+^*$ which converge to elements in $\widehat{SO(n)}$ have infinitely many different limit points.
Of course the subset $\widehat{SO(n)}$ has the discrete topology.

5 Convergence of co-adjoint orbits.

convorb

We have previously seen that the dual space of our motion group $M_n = SO(n)\ltimes\mathbb{R}^n$ consists of all induced representations $\pi_{(\mu,r)} := \mathrm{ind}_{\mathrm{SO(n-1)\ltimes\mathbb{R}^n}}^{\mathrm{SO(n)\ltimes\mathbb{R}^n}}\rho_\mu\otimes\chi_{\mathrm{r}}$ where r runs over $]0,+\infty[$ and $\rho_\mu\in\widehat{SO(n-1)}$, and all extensions of irreducible unitary representations τ_λ of $SO(n)$ on M_n. The subspace W_{ℓ_r} of Formula (9) is generated by the vectors $(E_{n,j}-E_{j,n})$ $1\le j\le n-1$, where $\{E_{i,j}\}_{1\le i,j\le n}$ is the canonical basis of the space of $n\times n$ real matrices. Then, by definition, the space $\mathfrak{m}_n^\ddagger/M_n$ is the set of all orbits

$$\mathcal{O}_{(\mu,r)} = \{(A(J_\mu+W_{\ell_r})A^t, A\ell_r)/A\in SO(n)\} \tag{30}$$

and all orbits

$$\mathcal{O}_\lambda = \{(AJ_\lambda A^t,0)/A\in SO(n)\}, \tag{31}$$

J_μ and J_λ are well-defined in the subsection 3.2. In this way we have

$$\mathfrak{m}_n^\ddagger/M_n \cong\ \mathbb{N}^d\cup\mathbb{N}^{d-1}\times\mathbb{Z}\times]0,+\infty[$$

if $n=2d+1$ is odd. If $n=2d$ is even we have

$$\mathfrak{m}_n^\ddagger/M_n \cong\ \mathbb{N}^{d-1}\times\mathbb{Z}\cup\mathbb{N}^{d-1}\times]0,+\infty[.$$

Theorem 5.1. *Let $(\mathcal{O}_{(\mu^k,r_k)})_{k\in\mathbb{N}}$ be a sequence of orbits in $\mathfrak{m}_n^\ddagger/M_n$. Then $(\mathcal{O}_{(\mu^k,r_k)})_k$ converges to $\mathcal{O}_{(\mu,r)}$ in $\mathfrak{m}_n^\ddagger/M_n$ if and only if $\lim\limits_{k\to\infty} r_k = r$ and $\mu^k=\mu$ for large k.*

Proof. If r_k tends to r an $J_{\mu^k}=J_\mu$ for k large enough, then of course $\lim\limits_{k\to\infty}(J_{\mu^k},\ell_{r_k}) = (J_\mu,\ell_r)$ and so $\lim\limits_{k\to\infty}\mathcal{O}_{(\mu^k,r_k)} = \mathcal{O}_{(\mu,r)}$.
Suppose now that $(\mathcal{O}_{(\mu^k,r_k)})_k$ converges to $\mathcal{O}_{(\mu,r)}$. If $n=2d+1$ is odd, there is then a sequence

$$B_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -b_1(k) \\ 0 & 0 & \dots & 0 & 0 & -b_2(k) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -b_{2d-1}(k) \\ 0 & 0 & \dots & 0 & 0 & -b_{2d}(k) \\ b_1(k) & b_2(k) & \dots & b_{2d-1}(k) & b_{2d}(k) & 0 \end{pmatrix} \tag{32}$$

in W_{ℓ_r} and $(A_k)_k\subset SO(n)$, such that $\lim\limits_{k\to\infty} A_k(J_{\mu^k}+B_k)A_k^t = J_\mu$ and $\lim\limits_{k\to\infty} A_k\ell_{r_k} = \ell_r$. Therefore, there exists a subsequence $(A_{k_j})_{j\in I}$ which converges to an element A_∞, which is necessarily contained in the stabilizer $SO(n-1)$ of the linear form ℓ_r. Then we obtain that

$\lim\limits_{j\to\infty}(J_{\mu^{k_j}} + B_{k_j}) = A_\infty^t J_\mu A_\infty$. In addition, we have for $\ell_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix}$ that $(J_{\mu^{k_j}} + B_{k_j})\ell_r = r\begin{pmatrix} -b_1(k_j) \\ -b_2(k_j) \\ \vdots \\ -b_{2d}(k_j) \\ 0 \end{pmatrix}$ and $(A_\infty^t J_\mu A_\infty)\ell_r = 0$ since $A_\infty^t J_\mu A_\infty = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$. Hence it follows that B_{k_j} converges to zero and $\lim\limits_{j\to\infty} J_{\mu^{k_j}} = A_\infty^t J_\mu A_\infty$. Since the matrices J_{μ^k} are diagonal, so is the matrix $A_\infty^t J_\mu A_\infty$ and the fact that $A \in SO(n-1)$ implies that $A_\infty^t J_\mu A_\infty = J_\mu$. By considering all possible converging subsequences $(A_{k_j})_j$, we have $\mu^k = \mu$ for k large enough. The argument for $n = 2d$ is similar. □

`conorb` **Theorem 5.2.** *Let* $(\mathcal{O}_{(\mu^k,r_k)})_{k\in\mathbb{N}}$ *be a sequence of orbits in* $\mathfrak{m}_n^\ddagger/M_n$. *Then* $(\mathcal{O}_{(\mu^k,r_k)})_k$ *converges to* $\mathcal{O}_\lambda$ *in* $\mathfrak{m}_n^\ddagger/M_n$ *if and only if* $\lim\limits_{j+\infty} r_k = 0$ *and* $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq \dots \geq \lambda_d \geq |\mu_d^k|$ *for* k *large enough (if* $n = 2d+1$ *is odd) resp.* $\lim\limits_{j+\infty} r_k = r$ *and* $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq \dots \geq \mu_{d-1}^k \geq |\lambda_d|$ *for* k *large enough (if* $n = 2d$ *is even).*

Before beginning the proof of this theorem, we need to show some technical lemmas.

Lemma 5.3. *For any integer* $n \geq 2$ *and any scalars* $X_1, ..., X_{n-1}, Y_1, ..., Y_n$ *with* $Y_i \neq Y_j$ *for every* $i \neq j$, *we have*

$$\sum_{j=1}^{n} \frac{\prod_{i=1}^{n-1}(X_i - Y_j)}{\prod_{i=1,i\neq j}^{n}(Y_i - Y_j)} = 1. \tag{33}$$

Proof. According to the Lagrange's interpolation theorem, if P is a polynomial of degree $\leq n-1$, then

$$P(X) = \sum_{j=1}^{n} P(Y_j) \prod_{i=1,i\neq j}^{n} \frac{(X - Y_i)}{(Y_j - Y_i)}. \tag{34}$$

In particular, for $P(X) = \prod_{i=1}^{n-1}(X - X_i)$ we have

$$\prod_{i=1}^{n-1}(X - X_i) = \sum_{j=1}^{n} \prod_{i=1}^{n-1}(Y_j - X_i) \prod_{i=1,i\neq j}^{n} \frac{(X - Y_i)}{(Y_j - Y_i)}. \tag{35}$$

By differentiating $(n-1)$ times the polynomial P, we obtain

$$(n-1)! = \sum_{j=1}^{n} \prod_{i=1}^{n-1}(Y_j - X_i) \frac{(n-1)!}{\prod_{i=1,i\neq j}^{n}(Y_j - Y_i)}.$$

□

Lemma 5.4. *Let $\mu_1 \geq ... \geq \mu_{d-1} \geq |\mu_d|$ and $\lambda_1 \geq ... \geq \lambda_d \geq 0$, where the λ's and μ's are integers. Then, we have $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|$ if only if there exists a skew-symmetric matrix*

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -b_1 \\ 0 & 0 & \dots & 0 & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -b_{2d-1} \\ 0 & 0 & \dots & 0 & 0 & -b_{2d} \\ b_1 & b_2 & \dots & b_{2d-1} & b_{2d} & 0 \end{pmatrix} \tag{36}$$

`matr`

such that spectrum($J_\mu + B$)=$\{0, \pm i\lambda_1, \pm i\lambda_2, ..., \pm i\lambda_d\}$.

Proof. It is easy to prove that, for all $x \in \mathbb{R}$, $\det(J_\mu + B - ix\mathbb{I}) = i(-1)^{d+1}xP(x)$ where

$$P(x) = \prod_{i=1}^{d}(x^2 - \mu_i^2) - \sum_{j=1}^{d}\Big((b_{2j-1}^2 + b_{2j}^2)\prod_{i=1, i\neq j}^{d}(x^2 - \mu_i^2)\Big). \tag{37}$$

Hence we remark so that zero is always an element of the spectrum and that

- $\lim\limits_{x \to +\infty} P(x) = +\infty$,
- $P(\mu_1) = -(b_1^2 + b_2^2)\prod_{i=2}^{d}(\mu_1^2 - \mu_i^2) \leq 0$,
- $P(\mu_2) = -(b_3^2 + b_4^2)\prod_{i=1, i\neq 2}^{d}(\mu_2^2 - \mu_i^2) \geq 0$,
- $P(\mu_3) = -(b_5^2 + b_6^2)\prod_{i=1, i\neq 3}^{d}(\mu_3^2 - \mu_i^2) \leq 0$,
- $P(\mu_4) = -(b_7^2 + b_8^2)\prod_{i=1, i\neq 4}^{d}(\mu_4^2 - \mu_i^2) \geq 0$,

and so on, i.e $P(\mu_i) \leq 0$ if i is odd and $P(\mu_i) \geq 0$, if i is even. We deduce that if $\pm i\lambda_1, \pm i\lambda_2, ..., \pm i\lambda_d$ are the elements of the spectrum of $J_\mu + B$, (i.e. $\pm\lambda_1, \pm\lambda_2, ..., \pm\lambda_d$ are all possible roots of the polynomial P), then we necessarily have

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|. \tag{38}$$

Conversely, assume first that all μ_j are two-by-two distinct. We can choose the skew matrix B such that

$$b_{2j-1}^2 + b_{2j}^2 = \frac{\prod_{i=1}^{i=j}(\lambda_i^2 - \mu_j^2)\prod_{i=j+1}^{i=d}(\mu_j^2 - \lambda_i^2)}{\prod_{i=1}^{i=j-1}(\mu_i^2 - \mu_j^2)\prod_{i=j+1}^{i=d}(\mu_j^2 - \mu_i^2)} = \frac{\prod_{i=1}^{i=d}(\lambda_i^2 - \mu_j^2)}{\prod_{i=1, i\neq j}^{i=d}(\mu_i^2 - \mu_j^2)} \tag{39}$$

for all $j = 1, ..., d$. It follows, by the preceding lemma, that for all $1 \leq k \leq d$

$$\begin{aligned} P(\pm\lambda_k) &= \prod_{i=1}^{d}(\lambda_k^2-\mu_i^2) - \sum_{j=1}^{d}\left(\frac{\prod_{i=1}^{i=d}(\lambda_i^2-\mu_j^2)}{\prod_{i=1,i\neq j}^{i=d}(\mu_i^2-\mu_j^2)}\prod_{i=1,i\neq j}^{d}(\lambda_k^2-\mu_i^2)\right)\\ &= \prod_{i=1}^{d}(\lambda_k^2-\mu_i^2)\left[1-\sum_{j=1}^{d}\frac{\prod_{i=1,i\neq k}^{i=d}(\lambda_i^2-\mu_j^2)}{\prod_{i=1,i\neq j}^{i=d}(\mu_i^2-\mu_j^2)}\right]=0. \end{aligned}$$

Then the spectrum of the matrix $J_\mu + B$ is equal to the set $\{0, \pm i\lambda_1, \pm i\lambda_2, ..., \pm i\lambda_d\}$.

Assume now that there exist two families of integers $\{p_l\}_{1\leq l\leq s}$ and $\{q_l\}_{1\leq l\leq s}$ such that $1 \leq p_1 < q_2 < p_2 < q_2 < ... < p_s < q_s \leq d$, and for all $1 \leq l \leq s$ $\mu_{p_l} = \mu_{p_l+1} = ... = \mu_{q_l-1} = \mu_{q_l}$, $\mu_{q_l} \neq \mu_{q_l+1}$ and $\mu_{p_l-1} \neq \mu_{p_l}$. Hence, if we denote by

$$Q(x) = \prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(x^2-\mu_i^2) \quad , \quad \tilde{Q}(x) = \prod_{i=1}^{p_1-1}\prod_{i=q_1+1}^{p_2-1}\cdots\prod_{i=q_s+1}^{d}(x^2-\mu_i^2)$$

$$\text{and } Q_j(x) = \prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\lambda_k^2-\mu_i^2),$$

then $\det(J_\mu + B - ix\mathbb{I}) = i(-1)^{d+1}x\prod_{l=1}^{s}(x^2-\mu_{p_l}^2)^{q_l-p_l}P(x)$ where

$$\begin{aligned} P(x) &= Q(x) - \Big(\sum_{j=p_1}^{q_1}\cdots\sum_{j=p_s}^{q_s} b_{2j-1}^2 + b_{2j}^2\Big)\tilde{Q}(x)\\ &- \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{d}\Big((b_{2j-1}^2+b_{2j}^2)Q_j(x)\Big). \end{aligned}$$

We can choose the skew-symmetric matrix B such that

$$b_{2j-1}^2 + b_{2j}^2 = \frac{\prod_{i=1}^{i=d}(\lambda_i^2-\mu_j^2)}{\prod_{i=1,i\neq j}^{i=d}(\mu_i^2-\mu_j^2)} = \frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}$$

for all $j = 1, ..., p_1 - 1, q_1 + 1, ..., p_s - 1, q_s + 1, ..., d$ and

$$b_{2p_l-1}^2 + ... + b_{2q_l-1}^2 + b_{2q_l}^2 = \frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2-\mu_{p_l}^2)}{\prod_{i=1}^{p_1-1}\prod_{i=q_1+1}^{p_2-1}\cdots\prod_{i=q_s+1}^{d}(\mu_i^2-\mu_{p_l}^2)}$$

for all $l = 1, ..., s$. It is easy to see that if $\lambda_k = \mu_{p_l}$ then $P(\pm\lambda_k) = Q(\pm\lambda_k) = 0$. On the other hand for all $\lambda_k \neq \mu_{p_l}$

$$\begin{aligned}
P(\pm\lambda_k) = Q(\pm\lambda_k) - \Big(\sum_{l=1}^{s} \frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2-\mu_{p_l}^2)}{\prod_{i=1}^{p_1-1}\prod_{i=q_1+1}^{p_2-1}\cdots\prod_{i=q_s+1}^{d}(\mu_i^2-\mu_{p_l}^2)}\Big)\tilde{Q}(\pm\lambda_k) & \\
- \ \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{d}\Big(\frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}Q_j(\pm\lambda_k)\Big) & \\
= \ Q(\pm\lambda_k)\Big[1-\Big(\sum_{l=1}^{s}\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{d}(\lambda_i^2-\mu_{p_l}^2)}{\prod_{i=1}^{p_1-1}\prod_{i=q_1+1}^{p_2-1}\cdots\prod_{i=q_s+1}^{d}(\mu_i^2-\mu_{p_l}^2)}\Big) & \\
- \ \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{d}\Big(\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}\Big)\Big] & \\
= \ Q(\pm\lambda_k)\Big[1-\sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{d}\Big(\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}\Big)\Big] & \\
= \ 0 &
\end{aligned}$$

by using the preceding lemma. Thus, $\det(J_\mu + B \pm \lambda_k \mathbb{I}) = 0$ for all $k = 1, ..., d$. ☐

Proof. (of the theorem (5.2)) Let $n = 2d + 1$ be odd. Suppose that $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_d \geq |\mu_d^k|$ for k large enough. So there is at least one subsequence $(\mu^{k_j})_{j\in I}$ such that $\mu^{k_j} = \mu$ for all j in I where μ depends on I and $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_d \geq |\mu_d|$. We have proved in the preceding Lemma that there exists an skew-symmetric matrix B defined in (36) such that the spectrum of the matrix $J_\mu + B$ is given by zero and the complex numbers $\pm i\lambda_1$, $\pm i\lambda_2, ..., \pm i\lambda_d$. On the other hand, there is an orthogonal matrix A such that $A(J_\mu + B)A^t = J_\lambda$ (c.f. [BGLR] paragraph 7). If $A \in SO(2d+1)$, then we can take $A_{k_j} = A$ and $B_{k_j} = B$ for all j in I.
Conversely, it is clear that $\lim\limits_{k\to\infty} r_k = \lim\limits_{k\to\infty} \|A_k \ell_{r_k}\| = 0$, and for all $j = 1, 2, ..., d$ one has $\lim\limits_{k\to\infty} \det(J_{\mu^k} + B_k \pm i\lambda_j \mathbb{I}) = \lim\limits_{k\to\infty} \det(A_k(J_{\mu^k} + B_k)A_k^t \pm i\lambda_j \mathbb{I}) = 0$. Then, by the preceding Lemma, $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_d \geq |\mu_d^k|$ for k large enough large.
If n is even i.e. $n = 2d$, then the same proof applies. The only exception is the choice of the matrix A in $O(2d)$ satisfying $A(J_\mu + B)A^t = J_\lambda$, if $\det(A) = -1$. In this situation, we multiply the last line of the matrix A by -1. Then we obtain $\det(A) = 1$ and $A(J_\mu + B)A^t = J_{\tilde{\lambda}}$ such that $\tilde{\lambda} = (\lambda_1, ..., \lambda_{d-1}, -\lambda_d)$. ☐

We have finished the proof of

`homdualorb` **Theorem 5.5.** *The dual space of the group $M_n = SO(n) \ltimes \mathbb{R}^n$ is homeomorphic with its space of admissible coadjoint orbits $\mathfrak{m}_n^{\ddagger}/M_n$.*

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