

DISINTEGRATING TENSOR REPRESENTATIONS OF NILPOTENT LIE GROUPS

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ABSTRACT. Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group and $H = \exp \mathfrak{h}$ be a closed connected subgroup of G . Given an irreducible unitary representation π of G , we present an explicit disintegration of the restriction $\pi|_H$ of π to H , which is based on a precise description of the space of double classes $H \backslash G / B$, where B is any closed connected subgroup of G , and the well known smooth disintegration of monomial representations of nilpotent Lie groups. As an application we get a concrete disintegration of tensor products of irreducible representations of G and a criterium for the irreducibility of these representations.

1. INTRODUCTION

In this paper, we present a concrete disintegration of the tensor product of two irreducible unitary representations of a simply connected, connected nilpotent Lie group $G = \exp \mathfrak{g}$ into irreducibles. "Concrete" here means that we write down an explicit integral of the spaces of the irreducible representations appearing in the disintegration and an explicit unitary intertwining operator between the Hilbert space of the tensor product and this direct integral.

In order to do that, we use the well known fact that a tensor product of two irreducible representations π, π' of G corresponds to the restriction to the diagonal group D of the irreducible representation $\pi \times \pi'$ of the group $G \times G$. Therefore we study first the general problem of describing a concrete disintegration of the restriction of an irreducible unitary representation of G to a closed connected subgroup $H = \exp \mathfrak{h}$. Since by Kirillov's theory every $\pi \in \hat{G}$ is of the form $\pi = \pi_{l, \mathfrak{b}} = \text{ind}_B^G \chi_l$, where $l \in \mathfrak{g}^*$ and $\mathfrak{b} \subset \mathfrak{g}$ is a polarization at l , we know from Mackey ([11]) that the restriction of π to H is disintegrated over the set of double classes $H \backslash G / B$ and that the integrands are of the form $\text{ind}_{B(x)}^H \chi_{l(x)}$, where $B(x) = H \cap \psi(x)B\psi(x)^{-1}$, $x \in H \backslash G / B$ and $l(x) = \text{Ad}^*(\psi(x))l|_{\mathfrak{b}}$, $x \in H \backslash G / B$ and $\psi : H \backslash G / B \rightarrow G$ is a section for the double classes. We present in Section 3 a description of an open dense subset of $H \backslash G / B$ and of such a section ψ which give us an explicit description of $\pi|_H$ in term of an integral over $H \backslash G / B$ of the representations $\text{ind}_{B(x)}^H \chi_{l(x)}$ (see 3.27). In the next section, we recall the results of Baklouti-Ludwig [2] concerning an explicit disintegration of monomial representations and we combine these results to obtain

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our concrete disintegration of the restriction of π to H . Let us recall that an "abstract" disintegration of irreducibles to connected closed subgroups of simply connected nilpotent Lie groups have been given in [5]. Abstract here means that the measure class in \hat{G} for the disintegration of the restriction and the multiplicities of the irreducibles appearing in the disintegration is given. In [7] these results are extended to the class of exponential solvable Lie groups. In [3], another concrete disintegration of the restriction of irreducibles to closed connected subgroups is given, but this disintegration is more complicated, since it does not use the tools of the double classes and does not give the necessary details for the application to tensor products.

In Section 5, we apply our construction to the disintegration of the tensor product of two irreducible representations π and π' . The disintegration is made over a subvariety of \mathfrak{g}^* , which is contained in the sum of the Kirillov orbits of π and π' . We show in an example that the formulas describing the disintegration depend very much on the choice of the polarizations used to realize the representations π and π' .

In the last section, we determine necessary and sufficient conditions on the representations π and π' for the irreducibility of $\pi \otimes \pi'$.

2. PRELIMINARIES AND NOTATIONS

Let \mathfrak{g} be a nilpotent Lie algebra, \mathfrak{b} any subalgebra, $G \subset B$ their simply connected Lie groups. Given a sequence of ideals

$$(2.1) \quad \mathfrak{g}_{n+1} := \{0\} \subsetneq \dots \subsetneq \mathfrak{g}_i \subsetneq \dots \subsetneq \mathfrak{g}_1 = \mathfrak{g}, \dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1,$$

we let for every $i = 1, \dots, n$ $G_i := \exp \mathfrak{g}_i$ and we choose a vector $Z_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$, so that $\mathfrak{g}_i = \text{span} \langle Z_i, \dots, Z_n \rangle$. We obtain in this way a Jordan-Hölder basis $\mathcal{Z} := (Z_1, \dots, Z_n)$ of \mathfrak{g} . Recall that the exponential mapping $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism. Now to simplify the notations, we put by convention for any k vectors $V_1, \dots, V_k \in \mathfrak{g}$,

$$V_1.V_2\dots V_k := \exp(V_1)\exp(V_2)\dots \exp(V_k) \in G.$$

We denote the Haar measure on G by dg . Using the basis \mathcal{Z} , we can express dg in the following way.

$$\int_G f(g) dg = \int_{\mathbb{R}^n} f(z_1 Z_1 \dots z_n Z_n) dz, \quad (f \in L^1(G)).$$

Since G is nilpotent, the quotient space G/B has a G -invariant measure which is unique up to a positif scalar multiple. To describe such a measure, which we shall denote by $d\dot{g}$, we proceed as follows. Let

$$(2.2) \quad \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{k \in \{1, \dots, n\}, Z_k \notin \mathfrak{b} + \mathfrak{g}_{k+1}\} =: \{k_1 < \dots < k_p\}.$$

We obtain the sequence of subalgebras

$$(2.3) \quad \mathfrak{b}^{p+1} := \mathfrak{b} \subsetneq \dots \subsetneq \mathfrak{b}^j := \mathbb{R}Z_{k_j} \oplus \mathfrak{b}^{j-1} \subsetneq \dots \subsetneq \mathfrak{b}^1 = \mathfrak{g}$$

and the *Malcev basis* $\mathcal{M} := (Z_{k_1}, \dots, Z_{k_p})$ of \mathfrak{g} relative to \mathfrak{b} . The invariant measure $d\dot{g}$ is then given by

$$(2.4) \quad \mu_{\mathcal{M}}(\varphi) = \mu_{\mathfrak{g}/\mathfrak{b}}(\varphi) = \int_{G/B} \varphi(\dot{g}) d\dot{g} := \int_{\mathbb{R}^p} \varphi(w_1 Z_{k_1} \dots w_p Z_{k_p} \cdot B) dw, \quad \varphi \in C_c(G/B),$$

where $C_c(G/B)$ denotes the space of complex valued continuous functions with compact support on G/B . This is a consequence of the fact that the mapping

$$(2.5) \quad \begin{array}{ccc} E_{\mathcal{M}} : & \mathbb{R}^p & \longrightarrow & G/B, \\ & w = (w_1, \dots, w_p) & \longmapsto & w_1 Z_{k_1} \dots w_p Z_{k_p} \cdot B =: E_{\mathcal{M}}(w) \end{array}$$

is a diffeomorphism (for detailed explanations see [6]).

If \mathfrak{h} is another subalgebra of \mathfrak{g} , then we denote by $\mathcal{I}^{\mathfrak{h}} \subset \{1, \dots, n\}$ the index set

$$(2.6) \quad \mathcal{I}^{\mathfrak{h}} := \{i \in \{1, \dots, n\}; \mathfrak{h} + \mathfrak{g}_{i+1} = \mathfrak{h} + \mathfrak{g}_i\} = \{1, \dots, n\} \setminus \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}.$$

We can then assume, if we want, that the vectors Z_i , $i \in \mathcal{I}^{\mathfrak{h}}$, are contained in \mathfrak{h} , so that

$$\mathfrak{h} = \text{span} \{Z_i, i \in \mathcal{I}^{\mathfrak{h}}\}.$$

Definition 2.1. Let $B = \exp \mathfrak{b}$ be a closed connected subgroup of G and let $(\rho, \mathcal{H}_{\rho})$ be a unitary representation of B . We define the induced representation $\tau_{\rho} := \text{ind}_B^G \rho$ of G in the following way. Let

$$\begin{aligned} C_c(G/B, \rho) &= \{\xi : G \rightarrow \mathcal{H}_{\rho}, \xi \text{ continuous with compact support mod. } B, \\ &\quad \xi(gb) = \rho(b)^{-1}(\xi(g)) \text{ for all } g \in G, b \in B\}. \end{aligned}$$

We define a scalar product $\langle \cdot, \cdot \rangle$ on $C_c(G/B, \rho)$ by

$$\langle \xi, \eta \rangle_{\mathcal{H}} := \int_{G/B} \langle \xi(x), \eta(x) \rangle dx, \quad \xi, \eta \in C_c(G/B, \rho)$$

and a Hilbert norm

$$\|\xi\|_{\mathcal{H}}^2 := \langle \xi, \xi \rangle_{\mathcal{H}} = \int_{G/B} \langle \xi(x), \xi(x) \rangle dx.$$

Let us define the left translation $\lambda(g)$ on $C_c(G/B, \rho)$ by an element g of G by

$$\lambda(g)\xi(g') = \xi(g^{-1}g'), \quad g' \in G, \xi \in C_c(G/B, \rho).$$

It is easy to see that the left translation is isometric on $C_c(G/B, \rho)$ and defines on the completion $L^2(G/B, \rho)$ of $C_c(G/B, \rho)$ a unitary representation of G , which will be denoted by τ_{ρ} , $\tau_{\rho, B}$ or by $\text{ind}_B^G \rho$. Let $\mathfrak{l} \in \mathfrak{g}^*$, such that $\langle \mathfrak{l}, [\mathfrak{b}, \mathfrak{b}] \rangle = \{0\}$. We can then define the unitary character $\chi_{\mathfrak{l}}(\exp X) := e^{-i\langle \mathfrak{l}, X \rangle}$, $X \in \mathfrak{b}$, of B and form the representation $\tau_{\mathfrak{l}, \mathfrak{b}} := \tau_{\chi_{\mathfrak{l}}, B}$ of G .

3. THE SET OF DOUBLE CLASSES

Let $G = \exp \mathfrak{g}$ be a connected, simply connected, nilpotent Lie group, \mathfrak{b} and \mathfrak{h} two subalgebras of \mathfrak{g} , $B = \exp \mathfrak{b}$ and $H = \exp \mathfrak{h}$ their Lie subgroups. We give a concrete description for $H \backslash G / B$, which will be used to describe later on in section 3.2 an appropriate measure $d\gamma$ which appears in the disintegration of $\pi|_H$ of π to H , where π is an irreducible unitary representation of G which is induced from a character χ_l of B .

3.1. Description of $H \backslash G / B$. For $g \in G$, we denote by \tilde{g} its double class which is the set $H \cdot g \cdot B = \{ hgb, (h, b) \in H \times B \}$. Our purpose is to find an open dense subset of $H \backslash G / B$ which will support the measure $d\gamma(\tilde{g})$ and which is diffeomorphic to a Zariski open subset of \mathbb{R}^d for some $d \in \mathbb{N}^*$. To illustrate this fact, we introduce the following example.

Example 3.1. Let \mathfrak{g} be the 7-dimensional Lie algebra spanned by its Jordan-Hölder basis $\mathcal{Z} = (Z_1, \dots, Z_7)$, equipped with the brackets

$$[Z_1, Z_4] = Z_6, \quad [Z_1, Z_5] = Z_7, \quad [Z_2, Z_3] = Z_7.$$

We consider its abelian subalgebras $\mathfrak{h} = \text{span} \langle Z_4, Z_5, Z_7 \rangle$ and $\mathfrak{b} = \text{span} \langle Z_3, Z_4, Z_7 \rangle$. Since many products commute, the element $g =: z_1 Z_1 \dots z_7 Z_7 \in G$, $(z_1, \dots, z_7) \in \mathbb{R}^7$, can be described in the following way:

$$\begin{aligned} g &= (z_1 Z_1 \cdot z_5 Z_5) \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot (z_3 Z_3 \cdot z_4 Z_4 \cdot z_7 Z_7) \\ &= (z_1 z_5 Z_7 \cdot z_5 Z_5 \cdot z_1 Z_1) \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot (z_3 Z_3 \cdot z_4 Z_4 \cdot z_7 Z_7) \\ &\in H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot B. \end{aligned}$$

This implies that $\tilde{g} = H \cdot g \cdot B = H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot B$.

But if $z_1 \neq 0$, the element $z_1 Z_1 \cdot z_2 Z_2 \cdot z_6 Z_6$ can also be written as $z_1 Z_1 \cdot z_2 Z_2$ conjugated by $\exp(-\frac{z_6}{z_1}) Z_4$, which is contained in $H \cap B$. So we deduce that if $z_1 \neq 0$, then $\tilde{g} = H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot B$. As a conclusion,

$$\begin{aligned} H \backslash G / B &= \{ H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot B, (z_1, z_2) \in \mathcal{V} \} \cup \{ H \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot B, (z_2, z_6) \in \mathbb{R}^2 \} \\ &= \mathbf{p}(G \setminus G_2) \cup \mathbf{p}(G_2) \end{aligned}$$

where $\mathbf{p} : g \mapsto \tilde{g}$, is the canonical projection of G on $H \backslash G / B$, and $\mathcal{V} := \mathbb{R}^* \times \mathbb{R}$.

Hence the space $H \backslash G / B$ is the disjoint union of two subsets, the first is the projection of a Zariski open subset of G and the second of a Zariski closed subset. We shall see later that the measure $d\gamma(\tilde{g})$ will be supported on the first set.

The general case. We shall prove that there is always a Zariski open \mathcal{V} of \mathbb{R}^d which is homeomorphic to a subset $\tilde{\mathcal{U}}$ of $H \backslash G / B$ containing the double classes of almost all (with respect to Haar measure) elements of G .

Definition 3.2.

- (1) We say that a subset \mathcal{W} of G
 - is symmetric if $\mathcal{W} = \mathcal{W}^{-1} := \{g^{-1}, g \in \mathcal{W}\}$,
 - has the property (\mathcal{P}) (with respect to H and B), if $H\mathcal{W}B \subseteq \mathcal{W}$.

(2) It is easy to see that for any $i \in \{1, \dots, n-1\}$, the subset

$$\{g \in G, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}$$

either contains a Zariski open subset or its complement in G does. We characterize the indices of subset $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ of $\{1, \dots, n\}$ by

$$i \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \iff \begin{array}{l} \text{the subset } \mathcal{W}_i := \{g \in G, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\} \\ = \{g \in G, Z_i \notin \text{Ad}(g^{-1})\mathfrak{h} + \mathfrak{b} + \mathfrak{g}_{i+1}\} \\ \text{contains a non empty Zariski open subset of } G. \end{array}$$

We denote by \mathcal{U}_i the maximal symmetric Zariski open subset contained in \mathcal{W}_i .

(3) Let also for $i \in \{1, \dots, n\} \setminus \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) =: \mathcal{I}(\mathfrak{h}, \mathfrak{b})$,

$$\mathcal{W}_i := \{g \in G, Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}.$$

For $i \in \mathcal{I}(\mathfrak{h}, \mathfrak{b})$, we consider the maximal symmetric Zariski open subset \mathcal{U}_i of \mathcal{W}_i , with the property that for every $g \in \mathcal{U}_i$, there exist rational mappings $G \ni g' \rightarrow h(g') \in \mathfrak{h}, g' \rightarrow b(g') \in \mathfrak{b}, g \rightarrow b_{i+1}(g) \in \mathfrak{g}_{i+1}$, with no singularities in a Zariski neighbourhood $\mathcal{U}_i(g)$ of g such that $Z_i = h(g') + \text{Ad}(g')b(g) + b_{i+1}(g')$ for all $g' \in \mathcal{U}_i(g)$.

It is easy to see that

Proposition 3.3.

- (1) The following two conditions are equivalent:
 - $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}$
 - $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{b} + \text{Ad}(g)\mathfrak{h} + \mathfrak{g}_{i+1}\}$
- (2) The index subset $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ is contained in $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}}$.
- (3) Every $\mathcal{W}_i, i \in \{1, \dots, n\}$, has the property (\mathcal{P}) . Furthermore, the maximal Zariski open subsets $\mathcal{U}_i, i \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$, of G have property (\mathcal{P}) .

Proof.

- (1) Suppose for instance that $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{b} + \text{Ad}(g)\mathfrak{h} + \mathfrak{g}_{i+1}\}$. Since \mathcal{U}_i is symmetric, for $g \in \mathcal{U}_i$, we have that $Z_i = X + \text{Ad}(g^{-1})Y + S_{i+1}$ for some $X \in \mathfrak{b}, Y \in \mathfrak{h}$ and $S_{i+1} \in \mathfrak{g}_{i+1}$. We obtain that

$$\begin{aligned} \text{Ad}(g)Z_i &= \text{Ad}(g)X + Y + \text{Ad}(g)S_{i+1} \\ &\text{and} \end{aligned}$$

$$\text{Ad}(g)Z_i = Z_i \pmod{\mathfrak{g}_{i+1}}.$$

By putting $T_{i+1} := (Z_i - \text{Ad}(g)Z_i) + \text{Ad}(g)S_{i+1} \in \mathfrak{g}_{i+1}$, we get $Z_i = \text{Ad}(g)X + Y + T_{i+1}$. Hence $Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}$, which means that $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}$.

- (2) If $i \in \{1, \dots, n\} \setminus \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \mathcal{I}^{\mathfrak{b}} \cup \mathcal{I}^{\mathfrak{h}}$, then $Z_i \in \mathfrak{b} + \mathfrak{g}_{i+1}$ or $Z_i \in \mathfrak{h} + \mathfrak{g}_{i+1}$. Therefore $\text{Ad}(g)(Z_i) \in \mathfrak{b} + \mathfrak{g}_{i+1}$ for every $g \in G$ or $\text{Ad}(g)(Z_i) \in \mathfrak{h} + \mathfrak{g}_{i+1}$ for every $g \in G$ and so $Z_i \in \text{Ad}(g^{-1})\mathfrak{b} + \mathfrak{g}_{i+1} \subset \text{Ad}(g^{-1})\mathfrak{b} + \mathfrak{h} + \mathfrak{g}_{i+1}$ for every $g \in G$ or $Z_i \in \text{Ad}(g^{-1})\mathfrak{h} + \mathfrak{g}_{i+1} \subset \text{Ad}(g^{-1})\mathfrak{h} + \mathfrak{b} + \mathfrak{g}_{i+1}$ for every $g \in G$. Hence $i \in \mathcal{I}(\mathfrak{h}, \mathfrak{b})$.

- (3) If $Z_i = \text{Ad}(g)X + Y + T_{i+1} = X + \text{Ad}(g^{-1})Y + S_{i+1}$ for some $X \in \mathfrak{b}, Y \in \mathfrak{h}$ and $T_{i+1}, S_{i+1} \in \mathfrak{g}_{i+1}$, then for every $b \in B, h \in H$, we have that

$$Z_i = \text{Ad}(gb)(\text{Ad}(b^{-1})X) + Y + Z_{i+1}$$

and

$$Z_i = X + \text{Ad}((hg)^{-1})(\text{Ad}(h)Y) + S_{i+1}.$$

This shows that the subset of G consisting of all elements g , such that $Z_i \in \text{Ad}(g)\mathfrak{b} + \mathfrak{h} + \mathfrak{g}_{i+1}$ is H -invariant on the right and left B -invariant on the left and therefore also its complement. □

Remark 3.4. Putting $\mathcal{U} := \bigcap_{1 \leq i \leq n} \mathcal{U}_i$, we obtain a "canonical" nonempty Zariski open of G having the property (\mathcal{P}) , which allows us to say that

$$(3.1) \quad \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{1 \leq i \leq n, \forall g \in \mathcal{U}, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}.$$

Definition 3.5. We order the set of indices $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1 < \dots < i_d\}$, and we put

(1)

$$\phi : \mathbb{R}^d \longrightarrow G, \quad (t_1, \dots, t_d) \longmapsto t_1 Z_{i_1} \dots t_d Z_{i_d}$$

(2)

$$\mathbf{p} : G \longrightarrow H \backslash G / B, \quad g \longmapsto \tilde{g} = H \cdot g \cdot B$$

(3)

$$\mathcal{V} = \phi^{-1}(\mathcal{U}) := \{t \in \mathbb{R}^d, \phi(t) \in \mathcal{U}\}, \quad \tilde{\mathcal{U}} = \mathbf{p}(\mathcal{U}) := \{\tilde{g}, g \in \mathcal{U}\}$$

(4)

$$\tilde{\phi} := \mathbf{p} \circ \phi|_{\mathcal{V}} : \begin{array}{ccc} \mathcal{V} & \longrightarrow & \tilde{\mathcal{U}}, \\ t = (t_1, \dots, t_d) & \longmapsto & H \cdot \phi(t) \cdot B \end{array}$$

- (5) Finally, we provide the space of the double classes $H \backslash G / B$ with the quotient topology, i.e a subset \tilde{U} in $H \backslash G / B$ is open if and only if the subset $U := \mathbf{p}^{-1}(\tilde{U})$ is open in G . The mapping \mathbf{p} is then open, since for every open subset U of G , we have that $\mathbf{p}^{-1}(\mathbf{p}(U)) = HUB = \bigcup_{h \in H, b \in B} hUb$ is open in G .

Lemma 3.6. *In the particular case where \mathfrak{b} is an ideal, we have that $\tilde{\phi}$ is a diffeomorphism between \mathbb{R}^d and $H \backslash G / B$.*

Proof. The closed subgroup $HB = BH$ has $\mathfrak{h} + \mathfrak{b}$ as Lie algebra, and it satisfies $HB \backslash G = H \backslash G / B$. Indeed for every $g \in G$,

$$(3.2) \quad H \cdot g \cdot B = \{hgb, h \in H, b \in B\} = \{hb'g, h \in H, gbg^{-1} =: b' \in B\} = HB \cdot g.$$

We also notice that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \mathcal{I}^{\mathfrak{g}/\mathfrak{h} + \mathfrak{b}} =: \{Z_{i_1}, \dots, Z_{i_d}\}$, since the condition $Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}$ is now equivalent to the condition $Z_i \notin \mathfrak{h} + \mathfrak{b} + \mathfrak{g}_{i+1}$. This tells us that \mathcal{U} is all of G . Hence $(Z_{i_1}, \dots, Z_{i_d})$ is a Malcev basis for \mathfrak{g} relative to $\mathfrak{h} + \mathfrak{b}$. Finally, by [5], we have the diffeomorphism: $t = (t_1, \dots, t_d) \longmapsto HB \cdot (-t_d Z_{i_d}) \dots (-t_1 Z_{i_1})$ between \mathbb{R}^d and the set $HB \backslash G$ of the right classes modulo HB . If we compose this mapping with the

mapping $HB \setminus G \ni HB \cdot g^{-1} \mapsto g \cdot HB \in G/HB$, which is also a diffeomorphism, then we obtain that the mapping $\mathbb{R}^d \ni t = (t_1, \dots, t_d) \mapsto t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot HB \in G/HB$, which is nothing else than $\tilde{\phi}$ by (3.5(4)), is also a diffeomorphism. \square

Remark 3.7. In theorem 3.10 we show that the canonical mapping $\tilde{\phi} : \mathcal{V} \rightarrow \tilde{\mathcal{U}}$ is a homeomorphism. We proceed by induction on the dimension of $\mathfrak{g}/\mathfrak{b}$. In order to do that, we consider the index

$$l := \max\{i \in \mathcal{I}^{\mathfrak{g}/\mathfrak{b}}\}$$

and we set

$$\mathfrak{a} := \mathfrak{g}_{l+1}, \quad \mathfrak{c} := \mathfrak{b} + \mathfrak{g}_l = \mathfrak{b} \oplus \mathbb{R}Z_l.$$

Since $Z_j \in \mathfrak{b}$ for every $j > l$, it follows that \mathfrak{a} is an ideal of \mathfrak{g} contained in \mathfrak{b} . Furthermore

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{c}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} \setminus \{l\} = \{k_1 < \dots < k_{p-1}\}.$$

Whence

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{c}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} \setminus \{l\}.$$

We encounter the following two cases which must be discussed in the proof of theorem 3.10.

- a):** $Z_l \notin \mathfrak{h} + \mathfrak{g}_{l+1}$, which means that $l \in \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$ and so $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{c}} \cup \{l\}$.
- b):** $Z_l \in \mathfrak{h} + \mathfrak{g}_{l+1}$, whence $Z_l \in \mathfrak{h}$ by remark 3.4, b). In this case we have $l \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$ and so $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{c}}$.

Since $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ for all $i < l$, it follows that

$$\mathfrak{h} + \text{Ad}(g)\mathfrak{c} + \mathfrak{g}_{i+1} = \mathfrak{h} + \text{Ad}(g)(\mathfrak{b} + \mathfrak{g}_l) + \mathfrak{g}_{i+1} = \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}, \quad \forall g \in G.$$

This tells us that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \setminus \{l\}$.

Hence in the case **b)**, we have that

$$\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}),$$

since $l \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \supset \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ by Proposition 3.3 (1).

In the first case we must distinguish between two possibilities.

- a-1)** $l \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$.: This case is characterized by the existence of a nonempty Zariski open subset \mathcal{U}_l of G , given by proposition 3.3, such that for every $g \in \mathcal{U}_l$, we have $Z_l \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b}$ (since $\mathfrak{g}_{l+1} \subseteq \text{Ad}(g)\mathfrak{b}$ for $g \in G$).
- a-2)** $l \notin \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$.: Here we have $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ as in the case b). Hence for every g in a nonempty Zariski open subset \mathcal{U}_l of G , we have that $Z_l \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b}$.

Remark 3.8. For every $(w, \tilde{g}) \in G_l \times H \setminus G/B$, we have $H \cdot wgw^{-1} \cdot B = \tilde{g}$. This is an immediate consequence of the fact that $wgw^{-1} \equiv g \pmod{[B]}$, ($g \in G$).

Let us first show that

Lemma 3.9. *the mapping*

$$\tilde{\phi} : \mathcal{V} \ni (t_1, \dots, t_d) \mapsto H \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B \in \tilde{\mathcal{U}}$$

is bijective.

Proof. We will proceed by induction on the codimension p of B . If $p = 1$, then the claim is trivial in view of lemma 3.6, since \mathfrak{b} is then an ideal of \mathfrak{g} . Let $l = \max \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{k_1 < \dots < k_p = l\}$ and $\mathfrak{c} := \mathbb{R}Z_l + \mathfrak{b}$ as in Remark 3.7. We apply the induction hypothesis to $\mathfrak{g}, \mathfrak{h}$ and \mathfrak{c} , giving us the bijection $\tilde{\phi}^\circ : \mathcal{V}^\circ \ni s \mapsto H \cdot \phi^\circ(s) \cdot C \in \tilde{\mathcal{U}}^\circ$. Let us introduce the projection

$$(3.3) \quad \begin{aligned} \Lambda^\circ : \quad H \backslash G / B &\longrightarrow H \backslash G / C \\ H \cdot g \cdot B = \tilde{g} &\longmapsto \tilde{g} = H \cdot g \cdot C \end{aligned}$$

We notice that the surjectivity of $\tilde{\phi}$ is equivalent to $H\phi(\mathcal{V})B = \mathcal{U}$. Since we have already the first inclusion " \subseteq " thanks to the property (\mathcal{P}) , it's sufficient for the surjectivity to prove that we have the other inclusion: $\{h\phi(t)b, h \in H, t \in \mathcal{V}, b \in B\} \supseteq \mathcal{U}$. We discuss now the three cases of Remark 3.7.

Case a-1) $l \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d = l\}$.

Let's recall that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \setminus \{l\} = \{i_1, \dots, i_{d-1}\}$. The map ϕ° goes from \mathbb{R}^{d-1} to G . We have also the inclusion $\bigcap_{i \in \mathcal{I}^{\mathfrak{b}, \mathfrak{c}}} \mathcal{U}_i = \mathcal{U}^\circ \supseteq \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l$.

By the induction hypothesis,

$$\tilde{\phi}^\circ : \mathcal{V}^\circ \ni s = (t_1, \dots, t_{d-1}) \mapsto H \cdot \phi^\circ(s) \cdot C \in \tilde{\mathcal{U}}^\circ$$

is bijective. As defined in 3.5, \mathcal{V}° is the nonempty Zariski open subset $\phi^{\circ-1}(\mathcal{U}^\circ)$ of \mathbb{R}^{d-1} and $\tilde{\mathcal{U}}^\circ$ is the open dense subset $\{\tilde{g} = H \cdot g \cdot C, g \in \mathcal{U}^\circ\}$ of $H \backslash G / C$.

• **Surjectivity of $\tilde{\phi}$:**

Let $u \in \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l$. As $\mathcal{U}^\circ = H\phi^\circ(\mathcal{V}^\circ)C$, we can write $u = h\phi^\circ(s)c$ for certain $h \in H, s = (s_1, \dots, s_{d-1}) \in \mathcal{V}^\circ$ and $c \in C$. We have $c = (\exp \gamma Z_l)b$ where $b \in B$ and $\gamma \in \mathbb{R}$. Denoting by $t = (s_1, \dots, s_{d-1}, t_d := \gamma) \in \mathbb{R}^d$, we obtain that $u = h\phi(t)b$.

But $t \in \mathcal{V}$ since $\phi(t) = h^{-1}ub^{-1}$ is in \mathcal{U} thanks to (\mathcal{P}) . Finally $u \in H\phi(\mathcal{V})B$.

• **Injectivity of $\tilde{\phi}$:**

At first, we remark that if $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ is in \mathcal{V} , then $s := (t_1, \dots, t_{d-1}) \in \mathbb{R}^{d-1}$ must be in \mathcal{V}° . Indeed, $\phi(t) \in \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l \subset \mathcal{U}^\circ$ and so, thanks to the property (\mathcal{P}) , the stability of \mathcal{U}° by right multiplication with elements of C gives us $\phi^\circ(s) = \phi(t) \exp(-t_d Z_l) \in \mathcal{U}^\circ$, hence $s \in \mathcal{V}^\circ$.

Let now $t = (t_1, \dots, t_d), t' = (t'_1, \dots, t'_d) \in \mathcal{V}$ such that $H \cdot \phi(t) \cdot B = H \cdot \phi(t') \cdot B$. Whence $H \cdot \phi(s) \cdot C = H \cdot \phi(s') \cdot C$, where $s = (t_1, \dots, t_{d-1}), s' = (t'_1, \dots, t'_{d-1}) \in \mathcal{V}^\circ$. Now since $\tilde{\phi}^\circ$ is injective, $s = s'$. Finally, to show that $t_d = t'_d$, we make the following calculation. Let $h = \exp Y \in H$ and $b = \exp X \in B$, such that $\phi(t) = h\phi(t')b$, and let $g = \phi^\circ(s) \in \mathcal{U}^\circ$.

Then $g \exp(t_d Z_l)$ and $g \exp(t'_d Z_l) \in \mathcal{U}$. We have then if $t'_d - t_d \neq 0$,

$$\begin{aligned}
g \exp(t_d Z_l) &= \exp(Y) g \exp(t'_d Z_l) \exp(X) \\
&\Leftrightarrow \\
g^{-1} \exp(-Y) g &= \exp(t'_d Z_l) \exp(X) \exp(-t_d Z_l) \\
&\Leftrightarrow \\
\exp(-\text{Ad}(g^{-1})Y) &= \exp(X + (t'_d - t_d)Z_l) \text{ mod } G_{l+1} \\
&\quad (\text{ by the Campbell-Baker-Hausdorff formula}) \\
&\Rightarrow Z_l - \frac{1}{t'_d - t_d} (\text{Ad}(g^{-1})(Y) + X) \in \mathfrak{g}_{l+1} \\
&\Rightarrow Z_l \in \mathfrak{b} + \text{Ad}(g^{-1})(\mathfrak{b}) + \mathfrak{g}_{l+1}.
\end{aligned}$$

This last line contradicts the fact that $g \in \mathcal{U}^\circ$. Hence $t_d = t'_d$ and $\tilde{\phi}$ is injective.

Case a-2) $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{c}} \cup \{l\}$ but $l \notin \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d\} = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c})$. Here, we have $\phi^\circ = \phi : \mathbb{R}^d \rightarrow G$, but the non empty Zariski subsets of \mathbb{R}^d defined by $\mathcal{V}^\circ = \phi^{-1}(\mathcal{U}^\circ)$ and $\mathcal{V} = \phi^{-1}(\mathcal{U})$ do not coincide in general. In fact, we know that $\mathcal{U}^\circ = \bigcap_{i \in \mathcal{I}^{\mathfrak{h}, \mathfrak{g}}} \mathcal{U}_i \supseteq \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l$, where the Zariski open subset \mathcal{U}_l of G satisfies $\forall g \in \mathcal{U}_l, Z_l \in$

$\mathfrak{h} + \text{Ad}(g)\mathfrak{b}$. By the induction hypothesis, $\tilde{\mathcal{U}}^\circ = \{\tilde{g} = H \cdot g \cdot C, g \in \mathcal{U}^\circ\}$ is in bijection with \mathcal{V}° thanks to $\tilde{\phi}^\circ$.

• Surjectivity of $\tilde{\phi}$:

Let $u \in \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l \subset \mathcal{U}^\circ$. Then u can be written as $h\phi^\circ(t)c = h\phi(t)c$, where $h \in H, c = \exp(\gamma Z_l)b \in C$ and $t \in \mathcal{V}^\circ$. Then also $v := \phi(t) \exp(\gamma Z_l) \in \mathcal{U}$, because of property \mathcal{P} . We shall show that

$$(3.4) \quad v = h'\phi(t)b'$$

for some $h' \in H, b' \in B$. This then shows that $t \in \mathcal{V} = \phi^{-1}(\mathcal{U})$ and so we see that $\tilde{\phi}$ is surjective.

Now since $v \in \mathcal{U}_l$, we have $Z_l \in \text{Ad}(v^{-1})\mathfrak{h} + \mathfrak{b}$. Hence there exist $Y =: \log h' \in \mathfrak{h}$ and $X =: \log x \in \mathfrak{b}$, such that

$$\begin{aligned}
(3.5) \quad \gamma Z_l &= \text{Ad}(v^{-1})Y + X \\
&\Leftrightarrow -X + \gamma Z_l = \text{Ad}(v^{-1})Y \\
&\Rightarrow \exp(\gamma Z_l)x^{-1}q = v^{-1}h'v \quad (\text{for some } q \in G_{l+1} \subset B) \\
&\Rightarrow \exp(\gamma Z_l) = v^{-1}h'vb'' \quad (\text{for some } b'' \in B) \\
&\Rightarrow v = \phi(t) \exp(\gamma Z_l) = \phi(t)(\exp(-\gamma Z_l)\phi(t)^{-1}h'\phi(t) \exp(\gamma Z_l)b'') \\
&\Rightarrow v = \phi(t)(\phi(t)^{-1}h'\phi(t)b') = h'\phi(t)b' \quad (\text{for some } b' \in B).
\end{aligned}$$

• Injectivity of $\tilde{\phi}$:

This is an immediate consequence from the injectivity of $\tilde{\phi}^\circ$, since $\mathcal{V} \subset \mathcal{V}^\circ$ and $\tilde{\phi} = \tilde{\phi}^\circ|_{\mathcal{V}}$

Case b) $l \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$.

This means that $Z_i \in \mathfrak{h}$. Now we have $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d\}$, $\phi^\circ = \phi$, $\mathcal{U}^\circ = \mathcal{U} = \bigcap_{i \in \mathcal{I}} \mathcal{U}_i$ and $\mathcal{V}^\circ = \phi^{\circ-1}(\mathcal{U}^\circ) = \phi^{-1}(\mathcal{U}) = \mathcal{V}$. But of course the spaces $H \backslash G / B$ and $H \backslash G / C$ are different, and $\tilde{\phi}^\circ$ and $\tilde{\phi}$ are related by the formula

$$\begin{aligned} \tilde{\phi}^\circ = \Lambda^\circ \circ \tilde{\phi} : \quad \mathcal{V} &\longrightarrow \tilde{\mathcal{U}}^\circ \\ t = (t_1, \dots, t_d) &\longmapsto H \cdot \phi(t) \cdot C \end{aligned}$$

• **Surjectivity of $\tilde{\phi}$:**

As we have already $\mathcal{U} = \mathcal{U}^\circ = H\phi^\circ(\mathcal{V}^\circ)C = H\phi(\mathcal{V})C$, it suffices to prove the inclusion $H\phi(\mathcal{V})C \subseteq H\phi(\mathcal{V})B$. Now if we take $u = h\phi(t)c$ where the elements $h, t = (t_1, \dots, t_d), c = \exp(\gamma Z_i)b$, γ and b are respectively in $H, \mathcal{V}, C, \mathbb{R}$ and B . Hence

$$u = h\phi(t)c = h\phi(t) \exp(\gamma Z_i)b = (h \exp(\gamma Z_i)) \exp(-\gamma Z_i)\phi(t) \exp(\gamma Z_i) b$$

which gives

$$\begin{aligned} (3.6) \quad H \cdot u \cdot B &= H \cdot h \exp(\gamma Z_i) \exp(-\gamma Z_i)\phi(t) \exp(\gamma Z_i)b \cdot B \\ &= H \cdot \exp(-\gamma Z_i)\phi(t) \exp(\gamma Z_i) \cdot B = H\phi(t)B, \end{aligned}$$

by remark 3.8.

• **Injectivity of $\tilde{\phi}$:**

This is exactly as in the case a-2), in fact $\tilde{\phi}^\circ = \Lambda^\circ \circ \tilde{\phi}$ is injective, and so is $\tilde{\phi}$. □

Theorem 3.10. *The mapping $\tilde{\phi} : \mathcal{V} \ni t = (t_1, \dots, t_d) \longmapsto H \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B \in \tilde{\mathcal{U}}$ is an homeomorphism.*

Proof. The mapping $\tilde{\phi} = \mathbf{p} \circ \phi|_{\mathcal{V}}$ is evidently continuous since it is the case for ϕ and \mathbf{p} . Hence it is enough to prove that the mapping

$$H \times \mathbb{R}^d \times B \rightarrow G, (h, t, b) \rightarrow h\phi(t)b$$

is open.

We do this again by induction on the dimension of G/B , using the notations of the proof of Lemma 3.9.

If the dimension of $G/B = 1$, then B is normal and we apply Lemma 3.6.

In **Case a-1)**, it suffices to consider subsets $W_H \subset H$, which are open in H , $W = W_0 \times]a, b[\subset \mathcal{V}$ which are open in $\mathbb{R}^{d-1} \times \mathbb{R}$ and subsets $W_B \subset B$, which are open in B and to prove that $W_H \phi(W) W_B$ is open in G . But by the induction hypothesis for the groups H and C , we know that

$$W_H \phi(W) W_B = W_B \phi^\circ(W_0) \exp(]a, b[Z_i) W_B$$

is open in G , since $\exp(]a, b[Z_i) W_B$ is open in B .

In case **Case a-2)** consider open subsets W_H of H , $W = W_0 \times]a, b[$ open in $\mathcal{V} \subset \mathbb{R}^{d-1} \times \mathbb{R}$ and W_B open in B and prove that $W_H \phi(W) W_B$ is open in G . Let $b \in W_B, h \in W_H, t =$

$(t_1, \dots, t_d) \in W$. We know that $\phi = \phi|_{\mathcal{V}}$ and that W is open in \mathcal{V}° . Since $l \in \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap I^{\mathfrak{g}/\mathfrak{b}} \setminus \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$, there exist rational mappings

$$g \mapsto Y(g) \in \mathfrak{h}, g \mapsto X(g) \in \mathfrak{b},$$

defined on a Zariski open neighborhood of $\phi(t)$, which we can assume to contain W , such that for every g in this neighborhood we have that

$$\begin{aligned} Z_l &= \text{Ad}(g^{-1})Y(g) + X(g) \\ &\Rightarrow \exp(t_d Z_l - t_d X(g)) = g^{-1} \exp(t_d Y(g))g, \quad t_d \in \mathbb{R}, \\ &\Rightarrow \exp(t_d Z_l) b(g, t_d)^{-1} = g^{-1} \exp(t_d Y(g))g \quad (\text{where } (g, t_d) \mapsto b(g, t_d) \in B \text{ is rational}), \\ &\Rightarrow \exp(t_d Z_l) = g^{-1} h(g, t_d) g b(g, t_d) \quad (\text{where } h(t_d) = \exp(t_d Y(g)) \in H). \end{aligned}$$

Hence, for $b' \in B, h' \in H, t' \in W, t_d \in \mathbb{R}$, we get

$$(3.7) \quad h' \phi(t') \exp(t_d Z_l) b' = h' h(\phi(t'), t_d) \phi(t') b(g, t_d) b'.$$

Since the mapping

$$\begin{aligned} H \times B \times \mathbb{R} \times W &\rightarrow H \times B \times \mathbb{R} \times W; \\ (h', b', t_d, t') &\mapsto (h' h(\phi(t'), t_d), b(g, t_d) b', t_d, t), \end{aligned}$$

is a diffeomorphism, we can find a small neighbourhood $W'_H \subset W_H$ of h , $W'_B \subset W_B$ of b and $W' \subset W$ of t , such that for every $t_d \in]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$, we have that

$$(h' h(\phi(t'), t_d), b(g, t_d) b') \subset W_H \times W_B$$

This implies by 3.7 that

$$h\phi(t)b \in W'_B \phi(W') \exp(] - \varepsilon, \varepsilon[Z_l) W'_B \subset W_H \phi(W) W_B.$$

By the induction hypothesis, we know that the subset $W'_B \phi(W') \exp(] - \varepsilon, \varepsilon[Z_l) W'_B$ is an open neighborhood of $h\phi(t)b$ and thus the set $W_H \phi(W) W_B$ is open in G .

In case **Case b)**, we have that Z_l is an element of \mathfrak{c} . We consider again open subsets W_H of H , $W = W_0 \times]a, b[$ open in $\mathcal{V} \subset \mathbb{R}^d$ and W_B open in B and prove that $W_H \phi(W) W_B$ is open in G . Let $b \in W_B, h \in W_H, t = (t_1, \dots, t_d) \in W$. For $h' \in H, b' \in B, s \in \mathbb{R}$ and $t \in W$, we have that

$$\begin{aligned} (3.8) \quad h' \phi(t) \exp(s Z_l) b' &= h' \exp(s Z_l) (\exp(-s Z_l) \phi(t) \exp(s Z_l)) b' \\ &= h' \exp(s Z_l) \phi(t) b(s, t) b' \\ &\quad (\text{where } b(s, t) \in B \text{ is polynomial in } (s, t)) \end{aligned}$$

This implies as in the case a-2), that there exist small neighbourhoods $W'_H \subset W_H$ of h , $W'_B \subset W_B$ of b and $W' \subset W$ of t , such that for every $s \in]-\varepsilon, \varepsilon[$ (for some $\varepsilon > 0$), $h' \in W'_H, b' \in W'_B$, we have that

$$(3.9) \quad (h' h, b(s, t) b') \subset W_H \times W_B$$

Therefore, by 3.8 and 3.9,

$$h\phi(t)b \in W'_B \phi(W') \exp(] - \varepsilon, \varepsilon[Z_l) W'_B \subset W_H \phi(W) W_B$$

and we conclude as in the case **a2)** □

Definition 3.11. We denote by

$$\psi : H \backslash G / B \rightarrow G,$$

the inverse of the mapping $\phi(\mathcal{V}) \rightarrow H \backslash G / B$. Then ψ is a continuous section of the open dense subset $\tilde{\mathcal{U}}$ of $H \backslash G / B$ into G .

Example 3.12. Returning to the previous example, we can see that $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{1, 2, 6\}$ and $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{1, 2\}$. In fact, if $g = z_1 Z_1 \dots z_7 Z_7 \in G$, then $\text{Ad}(g)\mathfrak{b} = \text{span} \langle Z_3, Z_4 + z_1 Z_6, Z_7 \rangle$, so in the Zariski open $\mathcal{U}_6 := G \setminus G_2$ of G , we have $\forall g \in \mathcal{U}_6, Z_6 \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_7 = \text{span} \langle Z_3, Z_4, Z_5, z_1 Z_6, Z_7 \rangle$, but for $i = 1, 2$, it's clear that $Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}, \forall g \in G =: \mathcal{U}_i$. In conclusion,

$$\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i \in \mathcal{I} = \{1, 2, 6\}, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1} \ \forall g \in \mathcal{U} = G \setminus G_2\} = \{1, 2\}.$$

Now for $H \backslash G / B$, thanks to $\mathcal{V} = \phi^{-1}(\mathcal{U}) = \{t \in \mathbb{R}^2, t_1 Z_1 \cdot t_2 Z_2 \in G \setminus G_2\} = \mathbb{R}^* \times \mathbb{R}$ and $\tilde{\mathcal{U}} = \mathbf{p}(\mathcal{U}) = \{H \cdot z_1 Z_1 \dots z_7 Z_7 \cdot B, z = (z_1, \dots, z_7) \in \mathbb{R}^* \times \mathbb{R}^6\}$ where these maps are given by

$$\begin{aligned} \phi : \mathbb{R}^2 &\longrightarrow G & \text{and} & \quad \mathbf{p} : G \longrightarrow H \backslash G / B, \\ (t_1, t_2) &\longmapsto t_1 Z_1 \cdot t_2 Z_2 & & \quad g \longmapsto H \cdot g \cdot B \end{aligned}$$

we have that

$$\begin{aligned} \tilde{\phi} : \mathbb{R}^* \times \mathbb{R} &\longrightarrow \tilde{\mathcal{U}} \\ (t_1, t_2) &\longmapsto H \cdot t_1 Z_1 \cdot t_2 Z_2 \cdot B \end{aligned}$$

which is an homeomorphism thanks to the previous theorem. In particular, this confirms the relation $\mathbf{p}(G \setminus G_2) = \tilde{\phi}(\mathbb{R}^* \times \mathbb{R})$ of the first example's calculation.

3.2. A measure on $H \backslash G / B$. Our aim is to disintegrate the restriction of the irreducible representation $\pi_{l, \mathfrak{b}}$. The first step is to realize the Hilbert space $L^2(G/B, \chi_l)$ as a direct integral. Therefore we need a measure $d\gamma$ on the space $H \backslash G / B$ such

$$L^2(G/B, \chi_l) \simeq \int_{H \backslash G / B}^{\oplus} L^2(H/B(x), \chi_{l(x)}) d\gamma(x),$$

where the linear functional

$$l(x) = \text{Ad}^*(\psi(x))l, x \in H \backslash G / B,$$

defines a character of the subalgebra $\mathfrak{b}(x) := \mathfrak{h} \cap \text{Ad}(\psi(x))\mathfrak{b}$ and where $B(x) := H \cap \psi(x)B\psi(x)^{-1}$.

Definition 3.13. Let $H = \exp(\mathfrak{h})$ be a closed subgroup of G . Let $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ be a Jordan-Hölder basis of \mathfrak{h} . As it was said in section 2, the mapping

$$E_{\mathcal{Y}} : \mathbb{R}^m \rightarrow H, \quad (y_1, \dots, y_m) \mapsto \exp(y_1 Y_1) \cdots \exp(y_m Y_m)$$

is a diffeomorphism and the measure $d_{\mathcal{Y}}$ defined on H by

$$\int_H f(h) d_{\mathcal{Y}}(h) := \int_{\mathbb{R}^m} f \circ E_{\mathcal{Y}}(y) dy, \quad f \in C_c(H),$$

is a Haar measure of H .

Furthermore, we know that for any subalgebra $\mathfrak{k} \subset \mathfrak{h}$, if we define the index set $\mathcal{I}^{\mathfrak{h}/\mathfrak{k}}$ by

$$\mathcal{I}^{\mathfrak{h}/\mathfrak{k}} := \{i \in \{1, \dots, m\}, Y_i \notin \mathfrak{k} + \mathfrak{h}_{i+1}\} = \{j_1 < \dots < j_r\},$$

then the subset

$$(3.10) \quad \mathcal{N} := \{Y_j, j \in \mathcal{I}^{\mathfrak{h}/\mathfrak{k}}\}$$

is a Malcev basis of \mathfrak{h} relative to \mathfrak{k} and the mapping

$$E_{\mathcal{N}} : \mathbb{R}^r \rightarrow H, \quad (s_1, \dots, s_r) \mapsto \exp(s_1 Y_{j_1}) \cdots \exp(s_r Y_{j_r}),$$

composed with the mapping $: H \ni h \mapsto \dot{h} = h \cdot K \in H/K$, gives the diffeomorphism

$$\dot{E}_{\mathcal{N}} : \mathbb{R}^r \rightarrow H/K, \quad (s_1, \dots, s_r) \mapsto \exp(s_1 Y_{j_1}) \cdots \exp(s_r Y_{j_r}) \cdot K.$$

In particular, similarly to (2.4), we obtain a H -invariant measure on H/K denoted $d_{\mathcal{N}}(\dot{h})$ or simply $d\dot{h}$, by putting

$$\int_{H/K} \varphi(\dot{h}) d_{\mathcal{N}}(\dot{h}) := \int_{\mathbb{R}^r} \varphi \circ \dot{E}_{\mathcal{N}}(s) ds, \quad \varphi \in C_c(H/K).$$

Proposition 3.14. There exists a Zariski open subset \mathcal{U}'' of G with property \mathcal{P} , and an index set $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}} \subset \mathcal{I}^{\mathfrak{h}}$, such that

$$\mathcal{J}^{\mathfrak{h}/\mathfrak{b}} = \mathcal{I}^{\mathfrak{h}/\mathfrak{h} \cap \text{Ad}(g)\mathfrak{b}} \quad \text{for all } g \in \mathcal{U}''.$$

Proof. For every $i \in \mathcal{I}^{\mathfrak{h}}$, the set

$$\mathcal{S}'_i = \{g \in G, Z_i \notin \mathfrak{h} \cap \text{Ad}(g)\mathfrak{b} + \mathfrak{h} \cap \mathfrak{g}_{i+1}\}$$

has property \mathcal{P} and is either contained in a Zariski closed subset of G or it contains a Zariski open subset \mathcal{S}_i with property \mathcal{P} .

The collection of all the i 's verifying the second eventuality gives us the subset $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}}$ of $\mathcal{I}^{\mathfrak{h}}$ and $\mathcal{U}'' = \bigcap_{i \in \mathcal{J}^{\mathfrak{h}/\mathfrak{b}}} \mathcal{S}_i$. \square

Definition 3.15. Let $\{j_1 < \dots < j_r\}$ be the elements of $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}}$, then $\mathcal{N} := \{Y_{j_1}, \dots, Y_{j_r}\}$ is a Malcev basis of \mathfrak{h} relative to $\mathfrak{b}(g) := \mathfrak{h} \cap \text{Ad}(g)\mathfrak{b}$ for all $g \in \mathcal{U}''$. Now we put

$$\mathcal{U}' := \mathcal{U} \cap \mathcal{U}'' \subset G, \quad \mathcal{V}' := \phi^{-1}(\mathcal{U}') \subset \mathbb{R}^d, \quad \tilde{\mathcal{U}}' := \mathfrak{p}(\mathcal{U}') \subset H \backslash G / B.$$

The subsets \mathcal{U}' and \mathcal{V}' respectively of G and \mathbb{R}^d are Zariski opens and $\tilde{\mathcal{U}}'$ is open and dense in $H \backslash G / B$.

Remark 3.16. Since the representations $\tau_{\mathfrak{l}, \mathfrak{b}}$ and $\tau_{\text{Ad}^*(g)\mathfrak{l}, \text{Ad}(g)\mathfrak{b}}$, $g \in G$, are equivalent, we can replace the subalgebra \mathfrak{b} by a conjugated algebra $\text{Ad}(g)\mathfrak{b}$, with $g \in \mathcal{U}'$. This allows us to assume, if we want, that e is contained in the Zariski open subset \mathcal{U}' . This means that we can suppose that 0 is an element of \mathcal{V}' .

Theorem 3.17. *The mapping*

$$\Phi : \mathbb{R}^r \times \mathcal{V}' \rightarrow G/B : (s, t) \mapsto E_{\mathcal{N}}(s)\phi(t) \cdot B$$

is injective and polynomial, its image is equal to $\mathcal{U}' \bmod B$ and it is regular on a Zariski open subset of $\mathbb{R}^r \times \mathcal{V}'$.

Proof. Let $g \in \mathcal{U}'$. Then $g = h\phi(t)b$ with $t \in \mathcal{V}'$, $h \in H$ and $b \in B$. We can write $h = E_{\mathcal{N}}(s)h_0$, where $h_0 = \phi(t)b_0\phi(t)^{-1}$ (with $b_0 \in B$), is an element of $H \cap \phi(t)B\phi(t)^{-1}$. Hence,

$$\Phi(s, t) = E_{\mathcal{N}}(s)\phi(t) \cdot B = E_{\mathcal{N}}(s)\phi(t)b_0 \cdot B = E_{\mathcal{N}}(s)\phi(t)b_0\phi(t)^{-1}\phi(t) \cdot B = g \cdot B.$$

If $\Phi(s, t) = \Phi(s', t')$ for two couples (s, t) and (s', t') in $\mathbb{R}^r \times \mathcal{V}'$, then $H \cdot \phi(t) \cdot B = H \cdot \phi(t') \cdot B$ and since $t \in \mathcal{V}$, it follows from the injectivity of $\tilde{\phi}$, that $t = t'$. Hence $E_{\mathcal{N}}(s) = E_{\mathcal{N}}(s')$ and the injectivity of the mapping $E_{\mathcal{N}}$ implies that $s = s'$.

Now, if we choose for instance the Malcev basis $\mathcal{M} = \{Z_{k_1}, \dots, Z_{k_p}\}$ of \mathfrak{g} relative to \mathfrak{b} defined in (2.3), then for

$$z = \exp\left(\sum_{j=1}^n z_j Z_j\right) \in G,$$

we have that

$$z \bmod B = E_{\mathcal{U}}(Q_1(z_1, \dots, z_n), \dots, Q_p(z_1, \dots, z_n)) \bmod B,$$

where $Q_1, \dots, Q_k : \mathbb{R}^n \rightarrow \mathbb{R}$ are real valued polynomials. This tells us that Φ is a polynomial mapping, since the mapping $(s, t) \rightarrow E_{\mathcal{N}}(s)\phi(t)$ is clearly polynomial. The differential of the mapping Φ in zero is given by the family of vectors

$$\{Y_{j_1}, \dots, Y_{j_r}, Z_{i_1}, \dots, Z_{i_d}\} \bmod \mathfrak{b}.$$

These vectors form a basis of \mathfrak{g} modulo \mathfrak{b} (but not necessarily a Malcev basis). Hence Φ is regular in $(0, 0)$ and since Φ is polynomial, Φ is regular on a Zariski open subset of $\mathbb{R}^r \times \mathbb{R}^d$. \square

Remark 3.18. As a consequence of the previous theorem, we have the following condition on the cardinals

$$\#\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \#\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) + \#\mathcal{J}^{\mathfrak{h}/\mathfrak{b}},$$

but in general,

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} \neq \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \cup \mathcal{J}^{\mathfrak{h}/\mathfrak{b}}.$$

This is shown by the following example.

Example 3.19. We use again the example 3.1. Clearly we have that $\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{1, 2, 5, 6\}$ and we have seen that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{1, 2\}$ and it follows easily that $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}} = \{4, 5\}$.

Corollary 3.20. We recall the definitions of the sets of indices

- $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d\}$ giving the directions $(Z_{i_1}, \dots, Z_{i_d})$ which characterize almost all elements of $H \backslash G / B$ by theorem 3.10
- $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}} = J = \{j_1, \dots, j_r\}$ giving the Malcev basis $\mathcal{N} = (Y_{j_1}, \dots, Y_{j_r})$ of \mathfrak{h} relative to $\mathfrak{b}(g)$ for every $g \in \mathcal{U}''$ defined in 3.15.
- $\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{k_1, \dots, k_p\}$, ($p = r + d$), giving the Malcev basis $\mathcal{M} = (Z_{k_1}, \dots, Z_{k_p})$ of \mathfrak{g} relative to \mathfrak{b} defined in 2.3.

As a consequence of Theorem 3.17, we have that the mapping

$$\begin{aligned} \Psi = \dot{E}_{\mathcal{M}}^{-1} \circ \Phi & : \mathbb{R}^r \times \mathcal{V}' & \longrightarrow & \mathbb{R}^p \\ E_{\mathcal{N}}(s)\phi(t) \cdot B & = \begin{matrix} (s = (s_1, \dots, s_r), t = (t_1, \dots, t_d)) \\ s_1 Y_{j_1} \dots s_r Y_{j_r} \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B \end{matrix} & \longmapsto & w = (w_1, \dots, w_p) / \\ & & & w_1 Z_{k_1} \dots w_p Z_{k_p} \cdot B = E_{\mathcal{M}}(w) \end{aligned}$$

is a polynomial diffeomorphism into its image \mathcal{W} which is a Zariski open subset of \mathbb{R}^p .

Definition 3.21. We denote for $t \in \mathcal{V}'$, $B(t) = H \cap \phi(t)B\phi(t)^{-1}$ the closed subgroup of H associated to the Lie algebra $\mathfrak{h} \cap \text{Ad}(\phi(t))\mathfrak{b}$. Then we have a diffeomorphism E_t between H and $\mathbb{R}^r \times B(t)$ such that for $h \in H$, and $(s, b) \in \mathbb{R}^r \times B(t)$,

$$E_t(h) = (s, b) \iff h = E_{\mathcal{N}}(s)b.$$

Write also for $t \in \mathcal{V}'$ and $h \in H$, the element $s(h, t)$ of \mathbb{R}^r verifying

$$(3.11) \quad h = E_{\mathcal{N}}(s(h, t)) \text{ mod } B(t).$$

We fix now $t \in \mathcal{V}'$ and $h \in H$ in order to define the polynomial mapping $q_{h,t} : \mathbb{R}^r \rightarrow \mathbb{R}^r$ by

$$(3.12) \quad q_{h,t}(s) = \dot{E}_{\mathcal{N}}^{-1}(hE_{\mathcal{N}}(s)) = s(hE_{\mathcal{N}}(s), t), s \in \mathbb{R}^r.$$

It follows then that

$$q_{hh',t} = q_{h,t} \circ q_{h',t}, \quad h, h' \in H, t \in \mathcal{V}'.$$

Proposition 3.22. The Jacobian $Jac(s, t)$ of the mapping Φ of Theorem 3.17 is constant in $s \in \mathbb{R}^r$.

Proof. Indeed, if $f \in C_c(\mathcal{U}'/B)$, $t \in \mathcal{U}'$, and $h' \in H$, using the left invariant measure $d\dot{g}$ on G/B , we get

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(\Phi(s, t)) |Jac(s, t)| ds dt & = \int_{G/B} f(g) d\dot{g} = \int_{G/B} f(h'g) d\dot{g} \\ & = \int_{\mathbb{R}^d} \int_{H/B(t)} f(h'h\phi(t)) |Jac(s((h, t), t))| d_{\mathcal{N}}(h) dt \\ & = \int_{\mathbb{R}^d} \int_{H/B(t)} f(h\phi(t)) |Jac(s(h'^{-1}h, t), t)| d_{\mathcal{N}}(h) dt \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(E_{\mathcal{N}}(s)\phi(t)) |Jac(q_{h'^{-1},t}(s), t)| ds dt \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(\Phi(s, t)) |Jac(q_{h'^{-1},t}(s), t)| ds dt. \end{aligned}$$

This shows that $|Jac(q_{h'^{-1},t}(s), t)| = |Jac(s, t)|$ for every $h' \in H, s \in \mathbb{R}^r$ and $t \in \mathcal{V}'$. Therefore $Jac \Phi(s, t) = Jac \Phi(0, t)$ for every $s \in \mathbb{R}^r, t \in \mathcal{V}'$, since Φ is a real valued polynomial function. \square

Definition 3.23. Let

$$F(t) := Jac \Phi(0, t), t \in \mathcal{V}'.$$

Then $F : \mathcal{V}' \rightarrow \mathbb{R}$ is a function on \mathcal{V}' which admits a polynomial extension on \mathbb{R}^d .

Definition 3.24. We can define now our measure on the topological space $H \backslash G / B$. We have the Zariski dense open subset \mathcal{U} of G , which has property \mathcal{P} and therefore its complement $G \setminus \mathcal{U}$ in G is Zariski closed and has also property \mathcal{P} . The projection $\mathbf{p} : G \rightarrow H \backslash G / B$ maps \mathcal{U} to the open dense subset $\tilde{\mathcal{U}}$ and \mathcal{C} to the closed subset $\tilde{\mathcal{C}}$. We define a measure $d\gamma$, which is supported on $\tilde{\mathcal{U}}$, by the rule

$$\int_{H \backslash G / B} f(x) d\gamma(x) := \int_{\mathcal{V}} f(\phi(t)) |F(t)| dt$$

Proposition 3.25. Let $f \in C_c(G/B)$. Then

$$\int_{G/B} f(g) d\dot{g} = \int_{H \backslash G / B} \int_{H/B(x)} f(h\psi(x)) d_{\mathcal{Y}}(h) d\gamma(x).$$

Proof. Indeed, by 3.17, 3.22 and Definition 3.23, we have that

$$\begin{aligned} \int_{G/B} f(g) d\dot{g} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(\Phi(s, t)) |Jac \Phi(s, t)| ds dt \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(E_{\mathcal{N}}(s)\phi(t)) |F(t)| ds dt \\ &= \int_{\mathbb{R}^d} \int_{H/B(t)} f(h\phi(t)) |F(t)| d_{\mathcal{Y}}(h) dt \\ &= \int_{H \backslash G / B} \int_{H/B(x)} f(h\psi(x)) d_{\mathcal{Y}}(h) d\gamma(x) \\ &\quad \text{(by Definition 3.24).} \end{aligned}$$

□

Definition 3.26. Let $H = \exp(\mathfrak{h})$ be a closed subgroup of the simply connected nilpotent Lie group $G = \exp(\mathfrak{g})$. Let $l \in \mathfrak{g}^*$, let $B = \exp(\mathfrak{b})$ be a polarization at l and $\pi_l = \pi_{l, \mathfrak{b}} = \text{ind}_B^G \chi_l$ be the irreducible representation associated to (l, \mathfrak{b}) . Let for $g \in \mathcal{U}'$,

$$l(g) = \text{Ad}^*(g)l|_{\mathfrak{h}}, \quad B(g) = gBg^{-1} \cap H, \quad \tau_{l(g), \mathfrak{b}(g)} = \text{ind}_{B(g)}^H \chi_{l(g)} =: \sigma(g).$$

Then we can show that the representations $\sigma(g')$ and $\sigma(g)$ of H , are equivalent whenever g' and g are "equivalent", i.e g and g' are in the same double class. Hence we can define $\sigma(\tilde{g}) := [\sigma(g)]$ as the equivalence class of the unitary representation $\sigma(g)$ for $\tilde{g} \in \tilde{\mathcal{U}}'$.

Proposition 3.27. Let for $x \in \tilde{\mathcal{U}}' \subset H \backslash G / B$, $l(x) := l(\psi(x))$, $B(x) := B(\psi(x))$, and the representation $\sigma(x) := \tau_{l(x), \mathfrak{b}(x)}$ whose Hilbert space is equal to $\mathcal{H}_{\sigma(x)} := L^2(H/B(x), \chi_{l(x)})$. Then

$$L^2(G/B, \chi_l) \simeq \int_{H \backslash G / B}^{\oplus} \mathcal{H}_{\sigma(x)} d\gamma(x),$$

$$\pi_{l, \mathfrak{b}|_H} \simeq \int_{H \backslash G / B}^{\oplus} \sigma(x) d\gamma(x).$$

Proof. Let L_0 be the space of all the vector fields $F = (f_x \in C_c(H/B(x)))_{x \in \tilde{\mathcal{U}}'}$, such that the function $\mathbb{R}^r \times \tilde{\mathcal{U}}' \ni (s, x) \mapsto f_x(E_{\mathcal{N}}(s))$ is continuous with compact support. On L_0 we have the Hilbert space norm

$$\|F\|_2^2 := \int_{H \setminus G/B} \int_{\mathbb{R}^r} |F_x(E_{\mathcal{N}}(s))|^2 ds d\gamma(x) = \int_{H \setminus G/B} \|F_x\|_{L^2(H/B(x), \chi_{l(x)})}^2 d\gamma(x).$$

The direct integral $\int_{H \setminus G/B} L^2(H/B(x), \chi_{l(x)}) d\gamma(x)$ can then be defined as the completion of L_0 for the norm $\|\cdot\|_2$ of L_0 . Define now for $\xi \in C_c(G/B, \chi_l)$ the function

$$\mathcal{T}(\xi)(x)(h) := \xi(h\psi(x)), x \in \tilde{\mathcal{U}}', h \in H.$$

For $x \in \tilde{\mathcal{U}}'$, the function $\mathcal{T}(\xi)(x)$ is then an element of $C_c(H/B(x), \chi_{l(x)})$, since it satisfies the covariance relation,

$$\mathcal{T}(\xi)(hb) = \xi(hh\phi(t)) = \xi(h\phi(t)\phi(t)^{-1}b\phi(t)) = \chi_{l(x)}\xi(h\phi(t)) = \chi_{l(x)}\mathcal{T}(\xi)(x)(h),$$

for $t = \psi(x)$, $h \in H$, and $b \in B(x)$ and it is obviously continuous with compact support modulo $B(x)$, for every x , since the mapping $\Phi : \mathbb{R}^r \times \mathcal{V}' \rightarrow \mathcal{U}'$ is a homeomorphism.

We obtain in this way a vector field $(\mathcal{T}(\xi)(x) \in C_c(H/B(x), \chi_{l(x)}))_{x \in \tilde{\mathcal{U}}'}$, which is contained in L_0 . The mapping $\mathcal{T} : C_c(G/B, \chi_l) \rightarrow L_0$ is isometric by 3.25, it is a linear isomorphism and it obviously intertwines the representation $\pi|_H$ and the direct integral representation $\int_{H \setminus G/B} \tau_{\chi_{l(x)}, B(x)} d\gamma(x)$. \square

4. DISINTEGRATION OF INDUCED REPRESENTATIONS

We have seen in the preceding section that the disintegration of the restriction of the irreducible representation $\pi_{l, \mathfrak{b}}$ to H reduces now to the disintegration of the monomial representations $\sigma(x) = \tau_{\chi_{l(x)}, B(x)}$ of H into irreducibles. We shall use the techniques developed in [2].

Let $\mathcal{N} = \{Y_{j_1}, \dots, Y_{j_r}\}$ be the Malcev basis defined in 3.10 and let us use the following notations:

$$\mathfrak{b}(t) := \mathfrak{b}(\tilde{\phi}(t)) = \mathfrak{b}_{r+1} \subset \dots \subset \mathfrak{b}_c(t) = \text{span} \langle Y_{j_c}, \dots, Y_{j_r}, \mathfrak{b}(t) \rangle \subset \dots \subset \mathfrak{b}_1(t) = \mathfrak{h}$$

$$B_c(t) := \exp \mathfrak{b}_c(t), \quad l(t) := \text{Ad}^*(\phi(t))|_{\mathfrak{h}}, \quad t \in \mathcal{V}', c = 1, \dots, r.$$

We have for every $t \in \mathcal{V}'$ a Jordan-Hölder basis $\mathcal{X}(t) = \{X_{r+1}(t), \dots, X_m(t)\}$ of $\mathfrak{b}(t)$, such that the vectors $X_i(t)$, $i = r+1, \dots, m$, vary rationally without singularities in $t \in \mathcal{V}'$. Let

$$\mathcal{Y}(t) = \{X_1(t) := Y_{j_1}, \dots, X_r(t) := Y_{j_r}, X_{r+1}(t), \dots, X_m(t)\}, \quad t \in \mathcal{V}'$$

be the corresponding Malcev basis of \mathfrak{h} and let

$$\mathcal{Y}(t)^* = \{X_1(t)^*, \dots, X_r(t)^*, X_{r+1}(t)^*, \dots, X_m(t)^*\}, \quad t \in \mathcal{V}'$$

be its dual basis in \mathfrak{h}^* .

For $t \in \mathcal{V}'$, let $\Gamma(t) = l(t) + \mathfrak{b}(t)^\perp$ and for any $c \in \{1, \dots, r\}$ let

$$d_c(t) := \max(\dim(\text{Ad}^*(B_c(t))(f))); f \in \Gamma(t), t \in \mathcal{V}'$$

and let $d_{r+1}(t) = 0$. Let also

$$(4.1) \quad I(t) := \{c_1 < \cdots < c_{q(t)}\} := \{c \in \{1, \dots, r\}, \quad d_c(t) = d_{c+1}(t)\}$$

and

$$(4.2) \quad \mathcal{R}(t) := \{l(t) + \sum_{c_i \in I(t)} v_i X_{c_i}(t)^*, \quad (v_1, \dots, v_{q(t)}) \in \mathbb{R}^{q(t)}\}, t \in \mathcal{V}'.$$

We can again assume, if necessary by shrinking \mathcal{V}' , that there exists a fixed index set I such that for every $t \in \mathcal{V}'$ we have that

$$(4.3) \quad I(t) = I = \{c_1, \dots, c_q\} \text{ and so } q(t) = q = |I|.$$

We shall also use the notation

$$(4.4) \quad l(v, t) := l(t) + \sum_{c \in I(t)} v_i Y_{j_c}(t)^* \in \Gamma(t), t \in \mathcal{V}', v \in \mathbb{R}^q.$$

It follows from the proofs used in the paper [2], that there exists a Zariski open subset \mathcal{S} of $\mathbb{R}^q \times \mathbb{R}^d$ such that for every $(u, t) \in \mathcal{S}$, we have a Malcev basis

$$(4.5) \quad \mathcal{X}(v, t) = \{X_1(v, t), \dots, X_d(v, t)\}$$

of \mathfrak{h} relative to the Vergne polarization $\mathfrak{b}(v, t)$ associated to the Jordan-Hölder basis

$$\mathcal{Y} = \{Y_1, \dots, Y_m\} = \{Z_j, j \in \mathcal{I}^{\mathfrak{h}}\}$$

at $l(v, t) \subset \Gamma(t)$ and a Malcev basis

$$(4.6) \quad \mathcal{Y}(v, t) = \{Y_1(v, t), \dots, Y_e(v, t)\}$$

of $\mathfrak{b}(v, t)$ relative to the subalgebra $\mathfrak{b}(v, t) \cap \mathfrak{b}(t)$ such that all the vectors $X_j(v, t)$ and $Y_i(v, t)$ vary rationally without singularities in $(v, t) \in \mathcal{S}$ and such that the operator

$$\mathcal{T}(v, t) : \mathcal{S}(H/B(t), \chi_{l(t)}) \rightarrow \mathcal{S}(H/B(v, t), \chi_{l(v, t)})$$

defined by

$$\mathcal{T}(v, t)(\eta)(h) = \int_{B(v, t)/B(v, t) \cap B(t)} \eta(hb) \chi_{l(v, t)(b)} d_{\mathcal{Y}(v, t)}(b), \quad h \in H,$$

(where $\mathcal{Y}(v, t)$ is as in 4.6) gives us an isometric intertwining operator

$$\mathcal{T}(t) : L^2(H/B(t), \chi_{l(t)}) \rightarrow \int_{\Gamma(t)} L^2(H/B(v, t), \chi_{l(v, t)}), \quad \mathcal{T}(t)(\eta) := \mathcal{T}(v, t)\eta,$$

(here the norm on $L^2(H/B(v, t), \chi_{l(v, t)})$ is defined via the measure $d_{\mathcal{X}(v, t)}$ on $H/B(v, t)$). Combining these manipulations with Theorem 3.3 of [2] and Proposition 3.27, we obtain the following concrete decomposition of $\pi_{l|_H}$

Theorem 4.1. *Let $G = \exp \mathfrak{g}$ be a connected simply connected Lie group, $H = \exp \mathfrak{h}$ be a closed connected subgroup of G . Let \mathcal{Z} be a Jordan-Hölder basis of \mathfrak{g} , let $l \in \mathfrak{g}^*$ and let \mathfrak{b} be the Vergne polarization at l with respect to \mathcal{Z} . Let for any $(v, t) \in \mathcal{R}$, for $\xi \in \mathcal{S}(G/B, \chi_l)$*

$$T_{v,t}\xi(h) = \int_{B(v,t)/B(v,t) \cap B(t)} \xi(hb\phi(t))\chi_{v,t}(b)d_{\mathcal{Y}(v,t)}(b),$$

where $\mathcal{Y}(v, t)$ is as in 4.6. Then

$$\int_{H \backslash G/B} \int_{H/B(v,\psi(x))} \|T_{v,t}\xi\|_{L^2(H/B(v,\psi(x)), \chi_{l(v,\psi(x))})}^2 d\gamma(x) = \|\xi\|_{L^2(G/B, \chi_l)}^2$$

and the operator

$$\mathcal{T} : L^2(G/B, \chi_l) \rightarrow \int_{H \backslash G/B} L^2(H/B(\psi(x)), \chi_{l(v,\psi(x))}) d\gamma(x)$$

defined by

$$\xi \mapsto \mathcal{T}_{v,\psi(x)}(\xi), \quad \xi \in \mathcal{S}(G/B, \chi_l), (x, \psi(x)) \in \mathcal{R},$$

extends to a unitary intertwining operator.

5. DISINTEGRATING TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATIONS

Let now $G = \exp \mathfrak{g}$ be a simply connected, connected Lie group and let $\pi = \pi_l, \pi' = \pi_{l'}$ be two irreducible unitary representations of G . We take a Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ of \mathfrak{g} and we obtain the corresponding Vergne polarizations \mathfrak{b} at l and \mathfrak{b}' at l' . We realize then π as $\pi_{l,\mathfrak{b}}$ resp. π' as $\pi_{l',\mathfrak{b}'}$.

Let $\mathbb{G} := G \times G$ and $\mathbb{B} := B \times B'$ be the direct products respectively of G with G and B with B' . Let also $\Pi = \pi \times \pi'$ be the outer tensor product of π and π' . We can realize Π as induced representation. It acts by left translation on the space

$$\mathcal{H}_{\Pi} := L^2(\mathbb{G}/\mathbb{B}, \chi_{(l,l')}) \simeq \mathcal{H}_{\pi} \hat{\otimes} \mathcal{H}_{\pi'},$$

where (l, l') is the linear functional defined on $\mathfrak{g} \times \mathfrak{g}$ by

$$(l, l')(Z, Z') := l(Z) + l'(Z'), \quad Z, Z' \in \mathfrak{g},$$

and so $\mathfrak{b} \times \mathfrak{b}'$ is a polarization at (l, l') . Hence $\Pi = \pi_{(l,l'), \mathfrak{b} \times \mathfrak{b}'}$.

Let $\mathbb{D} = \{(g, g); g \in G\}$ be the diagonal group of \mathbb{G} . This group is of course isomorphic to G and the tensor product $\pi \otimes \pi'$ is the representation of G acting on the space of Π through the rule

$$\pi \otimes \pi'(g) := \Pi(g, g), \quad g \in G.$$

We are led to investigate the structure of the space of double classes $\mathbb{D} \backslash \mathbb{G} / \mathbb{B}$.

Proposition 5.1. The mapping

$$\mathbb{M} : B \backslash G / B' \rightarrow \mathbb{D} \backslash \mathbb{G} / \mathbb{B}, \quad B \cdot g \cdot B' \mapsto \mathbb{D} \cdot (e, g) \cdot \mathbb{B} := \mathbb{M}(B \cdot g \cdot B'),$$

is well defined and is a bijection.

Proof. Indeed, if $g \in G$ and $g' \in B \cdot g \cdot B'$, then we have that $g' = bgb'$ for some $b \in B$ and $b' \in B'$ and so

$$\begin{aligned} \mathbb{D} \cdot (e, g') \cdot \mathbb{B} &= \mathbb{D} \cdot (e, bgb') \cdot \mathbb{B} = \{(ub_2, ubgb'b'_2), u \in G, b_2 \in B, b'_2 \in B'\} \\ &= \{(ub^{-1}b_2, ugb'_2), u \in G, b_2 \in B, b'_2 \in B'\} \\ &= \{(ub_2, ugb'_2), u \in G, b_2 \in B, b'_2 \in B'\} \\ &= \mathbb{D} \cdot (e, g) \cdot \mathbb{B}. \end{aligned}$$

Hence our mapping is well defined. Since

$$\mathbb{D} \cdot (g, g') \cdot \mathbb{B} = \mathbb{D} \cdot (g, g) \cdot (e, g^{-1}g') \cdot \mathbb{B} = \mathbb{D} \cdot (e, g^{-1}g') \cdot \mathbb{B} = \mathbb{M}(B \cdot (g^{-1}g') \cdot B'),$$

it is also surjective. If $\mathbb{M}(B \cdot g \cdot B') = \mathbb{M}(B \cdot g' \cdot B')$ for two elements g, g' in G , then $(e, g') = (u, u)(e, g)(b, b')$ for some $u \in G, b \in B$ and $b' \in B'$. Hence, $ub = e$ and $ugb' = g'$ which means that $g' = b^{-1}gb'$ is an element of $B \cdot g \cdot B'$. Therefore \mathbb{M} is injective and so it is a bijection. \square

According to the two preceding sections, we need first a Jordan-Hölder basis of $\mathfrak{g} \times \mathfrak{g}$ in order to perform our computations.

We make the following choice. Let

$$\mathcal{Z}_i := \begin{cases} (Z_j, Z_j) & \text{if } i = 2j - 1 \text{ is odd,} \\ (Z_j, -Z_j) & \text{if } i = 2j \text{ is even} \end{cases} \quad (j \in \{1, \dots, n\}).$$

The basis $\mathcal{A} = \{A_1, \dots, A_n\}$ of the Lie algebra $\mathfrak{d} = \{(Z, Z); Z \in \mathfrak{g}\}$ of the diagonal group \mathbb{D} is then given by the vectors

$$A_j := (Z_j, Z_j) = \mathcal{Z}_{2j-1}, \quad j = 1, \dots, n.$$

Hence

$$(5.1) \quad \mathcal{I}^\mathfrak{d} = \{2j - 1, j \in \{1, \dots, n\}\}, \quad \mathcal{I}^{\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}} = \{2j, j \in \{1, \dots, n\}\}.$$

We must determine the Index sets $\mathcal{I}(\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}, \mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}') \subset \mathcal{I}^{\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}} \cap \mathcal{I}^{\mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}'}$. By 5.1 we have that

$$\mathcal{I}(\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}, \mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}') \subset \{2j, j \in \{1, \dots, n\}\}.$$

Proposition 5.2.

$$(5.2) \quad \mathcal{I}(\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}, \mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}') = \{2j, j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')\}.$$

Proof. Let \mathcal{U} be a Zariski open subset of G with property \mathcal{P} , such that for every $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$,

$$Z_j \notin \mathfrak{b} + \text{Ad}(g)\mathfrak{b}' + \mathfrak{g}_{j+1} \text{ for every } g \in \mathcal{U}.$$

Let

$$\mathbb{U} := \{(g, g') \in \mathbb{G}, g^{-1}g' \in \mathcal{U}\}.$$

Then \mathbb{U} is a Zariski open subset of \mathbb{G} . Let us show that for every $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$, we have

$$(5.3) \quad \mathcal{Z}_{2j} = (Z_j, -Z_j) \notin \mathfrak{d} + \text{Ad}(g, g')(\mathfrak{b} \times \mathfrak{b}') + (\mathfrak{g} \times \mathfrak{g}')_{2j+1}, \forall (g, g') \in \mathbb{U}.$$

Then necessarily the index $2j$ is in $\mathcal{I}(\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}, \mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}')$. Suppose that for some elements j and (g, g') respectively of $\mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$ and \mathbb{U} , the relation 5.3 is false. Then we have that

$$(Z_j, -Z_j) = (Y, Y) + (\text{Ad}(g)(X), \text{Ad}(g')(X')) + (U, U')$$

for certain vectors Y, X, X', U and U' respectively in $\mathfrak{g}, \mathfrak{b}, \mathfrak{b}'$ and \mathfrak{g}_{j+1} , since $(\mathfrak{g} \times \mathfrak{g})_{2j+1} = \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1}$.

Hence

$$\begin{cases} Z_j &= Y + \text{Ad}(g)(X) + U \\ -Z_j &= Y + \text{Ad}(g')(X') + U' \end{cases}$$

which gives us the identity

$$Z_j = \text{Ad}(g)\left(\frac{X}{2}\right) + \text{Ad}(g')\left(\frac{-X'}{2}\right) + \frac{U - U'}{2}.$$

This implies that

$$\begin{aligned} Z_j &\equiv \frac{X}{2} + \text{Ad}(g^{-1}g')\left(\frac{-X'}{2}\right) \quad \text{mod } [\mathfrak{g}_{j+1}] \\ &\in \mathfrak{b} + \text{Ad}(g^{-1}g')(\mathfrak{b}') \quad \text{mod } [\mathfrak{g}_{j+1}] \end{aligned}$$

where $g^{-1}g' \in \mathcal{U}$ (by the definition of \mathbb{U}). This last relation contradicts the fact that $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$.

Let now $2j \in \mathcal{I}(\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}, \mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}')$. Then for every (g, g') in a Zariski open subset of \mathbb{G} , we have that

$$(Z_j, -Z_j) \notin \mathfrak{d} + \text{Ad}(g)(\mathfrak{b}) \times \text{Ad}(g')(\mathfrak{b}') + \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1}.$$

Let us show first that $j \in \mathcal{I}^{\mathfrak{a}/\mathfrak{b}} \cap \mathcal{I}^{\mathfrak{a}/\mathfrak{b}'}$. If not, we can assume that $j \notin \mathcal{I}^{\mathfrak{a}/\mathfrak{b}'}$. There exists $X' \in \mathfrak{b}'$, such that

$$Z_j = X' \text{ mod } \mathfrak{g}_{j+1} = \text{Ad}(g')X' \quad \text{mod } \mathfrak{g}_{j+1}, \text{ for all } g' \in G.$$

Hence

$$\begin{aligned} \mathcal{Z}_{2j} = (Z_j, -Z_j) &= (Z_j, Z_j) + (0, \text{Ad}(g')(-2X')) \quad \text{mod } \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1} \\ &\in \mathfrak{d} + \text{Ad}(g, g')(\mathfrak{b} \times \mathfrak{b}') \quad \text{mod } (\mathfrak{g} \times \mathfrak{g})_{2j+1}, \quad \forall (g, g') \in \mathbb{G}. \end{aligned}$$

This relation contradicts the fact that $2j \in \mathcal{I}(\mathfrak{g} \times \mathfrak{g}/\mathfrak{d}, \mathfrak{g} \times \mathfrak{g}/\mathfrak{b} \times \mathfrak{b}')$.

Suppose now that $j \notin \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$. Then in the Zariski open subset \mathcal{U} of G , we have that

$$Z_j \in \mathfrak{b} + \text{Ad}(g)\mathfrak{b}' + \mathfrak{g}_{j+1}, \quad \forall g \in \mathcal{U}.$$

Let (g, g') one element of the Zariski open subset \mathbb{U} of \mathbb{G} , this means that $g'' := g^{-1}g' \in \mathcal{U}$. There exists $X \in \mathfrak{b}$ and $X' \in \mathfrak{b}'$, such that

$$Z_j = X + \text{Ad}(g'')(X') \text{ mod } \mathfrak{g}_{j+1}.$$

Whence

$$\text{Ad}(g)Z_j = \text{Ad}(g)X + \text{Ad}(g')X' \text{ mod } \mathfrak{g}_{j+1}$$

and Z_j can then be written as

$$Z_j = \text{Ad}(g)X + \text{Ad}(g')X' \text{ mod } \mathfrak{g}_{j+1}.$$

Let

$$Y = -\text{Ad}(g)X + \text{Ad}(g')X'$$

It is easy to verify then that

$$\begin{aligned} (Z_j, -Z_j) &= (Y, Y) + (\text{Ad}(g)(2X), \text{Ad}(g')(-2X')) \text{ mod } \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1} \\ &\subset \mathfrak{d} + \text{Ad}(g, g')(\mathfrak{g} \times \mathfrak{g}') \text{ mod } \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1}. \end{aligned}$$

This is impossible since $(g, g') \in \mathbb{U}$. Hence $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$. □

Corollary 5.3. Let \mathcal{U} be the Zariski open subset of G defined in 3.4 for the subgroups B and B' and let $\mathbb{U} = \{(g, g') \in \mathbb{G}; g^{-1}g' \in \mathcal{U}\}$. Let $\tilde{\mathcal{U}}$ be the projection of \mathcal{U} into $B \backslash G / B'$ and $\tilde{\mathbb{U}}$ the projection of \mathbb{U} into $\mathbb{D} \backslash \mathbb{G} / \mathbb{B}$. Then the mapping \mathbb{M} of Proposition 5.1 maps $\tilde{\mathcal{U}}$ homeomorphically onto $\tilde{\mathbb{U}}$.

Definition 5.4. Let now \mathcal{U} be the Zariski open subset of G with property \mathcal{P} such that

$$j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}') \iff Z_j \notin \mathfrak{b} + \text{Ad}(g)\mathfrak{b}' + \mathfrak{g}_{j+1}, \text{ for all } g \in \mathcal{U}.$$

Let us write

$$\mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}') = \{i_1, \dots, i_d\}$$

and

$$a(t) := \exp(t_1 Z_{i_1}) \cdots \exp(t_d Z_{i_d}), \quad t \in \mathbb{R}^d.$$

Then the linear functionals $l(t)$ of \mathfrak{g} we need are given by

$$l(t) = \text{Ad}^*(a(t))l + \text{Ad}^*(a(-t))l' \in \text{Ad}^*(G)l + \text{Ad}^*(G)l', \quad t \in \mathbb{R}^d,$$

since using Notations 3.5, we have

$$\phi(t) = \exp(t_1(Z_{i_1}, -Z_{i_1})) \cdots \exp(t_d(Z_{i_d}, -Z_{i_d})) = (a(t), a(-t)),$$

and

$$\langle l(t), (Z, Z) \rangle = \langle \text{Ad}^*(\phi(t))(l, l'), (Z, Z) \rangle = \langle \text{Ad}^*(a(t))l + \text{Ad}^*(a(-t))l, (Z, Z) \rangle, \quad Z \in \mathfrak{g}, t \in \mathbb{R}^d.$$

We must determine now the index set I defined in 4.3 for the algebras \mathfrak{b} and \mathfrak{b}' and the elements $l(v, t)$ in $l(t) + \mathfrak{b}(t)^\perp$ defined in 4.4.

Let us remark that

$$(5.4) \quad \mathfrak{b}(t) = \mathfrak{d} \cap \text{Ad}(\phi(t))(\mathfrak{b} \times \mathfrak{b}') \simeq \text{Ad}(a(t))\mathfrak{b} \cap \text{Ad}(a(-t))\mathfrak{b}'.$$

Hence, since $\text{Ad}(a(t))\mathfrak{b}$ is a polarization at $\text{Ad}^*(a(t))l$ and similarly for $\text{Ad}^*(a(-t))\mathfrak{b}'$, we have that

$$\begin{aligned} l(t) + \mathfrak{b}(t)^\perp &= \text{Ad}^*(a(t))l + \text{Ad}^*(a(-t))l' + (\text{Ad}(a(t))\mathfrak{b} \cap \text{Ad}(a(-t))\mathfrak{b}')^\perp \\ &= \text{Ad}^*(a(t))l + \text{Ad}^*(a(-t))l' + \text{Ad}(a(t))\mathfrak{b}^\perp + \text{Ad}(a(-t))\mathfrak{b}'^\perp \\ &= \text{Ad}^*(a(t)Ba(t)^{-1})l + \text{Ad}^*(a(-t)B'a(-t)^{-1})l' \\ &\subset \text{Ad}^*(G)l + \text{Ad}^*(G)l'. \end{aligned}$$

Hence the affine subsets defined in 4.2

$$\mathcal{R}(t) = \{l(t) + \sum_{i \in \mathcal{I}} v_i Y_{j_i}(t)^*, (v_1, \dots, v_q) \in \mathbb{R}^q\}, t \in \mathcal{V}'.$$

are all contained in the sum of the orbits of l and of l' .

Definition 5.5. Let us also denote for $t \in \mathcal{U}$,

$$\mathfrak{p}(t) = \text{Ad}(a(t))\mathfrak{b}, \mathfrak{p}'(t) := \text{Ad}(a(-t))\mathfrak{b}'.$$

Then $\mathfrak{p}(t)$ is a polarization at $\text{Ad}^*(a(t))l$ and $\mathfrak{p}'(t)$ at $\text{Ad}^*(a(-t))l'$. Let also

$$P(t) = \exp \mathfrak{p}(t), P'(t) = \exp \mathfrak{p}'(t), t \in \mathcal{U}.$$

On the Zariski open subset $\mathcal{R} \subset \mathbb{R}^q \times \mathbb{R}^d$, define the measure $d\nu$ by

$$d\nu = dp \otimes |F(t)|dt,$$

where F is the function defined as in 3.23 (for the subgroups B and B').

For $\theta = (v, t) \in \mathcal{R}$, let $t(\theta) = t$ and $v(\theta) = v$.

We can now rephrase Theorem 4.1 for the special case of Π and \mathbb{D} .

Theorem 5.6. *Let $G = \exp \mathfrak{g}$ be a connected simply connected Lie group, let \mathcal{Z} be a Jordan-Hölder basis of \mathfrak{g} , and let $l, l' \in \mathfrak{g}^*$ and let \mathfrak{b} res. \mathfrak{b}' be the Vergne polarization at l , resp. at l' , with respect to \mathcal{Z} . Let for any $\theta \in \mathcal{R} \subset \Omega_\pi + \Omega'_\pi$, for $\xi \in \mathcal{S}(\mathbb{G}/\mathbb{B}, \chi_{(l, l')})$*

$$T_\theta \xi(g) = \int_{B(\theta)/B(\theta) \cap P(t(\theta)) \cap P'(t(\theta))} \xi(gba(t(\theta)), gba(-t(\theta))) \chi_\theta(b) d_{\mathcal{Y}(\theta)}(b),$$

where $\mathcal{Y}(\theta)$ is as in 4.6. Then

$$\int_{\mathcal{R}} \|T_\theta \xi\|_{L^2(G/B(\theta), \chi_{l(\theta)})}^2 d\nu(\theta) = \|\xi\|_{L^2(\mathbb{G}/B, \chi_{(l, l')})}^2$$

and the operator

$$\mathcal{T} : L^2(G/B, \chi_l) \rightarrow \int_{\mathcal{R}} L^2(H/B(\theta), \chi_{l(\theta)}) d\nu(\theta)$$

defined by

$$\xi \mapsto \mathcal{T}_\theta(\xi), \quad \xi \in \mathcal{S}(G/B, \chi_l), \theta \in \mathcal{R},$$

extends to a unitary intertwining operator.

6. AN EXAMPLE

We show in the following example, that the choice of a polarization for a given linear functional has a strong influence on the concrete form of the disintegration of the representation $\pi \otimes \pi'$.

Let \mathfrak{g} be the Lie algebra spanned by the vectors $\langle Z_1, \dots, Z_7 \rangle$ and equipped with the non trivial brackets

$$\begin{aligned} [Z_1, Z_4] &= Z_7 & [Z_2, Z_4] &= Z_5 & [Z_2, Z_6] &= Z_7 \\ [Z_3, Z_4] &= Z_6 & [Z_3, Z_5] &= Z_7 \end{aligned}$$

Then $\mathcal{Z} = \{Z_1, \dots, Z_7\}$ is a Jordan-Hölder basis of \mathfrak{g} and one can easily verify that the coadjoint orbit Ω_l of any $l \in \mathfrak{g}^*$ with $l(Z_7) \neq 0$ is equal to $l + Z_7^\perp$. Let $\mathcal{Z}^* = \{Z_1^*, \dots, Z_7^*\}$ be the dual basis of \mathcal{Z} .

Choose now $l = Z_7^* + Z_6^* \in \mathfrak{g}^*$. Its stabilizer in \mathfrak{g} is reduced to the center $\mathbb{R}Z_7$ of \mathfrak{g} . There are several possibilities for polarizations at l . We take 4 such polarizations.

$$\begin{aligned}\mathfrak{b}_1 &:= \text{span} \langle Z_7, Z_6, Z_5, Z_4 \rangle \\ \mathfrak{b}_2 &:= \text{span} \langle Z_7, Z_3, Z_2, Z_1 \rangle \\ \mathfrak{b}_3 &:= \text{span} \langle Z_7, Z_6, Z_4 - Z_5, Z_3 \rangle \\ \mathfrak{b}_4 &:= \text{span} \langle Z_7, Z_6, Z_4, Z_1 \rangle\end{aligned}$$

Then \mathfrak{b}_1 is an abelian ideal of \mathfrak{g} , \mathfrak{b}_2 and \mathfrak{b}_4 are abelian subalgebras and \mathfrak{b}_3 is non-abelian.

Definition 6.1. Let now

$$\pi_j := \text{ind}_{B_j}^G \chi_l, \quad j = 1, 2, 3, 4.$$

6.1. Disintegration of $\pi_1 \otimes \pi_2$. We have that \mathfrak{b}_1 is an ideal of \mathfrak{g} and $\mathfrak{b}_1 + \mathfrak{b}_2 = \mathfrak{g}$. Hence $\mathcal{I}(\mathfrak{b}_1, \mathfrak{b}_2) = \emptyset$. Hence by 3.27, we have that

$$\pi_1 \otimes \pi_2 \simeq \text{ind}_{B_1 \cap B_2}^G \chi_{2l}.$$

We must determine the index set I of 4.1. We have now that $2l + (\mathfrak{b}_1 \cap \mathfrak{b}_2)^\perp = 2l + Z_7^\perp$ and the family $\{Z_1, \dots, Z_6\}$ is a Malcev basis of \mathfrak{g} relative $\mathfrak{b}_1 \cap \mathfrak{b}_2$. We must determine the indices set I . We have for a linear form $f = 2Z_7^* + \sum_{j=2}^6 \beta_j Z_j^* \in 2l + Z_7^\perp$ that

$$\begin{array}{llll} \mathfrak{g}_6(f_6) = \mathfrak{g}_6 & \not\subset \mathfrak{g}_7 & \implies & 6 \in I, \\ \mathfrak{g}_5(f_5) = \mathfrak{g}_5 & \not\subset \mathfrak{g}_6 & \implies & 5 \in I, \\ \mathfrak{g}_4(f_4) = \mathfrak{g}_4 & \not\subset \mathfrak{g}_5 & \implies & 4 \in I, \\ \mathfrak{g}_3(f_3) = \text{span} \langle Z_7, Z_6, Z_4 - \frac{\beta_6}{2} Z_5 \rangle & \subset \mathfrak{g}_4 & \implies & 3 \notin I, \\ \mathfrak{g}_2(f_2) = \text{span} \langle Z_7, Z_4 - \frac{\beta_6}{2} Z_5 + \frac{\beta_6}{2} Z_6 \rangle & \subset \mathfrak{g}_3 & \implies & 2 \notin I, \\ \mathfrak{g}(f) = \mathfrak{g}_7 & \subset \mathfrak{g}_2 & \implies & 1 \notin I. \end{array}$$

where $f_j := f|_{\mathfrak{g}_j}$. This implies that $I = \{k_1 = 4, k_2 = 5, k_3 = 6\}$ and the space of disintegration of $\pi_1 \otimes \pi_2$ is given by $\mathcal{R} = 2Z_7^* + \mathbb{R}Z_6^* + \mathbb{R}Z_5^* + \mathbb{R}Z_4^*$. We can take for every $\theta \in \mathcal{R}$ the Vergne polarization $\mathfrak{b} = \mathfrak{b}_1$ associated to Jordan-Hölder basis \mathcal{Z} . The mapping $T_\theta : L^2(G/B_1 \times B_2, \chi_{l,l})$ of Theorem 5.6 is then given by

$$\mathcal{I}_\theta(\xi)(g) = \int_{\mathbb{R}^3} \xi(g, g(u_4 Z_4)(u_5 Z_5)(u_6 Z_6)) e^{-2\pi i(v_4 u_4 + v_5 u_5 + (v_6 + 1)u_6 + 2Z_7^*)} du_4 du_5 du_6, \quad g \in G.$$

and

$$\mathcal{H}_{\pi_1} \hat{\otimes} \mathcal{H}_{\pi_2} \simeq \int_{\mathbb{R}^3} L^2(G/B, \chi_{v_4 Z_4^* + v_5 Z_5^* + (v_6 + 1)Z_6^* + 2Z_7^*}) du_4 du_5 du_6.$$

6.2. **Disintegration of $\pi_3 \otimes \pi_4$.** We have here that

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}_3} = \{1, 2, 5\}, \mathcal{I}^{\mathfrak{g}/\mathfrak{b}_4} = \{2, 3, 5\}$$

and so $I^{\mathfrak{g}/\mathfrak{b}_3} \cap I^{\mathfrak{g}/\mathfrak{b}_4} = \{2, 5\}$.

Since for every $g = \prod_{j=1}^7 (g_j Z_j)$, $g_2 \neq 0$,

$$Z_5 = Z_5 - Z_4 + \text{Ad}((g_1 Z_1)(g_2 Z_2)(g_3 Z_3)(g_4 Z_4))\left(\frac{1}{g_2}(-Z_4)\right) \text{mod } \mathfrak{g}_6 \subset \mathfrak{b}_3 + \text{Ad}(g)(\mathfrak{b}_4) \text{mod } \mathfrak{g}_6,$$

it follows that $5 \notin \mathcal{I}(\mathfrak{b}_3, \mathfrak{b}_4)$. Hence, since $[\mathfrak{g}, \mathfrak{g}] = \text{span} \langle Z_5, Z_6, Z_7 \rangle$, we have that

$$\mathcal{I}(\mathfrak{b}_3, \mathfrak{b}_4) = \{2\}.$$

We must compute the function F of Reference 3.23. Let $\Phi : \mathbb{R}^2 \times \mathbb{R} \rightarrow G/B_4$ the mapping such that

$$\Phi(s_1, s_2, t) = (s_1 Z_1)(s_2(Z_4 - Z_5))(t Z_2) \cdot B_4 = (s_1 Z_1)(t Z_2)((s_2 t + 1) Z_5) \cdot B_4.$$

Therefore

$$\Phi(s_1, s_2, t) \equiv (s_1, t, 1 + s_2 t)$$

and so

$$F(t) = t.$$

We have that

$$\text{Ad}^*(t Z_2)l = l - t Z_6^* \Rightarrow l(t) = \text{Ad}^*(t Z_2)l + \text{Ad}^*(-t Z_2)l = 2l, t \in \mathbb{R}.$$

The Lie algebra $\mathfrak{b}(t)$ of Reference 5.4 is given for $t \in \mathbb{R}$ by

$$\mathfrak{b}(t) = \text{Ad}(t Z_2)\mathfrak{b}_3 \cap \text{Ad}(-t Z_2)\mathfrak{b}_4 = \text{span}\{Z_6, Z_7\}.$$

Hence the space of disintegration is this time $\mathcal{R} = \mathbb{R}^2 \times \mathbb{R}$ and the mapping \mathcal{T}_θ , $\theta \in \mathcal{R}$ is defined by

$$T_{(v_1, v_2), t} \xi(g) = \int_{\mathbb{R}^2} \xi(g(u_4 Z_4)(u_5 Z_5)(t Z_2), g(u_4 Z_4)(u_5 Z_5)(-t Z_2)) e^{-2\pi i(u_4 v_4 + u_5 v_5)} du_4 du_5$$

and

$$\mathcal{H}_{\pi_3} \hat{\otimes} \mathcal{H}_{\pi_4} \simeq \int_{\mathbb{R}} \int_{\mathbb{R}^2} L^2(G/B, \chi_{v_4 Z_4^* + v_5 Z_5^* + 2Z_7^*}) |t| du_4 du_5 dt$$

7. CRITERIA FOR IRREDUCIBILITY

In this section we shall determine a necessary and sufficient condition for the irreducibility of the tensor representation $\pi \otimes \pi'$. We shall prove the following theorem

Theorem 7.1. *Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group, let $\pi = \pi_l$ and $\pi' = \pi_{l'}$ be two irreducible unitary representations of G . The following conditions are equivalent*

$$(1) \pi \otimes \pi' \in \hat{G}$$

(2) For every polarization \mathfrak{b} at l and for every polarization \mathfrak{b}' at l' , we have that

$$\begin{cases} \mathfrak{b} + \mathfrak{b}' = \mathfrak{g}, \\ \mathfrak{b} \cap \mathfrak{b}' \text{ is a polarization at } l + l'. \end{cases}$$

(3) there exist polarizations \mathfrak{b} at l and \mathfrak{b}' at l' , such that

$$\begin{cases} \mathfrak{b} + \mathfrak{b}' = \mathfrak{g}, \\ \mathfrak{b} \cap \mathfrak{b}' \text{ is a polarization at } l + l'. \end{cases}$$

(4)

$$\text{Ad}^*(G)l + \text{Ad}^*(G)l' = \text{Ad}^*(G)(l + l')$$

and

$$\mathfrak{g}(l) + \mathfrak{g}(l') = \mathfrak{g}.$$

Proof. 1) \implies 2)

By Theorem 4.1, the space of double classes $\mathbb{D} \backslash \mathbb{G} / \mathbb{B} \simeq B \backslash G / B'$ must be reduced to one point, which implies by Lemma 3.9 that $\mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}') = \emptyset$ and $G = BB'$. In particular this tells us that $L^2(\mathbb{G}/\mathbb{B}, \chi_{(l,l')})$ is isometrically isomorphic to $L^2(G/B \cap B', \chi_{l+l'})$ and so $\pi \otimes \pi' \simeq \text{ind}_{B \cap B'}^G \chi_{l+l'}$. Hence the representation $\text{ind}_{B \cap B'}^G \chi_{l+l'}$ is irreducible, which forces $\mathfrak{b} \cap \mathfrak{b}'$ to be a polarization at $l + l'$ by Kirillov's theorem.

2) \implies 3) is trivial.

3) \implies 4)

Let us first show that $\text{Ad}^*(G)l + \text{Ad}^*(G)l' = \text{Ad}^*(G)(l + l')$. We do this by an induction on $\dim(\mathfrak{g})$. If this dimension is 1, the implication is evident, since we work then with characters. If l' is a character, then we always have that $\text{Ad}^*(G)l + \text{Ad}^*(G)l' = \text{Ad}^*(G)l + l' = \text{Ad}^*(G)(l + l')$. We can thus assume that \mathfrak{b}' is a proper subalgebra of \mathfrak{g} . Let \mathfrak{h} be again an ideal of codimension 1 in \mathfrak{g} containing \mathfrak{b}' . Then $\mathfrak{b} \not\subset \mathfrak{h}$ by the condition $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$. Hence $\mathfrak{b} \cap \mathfrak{h}$ is a polarization at $l|_{\mathfrak{h}}$ and \mathfrak{b}' is a polarization at $l'|_{\mathfrak{h}}$ and also $\mathfrak{b} \cap \mathfrak{h} + \mathfrak{b}'$ is a polarization at $l|_{\mathfrak{h}} + l'|_{\mathfrak{h}}$. It follows from the induction hypothesis that

$$\text{Ad}^*(H)l|_{\mathfrak{h}} + \text{Ad}^*(H)l'|_{\mathfrak{h}} = \text{Ad}^*(H)(l|_{\mathfrak{h}} + l'|_{\mathfrak{h}}).$$

Therefore

$$\text{Ad}^*(H)l + \text{Ad}^*(H)l' + \mathfrak{h}^\perp = \text{Ad}^*(H)(l + l') + \mathfrak{h}^\perp \subset \text{Ad}(G)(l + l'),$$

since the G -orbit $\text{Ad}(G)(l + l')$ is saturated with respect to \mathfrak{h} . But since $\mathfrak{b}' \subset \mathfrak{h}$, we also have that $\text{Ad}^*(H)l' + \mathfrak{h}^\perp = \text{Ad}^*(G)l'$. Finally

$$\text{Ad}^*(G)l + \text{Ad}^*(G)l' \subset \text{Ad}(G)(l + l').$$

Since $l + l' \in \text{Ad}^*(G)l + \text{Ad}^*(G)l'$ it follows immediately that

$$\text{Ad}(G)(l + l') \subset \text{Ad}^*(G)l + \text{Ad}^*(G)l'$$

Let us show now that $\mathfrak{g}(l) + \mathfrak{g}(l') = \mathfrak{g}$.

We have for any polarization \mathfrak{b} at l that

$$\dim(\mathfrak{b}) = \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l))}{2}.$$

Hence, since $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$ and $\mathfrak{b} \cap \mathfrak{b}'$ is a polarization at $l + l'$ and since

$$\dim(\mathfrak{b}) + \dim(\mathfrak{b}') = \dim(\mathfrak{g}) + \dim(\mathfrak{b} \cap \mathfrak{b}')$$

we have that

$$\frac{2\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l)) + \dim(\mathfrak{g}(l'))}{2} = \dim(\mathfrak{g}) + \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l + l'))}{2}.$$

Therefore, since $\mathfrak{g}(l) \cap \mathfrak{g}(l') \subset \mathfrak{g}(l + l')$,

$$\begin{aligned} \dim(\mathfrak{g}(l) + \mathfrak{g}(l')) + \dim(\mathfrak{g}(l + l')) &\geq \dim(\mathfrak{g}(l) + \mathfrak{g}(l')) + \dim(\mathfrak{g}(l) \cap \mathfrak{g}(l')) \\ &= \dim(\mathfrak{g}(l)) + \dim(\mathfrak{g}(l')) \\ &= \dim(\mathfrak{g}(l + l')) + \dim(\mathfrak{g}), \end{aligned}$$

and so $\dim(\mathfrak{g}(l) + \mathfrak{g}(l')) \geq \dim(\mathfrak{g})$ which implies that $\mathfrak{g} = \mathfrak{g}(l) + \mathfrak{g}(l')$.

4) \implies 1)

Let \mathfrak{b} (resp. \mathfrak{b}') be a polarization at l (resp. at l'). Then $\mathfrak{g}(l) \subset \mathfrak{b}, \mathfrak{g}(l') \subset \mathfrak{b}'$ and hence $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$. Therefore BB' contains an open subset of G , whence it is itself open. Since the product of two closed connected subgroups of G is always closed, it follows that BB' is open and closed which implies that $BB' = G$. Therefore $\pi_{l,\mathfrak{b}} \otimes \pi_{l',\mathfrak{b}'} \simeq \text{ind}_{B \cap B'}^G \chi_{l+l'}$. If we can show that $\mathfrak{b} \cap \mathfrak{b}'$ is a polarization at $l + l'$, then we know that $\pi \otimes \pi'$ is irreducible. The condition $\text{Ad}^*(G)l + \text{Ad}^*(G)l' = \text{Ad}^*(G)(l + l')$ implies that we have an injective mapping

$$E : G/G(l) \times G/G(l') \ni (g \cdot G(l), g' \cdot G(l')) \mapsto \text{Ad}^*(g)l + \text{Ad}^*(g')l' \in \text{Ad}^*(G)(l + l').$$

Whence

$$\dim(\mathfrak{g}/\mathfrak{g}(l)) + \dim(\mathfrak{g}/\mathfrak{g}(l')) \leq \dim(\mathfrak{g}/\mathfrak{g}(l + l')).$$

Hence

$$\begin{aligned} 2\dim(\mathfrak{g}) - \dim(\mathfrak{g}(l)) - \dim(\mathfrak{g}(l')) &\leq \dim(\mathfrak{g}) - \dim(\mathfrak{g}(l + l')) \\ &\implies \\ \dim(\mathfrak{g}(l) + \mathfrak{g}(l')) - \dim(\mathfrak{g}(l)) - \dim(\mathfrak{g}(l')) &\leq -\dim(\mathfrak{g}) - \dim(\mathfrak{g}(l + l')) \\ &\implies \\ \dim(\mathfrak{g}(l + l')) &\leq \dim(\mathfrak{g}(l) \cap \mathfrak{g}(l')). \end{aligned}$$

Therefore $\mathfrak{g}(l) \cap \mathfrak{g}(l') = \mathfrak{g}(l + l')$, since $\mathfrak{g}(l) \cap \mathfrak{g}(l') \subset \mathfrak{g}(l + l')$.

This tells us then that

$$\begin{aligned}
\dim(\mathfrak{b} \cap \mathfrak{b}') &= \dim(\mathfrak{b}) + \dim(\mathfrak{b}') - \dim(\mathfrak{b} + \mathfrak{b}') \\
&= \dim(\mathfrak{b}) + \dim(\mathfrak{b}') - \dim(\mathfrak{g}) \\
&= \frac{2\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l)) + \dim(\mathfrak{g}(l'))}{2} - \dim(\mathfrak{g}) \\
&= \frac{\dim(\mathfrak{g}(l)) + \dim(\mathfrak{g}(l'))}{2} \\
&= \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l)) \cap \dim(\mathfrak{g}(l'))}{2} \\
&= \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l + l'))}{2}.
\end{aligned}$$

Hence $\mathfrak{b} \cap \mathfrak{b}'$ is a polarization at $l + l'$. □

Example 7.2. Let \mathfrak{g} be the Lie algebra spanned by the basis $\langle Z_1, \dots, Z_6 \rangle$ and equipped with the brackets

$$[Z_2, Z_3] = Z_5 \quad [Z_1, Z_4] = Z_6$$

Let $l = Z_5^*, l' = Z_6^* \in \mathfrak{g}^*$. Take the polarizations

$$\mathfrak{b} := \text{span} \langle Z_6, Z_5, Z_4, Z_3, Z_1 \rangle$$

at l and

$$\mathfrak{b}' := \text{span} \langle Z_6, Z_5, Z_4, Z_3, Z_2 \rangle$$

at l' .

Then $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$ and

$$\mathfrak{b} \cap \mathfrak{b}' = \text{span} \langle Z_6, Z_5, Z_4, Z_3 \rangle,$$

which is a polarization at $l + l'$. Hence $\pi_l \otimes \pi_{l'} \in \hat{G}$.

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