

On Fourier's inversion theorem in the context of nilpotent Lie groups

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Abstract

We generalize some aspects of the classical Fourier inversion theorem to the class of connected, simply connected, nilpotent Lie groups. In this setting, the generalized Fourier transform is the operator valued map $f \mapsto (\pi_l(f))_{l \in \mathfrak{g}^*/Ad^*G}$. These operators are characterized by operator kernels. We construct a retract to the generalized Fourier transform which maps into the Schwartz space $\mathcal{S}(G)$, by limiting ourselves to a suitable set of families of operator kernels. This is done via variable Lie structures. ¹

Introduction

This paper participates at the big effort which consists in trying to adapt and generalize results from harmonic analysis on \mathbb{R}^n to general locally compact groups. Among the groups for which we can hope for the best results are the connected, simply connected, nilpotent Lie groups. One of the most famous and useful results from harmonic analysis on \mathbb{R}^n is the Fourier inversion theorem. So it is quite natural to look for its counterpart for nilpotent Lie groups. This problem may be formulated in the following way:

Let $G = \exp \mathfrak{g}$ be a connected, simply connected, nilpotent Lie group. Let's consider the map that associates to each $f \in L^1(G)$ the operator field $(\pi_l(f))_{l \in \mathfrak{g}^*/Ad^*G}$ of the images of f under the unitary irreducible representations. This map may be considered as a generalization of the Fourier transform to the non abelian case. It has been studied and used by a certain number of authors, among others by Lipsman and Rosenberg ([Li-Ro]). The question of a Fourier inversion theorem means in this context, whether it is possible, under suitable hypotheses, to reconstruct the function f from its "Fourier transform" $(\pi_l(f))_{l \in \mathfrak{g}^*/Ad^*G}$. Some partial results are known. If $f \in \mathcal{S}(G)$, then the Plancherel formula gives

$$f(x) = \int_{\hat{G}} \text{tr}(\pi(x)^{-1} \circ \pi(f)) d\mu(\pi),$$

where $d\mu(\pi)$ is the Plancherel measure on \hat{G} . But what if we just have the operator field $(\pi_l(f))_{l \in \mathfrak{g}^*/Ad^*G}$ without knowing that $f \in \mathcal{S}(G)$? Or, more generally, if we just have an arbitrary operator field $(A(\pi))_{\pi \in \hat{G}}$? What assumptions are needed for this operator field to represent the Fourier transform of an L^1 -function, of a Schwartz function?

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A first, abstract result is the following: Let $(A(\pi))_{\pi \in \hat{G}}$ be an operator field such that

$$\begin{aligned} A(\pi) \text{ is a bounded operator on } \mathfrak{H}_\pi, \quad \forall \pi \in \hat{G} \\ \pi \mapsto \operatorname{tr}|A(\pi)| \text{ is measurable} \\ \int_{\hat{G}} \operatorname{tr}|A(\pi)| d\mu(\pi) < +\infty \\ \|A(\pi)\|_{op} \leq C, \quad \forall \pi \in \hat{G} \end{aligned}$$

for some constant $C > 0$, where $|A(\pi)| = \sqrt{A(\pi)^* A(\pi)}$. Then

$$\int_{\hat{G}} \|A(\pi)\|_{HS}^2 d\mu(\pi) \leq C \int_{\hat{G}} \operatorname{tr}|A(\pi)| d\mu(\pi) < +\infty$$

and, by the Plancherel theorem, there exists a unique $f \in L^2(G)$ such that $\pi(f) = A(\pi)$ for almost all π ($\pi(f)$ being taken in the L^2 sense). Moreover,

$$f(x) = \int_{\hat{G}} \operatorname{tr}(\pi(x)^{-1} \circ \pi(f)) d\mu(\pi) = \int_{\hat{G}} \operatorname{tr}(\pi(x)^{-1} \circ A(\pi)) d\mu(\pi)$$

for almost all $x \in G$. In fact, given the hypotheses on $(A(\pi))_{\pi \in \hat{G}}$, the function

$$g(x) := \int_{\hat{G}} \operatorname{tr}(\pi(x)^{-1} \circ A(\pi)) d\mu(\pi)$$

exists for all $x \in G$ and is continuous. Moreover, for any arbitrary $h \in \mathcal{C}_c(G)$, $\langle g, h \rangle = \langle f, h \rangle$, as can be checked easily. Hence $g = f$ almost everywhere. In particular, we may assume that f is continuous and $f = g$ everywhere.

Let now $f \in L^1(G)$ such that

$$\int_{\hat{G}} \operatorname{tr}|\pi(f)| d\mu(\pi) < +\infty.$$

As $\|\pi(f)\|_{op} \leq \|f\|_1$ for all π , the operator field $(\pi(f))_{\pi \in \hat{G}}$ satisfies the previous hypotheses. By the preceding arguments, the continuous function g given by $g(x) := \int_{\hat{G}} \operatorname{tr}(\pi(x)^{-1} \circ \pi(f)) d\mu(\pi)$ belongs to $L^2(G)$ and is such that $\pi(f) = \pi(g)$ for almost all $\pi \in \hat{G}$. Hence, for every $h \in \mathcal{C}_c(G)$, $f * h^* = g * h^*$ and $\langle f, h \rangle = \langle g, h \rangle$ by continuity of $f * h^*$ and $g * h^*$. This proves that $f = g$ almost everywhere and that $f \in L^1(G) \cap L^2(G)$. Examples for functions $f \in L^1(G)$ for which the previous result holds are for instance functions of the form $f = f_1^* * f_1$ with $f_1 \in L^1(G) \cap L^2(G)$, or, by the polarization identity, $f = f_1 * f_2$ with $f_1, f_2 \in L^1(G) \cap L^2(G)$.

But a lot of open questions remain, such as: Find a characterization of the image of the L^1 -Fourier transform $f \mapsto (\pi(f))_{\pi \in \hat{G}}$. Which operator fields in the image come from Schwartz functions? The latter question is particularly important, as Schwartz functions play a major role in analysis.

In this paper we shall give partial answers to this last question. In order to do this, let's recall that for $f \in \mathcal{S}(G)$, the operator $\pi_l(f)$ is completely characterized by its operator kernel

$$F(l, x, y) = \int_{P_l} f(xuy^{-1}) e^{-i\langle l, \log u \rangle} du,$$

where $P_l = \exp \mathfrak{p}_l$ for a polarization \mathfrak{p}_l of l , $\pi_l = \operatorname{ind}_{P_l}^G \chi_l$ and $\chi_l(u) = e^{-i\langle l, \log u \rangle}$. So the question of the image of the Fourier transform may be formulated like this: Given such a function $F(l, x, y)$ satisfying certain hypotheses, does there exist $f \in L^1(G)$ such that $\pi_l(f)$ admits $F(l, \cdot, \cdot)$ as a kernel for almost all l ? When is this function f a Schwartz function?

For one fixed $l_0 \in \mathfrak{g}^*$, we have the result of Howe ([How]): Let $\mathfrak{p}_0 = \mathfrak{p}(l_0)$ be a polarization for l_0 , let $P_0 = \exp \mathfrak{p}_0$, let χ_{l_0} be the character of P_0 defined by $\chi_{l_0}(h) = e^{-i\langle l_0, \log h \rangle}$ for all $h \in P_0$. Let $F \in \mathcal{S}((G/P_0)^2, \chi_{l_0})$, i. e. let's assume that F is a Schwartz function on $G/P_0 \times G/P_0$ which satisfies the covariance condition

$$F(xh, yh') = \overline{\chi_{l_0}(h)} \chi_{l_0}(h') F(x, y), \quad \forall x, y \in G, \forall h, h' \in P_0.$$

Then there exists $f \in \mathcal{S}(G)$, the Schwartz space of G , such that $\pi_{l_0}(f)$ has F as an operator kernel. This result has been generalized to exponential solvable Lie groups by Ludwig ([Lu1]) and Andele ([A]). But these results deal only with one fixed chosen $l_0 \in \mathfrak{g}^*$ and can therefore not be qualified as a Fourier inversion theorem, contrary to our main result, which is the following:

Theorem (4.2): *Let $G = \exp \mathfrak{g}$ be a connected, simply connected, nilpotent Lie group with a fixed Jordan-Hölder basis. For every $l \in \mathfrak{g}^*$, let \mathfrak{p}_l be the corresponding Vergne polarization and let $P_l = \exp \mathfrak{p}_l$. Then there exists a Zariski open subset \mathfrak{g}_{gen}^* of \mathfrak{g}^* (set of generic elements of \mathfrak{g}^*), such that for every \mathcal{C}^∞ function $F : \mathfrak{g}_{gen}^* \times G \times G \rightarrow \mathbb{C}$ which, for fixed $l \in \mathfrak{g}_{gen}^*$, is Schwartz on $G/P_l \times G/P_l$, with compact support in l (if we restrict to an orbit section) and which satisfies the covariance relation*

$$F(l, xh, yh') = \overline{\chi_l(h)} \chi_l(h') F(l, x, y),$$

there exists a unique Schwartz function $f \in \mathcal{S}(G)$ such that $\pi_l(f)$ has $F(l, \cdot, \cdot)$ as an operator kernel, for every $l \in \mathfrak{g}_{gen}^$. The map $F \mapsto f$ is continuous with respect to the appropriate function space topologies.*

This Fourier inversion theorem is then used to construct functions $f \in \mathcal{S}(G)$ whose generalized Fourier transform $(\pi_l(f))_{l \in \mathfrak{g}^*/Ad^*G}$ has certain required properties, as, for instance, a generalized Domar property ([D]). It is also useful to construct Schwartz functions which satisfy certain conditions as, for instance, $f * g = f$ for some $f, g \in \mathcal{S}(G)$, or, in the case of the Heisenberg group H_n , $f * (X_k + iY_k) = 0$, if $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ denote the generators of the corresponding Lie algebra ([Lu-M1]). Finally, it may be used in the construction of minimal ideals of a given hull. Let's insist on the fact that in these applications it is necessary to know that the function f is Schwartz. So an abstract result as the one presented in the beginning of the introduction is not sufficient.

The proof of the inverse Fourier transform is a proof by induction. In each induction step, new parameters appear and they are handled by the use of variable Lie structures. Such variable structures were already used in ([Le-Lu]) and ([Lu-Mu]). But for the purpose of the proof of the Fourier inversion theorem, a certain number of refinements of this theory are necessary. The corresponding results are developed in this paper.

1 Generalities and notations

1.1

Let $G = \exp \mathfrak{g}$ be a connected, simply connected, nilpotent Lie group. We denote by $\mathcal{S}(G)$ the Schwartz algebra of G (see [Lu-M]).

For any linear form $l \in \mathfrak{g}^*$ on \mathfrak{g} , we write $\mathfrak{g}(l)$ for the stabilizer of l in \mathfrak{g} defined by

$$\mathfrak{g}(l) = \{U \in \mathfrak{g} \mid \langle l, [U, \mathfrak{g}] \rangle \equiv 0\},$$

and $\mathfrak{p} = \mathfrak{p}_l$ for an arbitrary polarization of l in \mathfrak{g} , i. e. \mathfrak{p}_l is a subalgebra of \mathfrak{g} such that $\langle l, [\mathfrak{p}_l, \mathfrak{p}_l] \rangle \equiv 0$ and such that $\dim \mathfrak{p}_l = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}(l))$ (which means that $\dim \mathfrak{p}_l$ is

maximal among the dimensions of all the subalgebras \mathfrak{p} satisfying $\langle l, [\mathfrak{p}, \mathfrak{p}] \rangle \equiv 0$). We write $P = P_l = \exp \mathfrak{p}_l$ for the corresponding subgroup and let χ_l be the character defined on P_l by

$$\chi_l(u) = e^{-i\langle l, \log u \rangle}, \quad \forall u \in P_l.$$

Let π_l denote the induced representation $\pi_l = \text{ind}_{P_l}^G \chi_l$. Then two different polarizations \mathfrak{p}_1 and \mathfrak{p}_2 of l define unitary equivalent representations $\text{ind}_{P_1}^G \chi_l$ and $\text{ind}_{P_2}^G \chi_l$, a fact which justifies the notation π_l (see [C-G] for more details).

In this paper we shall use the following notation: If X_1, \dots, X_r are any vectors of \mathfrak{g} , $\langle X_1, \dots, X_r \rangle$ will denote the subspace generated by X_1, \dots, X_r . Let's now assume that \mathfrak{g} is equipped with a fixed Jordan-Hölder basis Z_1, \dots, Z_n , i. e. such that $[Z_i, Z_j] \subset \langle Z_{r+1}, \dots, Z_n \rangle$ where $r = \max(i, j)$, for all $i, j \in \{1, \dots, n\}$. Let's denote $\mathfrak{g}_k = \langle Z_k, \dots, Z_n \rangle$ and $l_k = l|_{\mathfrak{g}_k}$ for all $l \in \mathfrak{g}^*$. For any $l \in \mathfrak{g}^*$, the Vergne polarization \mathfrak{p}_l of l with respect to the given basis is defined by

$$\mathfrak{p}_l := \sum_{k=1}^n \mathfrak{g}_k(l_k)$$

where

$$\mathfrak{g}_k(l_k) := \{U \in \mathfrak{g}_k \mid \langle l, [U, Z_j] \rangle = 0, \text{ for } j = k, \dots, n\}.$$

If not otherwise stated, the polarizations used in this text will always be Vergne polarizations. Let's also recall that G acts on \mathfrak{g} and \mathfrak{g}^* by Ad , respectively Ad^* . Let $\Omega_l = (\text{Ad}^*G)(l) = G \cdot l$ be the orbit of l under the action of G on \mathfrak{g}^* . One knows that π_{l_1} and π_{l_2} are unitary equivalent if and only if the corresponding orbits Ω_{l_1} and Ω_{l_2} coincide. Let $\mathfrak{g}^*/\text{Ad}^*G = \mathfrak{g}^*/G$ denote the orbit space with the quotient topology and let \hat{G} denote the set of equivalence classes of unitary topologically irreducible representations of G . If \hat{G} is endowed with an appropriate topology (Fell topology), then $\mathfrak{g}^*/\text{Ad}^*G$ and \hat{G} are homeomorphic under the map $\Omega_l \mapsto [\pi_l]$ where $[\pi_l]$ denotes the equivalence class of π_l (see [B-al], [C-G]).

1.2

If π_l is a unitary irreducible representation of G , we may define the corresponding representation of $L^1(G)$, also denoted by π_l , by the formula

$$\pi_l(f)\xi := \int_G f(x)(\pi_l(x)\xi)dx$$

for all $\xi \in \mathfrak{H}_{\pi_l}$ (the representation space of π_l). If $f \in \mathcal{S}(G)$, then $\pi_l(f)$ is given by

$$(\pi_l(f)\xi)(g) = \int_{G/P_l} f_{\pi_l}(g, u)\xi(u)du,$$

where the operator kernel f_{π_l} is obtained by the formula

$$F(l, g, u) := f_{\pi_l}(g, u) = \int_{P_l} f(ghu^{-1})\chi_l(h)dh, \quad \forall g, u \in G.$$

The function f_{π_l} is \mathcal{C}^∞ , satisfies the covariance relation

$$f_{\pi_l}(gh, g'h') = \overline{\chi_l(h)}\chi_l(h')f_{\pi_l}(g, g'), \quad \forall h, h' \in P_l, \forall g, g' \in G,$$

and is a Schwartz function on $G/P_l \times G/P_l$ (i. e. is a Schwartz function in the coordinates if we work with a fixed basis of $\mathfrak{g}/\mathfrak{p}_l$). If f is given, let's write $F(l, g, g')$ for $f_{\pi_l}(g, g')$.

1.3

The map $f \mapsto (\pi_l(f))_{l \in \mathfrak{g}^*/Ad^*G}$, or equivalently $f \mapsto F$, where $F(l, \cdot, \cdot) = f_{\pi_l}(\cdot, \cdot)$ is the operator kernel of $\pi_l(f)$, may be considered as a generalized Fourier transform, because, in the abelian case, it coincides with the usual Fourier transform. This then raises the question of a Fourier inversion theorem as in the abelian case.

1.4

Let's now assume that $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_1$, where \mathfrak{g}_1 is an ideal of codimension one in \mathfrak{g} . Let's write $\tilde{l} = l|_{\mathfrak{g}_1}$. Let's assume that $\mathfrak{g}(l) \subset \mathfrak{g}_1$. In that case, any polarization \mathfrak{p} for \tilde{l} in \mathfrak{g}_1 is also a polarization for l in \mathfrak{g} . Let's write $P = \exp \mathfrak{p}$, $\pi_l := \text{ind}_P^G \chi_l$ and $\pi_{\tilde{l}} := \text{ind}_P^{G_1} \chi_{\tilde{l}}$ for the corresponding induced representations of G and $G_1 = \exp \mathfrak{g}_1$ respectively. Then π_l is equivalent to $\text{ind}_{G_1}^G \pi_{\tilde{l}}$, which may be expressed as follows: Let's write $\xi(t, g_1) = \xi(t)(g_1) := \xi(\exp(tX)g_1)$ for $\xi \in \mathfrak{H}_{\pi_l}$ (the Hilbert space on which π_l acts). Then, for almost all t , $\xi(t, \cdot) = \xi(t)(\cdot) \in \mathfrak{H}_{\pi_{\tilde{l}}}$ (the Hilbert space on which $\pi_{\tilde{l}}$ acts). The map $\xi \in \mathfrak{H}_{\pi_l} \mapsto \xi(\cdot)(\cdot) \in L^2(\mathbb{R}, \mathfrak{H}_{\pi_{\tilde{l}}})$ (endowed with the correct covariance relation) realizes the unitary equivalence between π_l and $\text{ind}_{G_1}^G \pi_{\tilde{l}}$. For $f \in \mathcal{S}(G)$, let's also define $f(s)(\cdot) = f(s, \cdot) \in \mathcal{S}(G_1)$ by $f(s)(g_1) = f(s, g_1) := f(\exp(sX)g_1)$. We have

$$\begin{aligned}
(\pi_l(f)\xi)(s)(u) &= \int_G f(g) (\pi_l(g)\xi)(s)(u) dg \\
&= \int_G f(\exp(sX)g^{-1}) \xi(gu) dg \\
&= \int_{\mathbb{R}} \left[\int_{G_1} f(\exp((s-t)X) [\exp(tX)g_1 \exp(-tX)]^{-1}) \xi(\exp(tX)g_1 u) dg_1 \right] dt \\
&= \int_{\mathbb{R}} \left[\int_{G_1} f(s-t, \exp(tX)g_1 \exp(-tX)) \xi(t)(g_1^{-1}u) dg_1 \right] dt \\
&= \int_{\mathbb{R}} \left[\int_{G_1} {}^t(f(s-t))(g_1) \xi(t)(g_1^{-1}u) dg_1 \right] dt \\
&= \int_{\mathbb{R}} \pi_{\tilde{l}}({}^t(f(s-t))) \xi(t)(u) dt
\end{aligned}$$

for all $u \in G_1$, where ${}^t\varphi(v)$ is defined by ${}^t\varphi(v) = \varphi(\exp(tX)v \exp(-tX))$. So

$$(\pi_l(f)\xi)(s) = \int_{\mathbb{R}} \pi_{\tilde{l}}({}^t(f(s-t))) \xi(t) dt,$$

i. e. $\pi_l(f)$ may be regarded as the operator defined by the operator kernel $\pi_{\tilde{l}}({}^t(f(s-t)))$ acting on $L^2(\mathbb{R}, \mathfrak{H}_{\pi_{\tilde{l}}})$ (endowed with the appropriate covariance relation), as the map $t \mapsto \xi(t)$ is in $L^2(\mathbb{R}, \mathfrak{H}_{\pi_{\tilde{l}}})$.

In particular, we have the following relation for the Hilbert-Schmidt norms:

$$\|\pi_l(f)\|_{HS}^2 = \int_{\mathbb{R}^2} \|\pi_{\tilde{l}}({}^t f(s-t))\|_{HS}^2 ds dt$$

1.5

We shall also need the following result: Let \mathfrak{w} be a closed subspace of the center $\mathfrak{z}(\mathfrak{g})$ of \mathfrak{g} and let $W = \exp \mathfrak{w}$. Let $\lambda \in \mathfrak{w}^*$. Then χ_λ given by $\chi_\lambda(w) = e^{-i\langle \lambda, \log w \rangle}$ for all $w \in W$, defines a unitary character of W . Let $L^1(G)_\lambda$ denote the set of all measurable functions f on G such that

$$f(xw) = \overline{\chi_\lambda(w)} f(x) = e^{i\langle \lambda, \log w \rangle} f(x), \quad \forall x \in G, \forall w \in W$$

and such that $|f| \in L^1(G/W)$. The map

$$\begin{aligned} L^1(G) &\rightarrow L^1(G)_\lambda \\ f &\mapsto f_\lambda \end{aligned}$$

defined by

$$f_\lambda(x) = \int_W f(xw) \chi_\lambda(w) dw = \int_W f(xw) e^{-i\langle \lambda, \log w \rangle} dw$$

is a surjection.

Let $(\widehat{L^1(G)_\lambda})$ be the set of topologically irreducible $*$ -representations of $L^1(G)_\lambda$ and

$$\hat{G}_\lambda = \{\pi_l \in \hat{G} \mid l|_{\mathfrak{w}} \equiv \lambda\}.$$

Then $\hat{G}_\lambda \equiv (\widehat{L^1(G)_\lambda})$, the identification map being given by

$$\begin{aligned} \hat{G}_\lambda &\rightarrow (\widehat{L^1(G)_\lambda}) \\ \pi_l &\mapsto \tilde{\pi}_l \end{aligned}$$

defined by

$$\tilde{\pi}_l(f) = \int_{G/W} f(g) \pi_l(g) dg.$$

This map is well defined and realizes a bijection between \hat{G}_λ and $(\widehat{L^1(G)_\lambda})$. In particular, if $f \in L^1(G)_\lambda$ such that $\tilde{\pi}_l(f) = 0$ for all $\pi_l \in \hat{G}_\lambda$, then $f = 0$ almost everywhere (see [C-G]).

2 Variable Lie structures

2.1

In our proofs by induction new parameters and new variations will appear in the formulation of the problems. This will be handled most easily by the concept of variable Lie structures. Such structures were already considered by Ludwig and Müller ([Lu-Mu]), Leptin and Ludwig ([Le-Lu]) among others and we shall heavily rely on their constructions and results. Some of the constructions we use appear already in the work of Ludwig and Zahir ([Lu-Z]). They will be recalled and further developed in the next paragraphs. We will among others give a new, useful characterization of the different indices that appear in the construction. The coexponential bases to the Vergne polarization will then be constructed by an easy algorithm.

2.2 Definition

a) Let \mathfrak{g} be a real vector space of finite dimension n and \mathcal{B} an arbitrary set. If \mathcal{B} is non empty, we say that $(\mathfrak{g}, \mathcal{B})$ is a *variable Lie algebra* if:

(i) For every $b \in \mathcal{B}$, there exists a Lie bracket $[\cdot, \cdot]_b$ defined on \mathfrak{g} such that $(\mathfrak{g}, [\cdot, \cdot]_b)$ is a nilpotent Lie algebra.

(ii) There exists a fixed basis $\{Z_1, \dots, Z_n\}$ of \mathfrak{g} such that the structure constants $a_{ij}^k(b)$ defined by

$$[Z_i, Z_j]_b = \sum_{k=1}^n a_{ij}^k(b) Z_k$$

satisfy the following property:

For all $b \in \mathcal{B}$, for $k \leq \max(i, j)$, $a_{ij}^k(b) = 0$. This means that $\{Z_1, \dots, Z_n\}$ is a Jordan-Hölder basis for every $(\mathfrak{g}, [\cdot, \cdot]_b)$.

We shall write \mathfrak{g}_b for the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_b)$.

If $\mathcal{B} = \emptyset$, we assume that \mathfrak{g} is endowed with a fixed Lie bracket $[\cdot, \cdot]$ such that $(\mathfrak{g}, [\cdot, \cdot])$ is a nilpotent Lie algebra. In this case the structure constants a_{ij}^k are really constants.

b) If \mathcal{B} is a non empty open subset of a real finite dimensional vector space, we say that $(\mathfrak{g}, \mathcal{B})$ is a *polynomially variable Lie algebra* provided the structure constants $a_{ij}^k(b)$ are restrictions of polynomial functions in b to \mathcal{B} . Similarly, we say that $(\mathfrak{g}, \mathcal{B})$ is a *rationally variable Lie algebra* provided the structure constants $a_{ij}^k(b)$ are restrictions of rational functions in b to \mathcal{B} . In that case there exists a Zariski open dense subset \mathcal{B}_1 of \mathcal{B} such that the a_{ij}^k 's have no singularities in \mathcal{B}_1 , i. e. $(\mathfrak{g}, \mathcal{B}_1)$ is a rationally variable Lie algebra without singularities. For the rest of this paper we will assume that all the rationally variable Lie algebras are without singularities.

c) In ([Lu-Mu]) the authors assume that \mathcal{B} is an algebraic subset of a real finite dimensional vector space. But for our purposes, we need \mathcal{B} to be open.

d) We then define the associated *variable Lie group* (G, \mathcal{B}) to be the family of connected, simply connected, nilpotent Lie groups (G, \cdot_b) , where $G = \mathfrak{g}$ as a set and where \cdot_b is the group product on \mathfrak{g} constructed by the Campbell-Baker-Hausdorff formula for $[\cdot, \cdot]_b$. We shall write G_b for the Lie group (G, \cdot_b) . With these choices the exponential map $\exp_b : \mathfrak{g}_b \rightarrow G_b$ is of course the identity mapping.

2.3

In this section we shall essentially recall the constructions of Ludwig and Müller ([Lu-Mu]). First we construct several families of indices.

For every $(b, l) \in \mathcal{B} \times \mathfrak{g}^*$, let $\mathfrak{a}(b, l)$ be the largest ideal of \mathfrak{g}_b contained in the stabilizer $\mathfrak{g}_b(l)$ of l in \mathfrak{g}_b . If $\mathfrak{a}(b, l) = \mathfrak{g}_b$, the corresponding set of indices will be empty. In that case, $\mathfrak{g}_b(l) = \mathfrak{g}_b$ and l defines a character on all of G_b . The linear form l will not be generic (see 2.8 for the definition of generic), except if \mathfrak{g}_b and G_b are abelian. Otherwise we define indices $j_1(b, l)$ and $k_1(b, l)$ by

$$\begin{aligned} j_1(b, l) &:= \max\{j \in \{1, \dots, n\} \mid Z_j \notin \mathfrak{a}(b, l)\} \\ k_1(b, l) &:= \max\{k \in \{1, \dots, n\} \mid \langle l, [Z_{j_1(b, l)}, Z_k]_b \rangle \neq 0\}. \end{aligned}$$

By construction, $k_1(b, l) < j_1(b, l)$. Indices $j_2(b, l), \dots, j_d(b, l)$ and $k_2(b, l), \dots, k_d(b, l)$ will be constructed later in a similar way.

Let's also define indices which are independent of l : For all $b \in \mathcal{B}$, we put

$$\begin{aligned} j_1(b) &:= \max\{j \mid Z_j \notin \text{center}(\mathfrak{g}_b)\} \\ k_1(b) &:= \max\{k \mid [Z_{j_1(b)}, Z_k]_b \neq 0\} \end{aligned}$$

and

$$\begin{aligned} j_1 &:= \max\{j_1(b) \mid b \in \mathcal{B}\} \\ k_1 &:= \max\{k_1(b) \mid b \in \mathcal{B} \text{ and } j_1(b) = j_1\}. \end{aligned}$$

Again, $k_1(b) < j_1(b)$ and $k_1 < j_1$. Indices $j_2(b), \dots, j_d(b); k_2(b), \dots, k_d(b); j_2, \dots, j_d; k_2, \dots, k_d$ will be constructed in a similar way.

Let's make a first restriction for the (b, l) 's. We put $\mathcal{D}_0 := \mathcal{B} \times \mathfrak{g}^*$ and

$$\mathcal{D}_1 := \{(b, l) \in \mathcal{D}_0 \mid j_1(b, l) = j_1 \text{ and } k_1(b, l) = k_1\}.$$

If \mathcal{B} is an open subset of a real finite dimensional vector space, \mathcal{D}_1 is a (dense) Zariski open subset of \mathcal{D}_0 as

$$\mathcal{D}_1 = \{(b, l) \in \mathcal{D}_0 \mid \langle l, [Z_{j_1}, Z_{k_1}]_b \rangle \neq 0\}.$$

Let's put $I_0 := \{1, \dots, n\}$ and $I_1 := I_0 \setminus \{k_1\}$. For $(b, l) \in \mathcal{D}_1$, we define

$$\mathfrak{p}_1(b, l) := \{X \in \mathfrak{g} \mid \langle l, [Z_{j_1(b, l)}, X]_b \rangle = 0\} = \{X \in \mathfrak{g} \mid \langle l, [Z_{j_1}, X]_b \rangle = 0\}$$

and

$$Z_i^1(b, l) := Z_i - \frac{\langle l, [Z_{j_1(b, l)}, Z_i]_b \rangle}{\langle l, [Z_{j_1(b, l)}, Z_{k_1(b, l)}]_b \rangle} Z_{k_1(b, l)} = Z_i - \frac{\langle l, [Z_{j_1}, Z_i]_b \rangle}{\langle l, [Z_{j_1}, Z_{k_1}]_b \rangle} Z_{k_1}, \quad \forall i \in I_1$$

(as $(b, l) \in \mathcal{D}_1$). It is easy to see that $\mathfrak{p}_1(b, l)$ is an ideal of codimension 1 in \mathfrak{g}_b and that $\{Z_i^1(b, l) \mid i \in I_1\}$ is a Jordan-Hölder basis for $\mathfrak{p}_1(b, l)$ (see [Lu-Z]). Moreover, $Z_i^1(b, l) = Z_i$ if $i > k_1$.

Step by step we construct the indices $j_i(b, l), k_i(b, l), j_i(b), k_i(b), j_i, k_i$ in the following way:

We use the algebra $\mathfrak{p}_1(b, l)$, the basis $Z_i^1(b, l), l|_{\mathfrak{p}_1(b, l)}$ to construct the indices $j_2(b, l), k_2(b, l)$, the algebra $\mathfrak{p}_2(b, l)$ (which is an ideal of codimension one in $\mathfrak{p}_1(b, l)$), the basis $Z_i^2(b, l)$ of $\mathfrak{p}_2(b, l)$ and continue in that manner until the process stops after a finite number of steps.

Recursively we also define $\mathcal{D}_i, j_i(b), k_i(b), j_i, k_i, I_i$ by

$$\begin{aligned} j_i(b) &:= \max\{j \in I_{i-1} \mid Z_j^{i-1}(b, l) \notin \text{center}(\mathfrak{p}_{i-1}(b, l)), \forall l \text{ s. t. } (b, l) \in \mathcal{D}_{i-1}\} \\ k_i(b) &:= \max\{k \in I_{i-1} \mid [Z_{j_i^{i-1}}(b, l), Z_k^{i-1}(b, l)]_b \neq 0, \forall l \text{ s. t. } (b, l) \in \mathcal{D}_{i-1}\} \\ j_i &:= \max\{j_i(b) \mid b \in \mathcal{B}\} \\ k_i &:= \max\{k_i(b) \mid b \in \mathcal{B} \text{ and } j_i(b) = j_i\} \\ \mathcal{D}_i &:= \{(b, l) \in \mathcal{D}_{i-1} \mid j_i(b, l) = j_i \text{ and } k_i(b, l) = k_i\} \\ I_i &:= I_{i-1} \setminus \{k_i\} \end{aligned}$$

This process stops after a finite number d of steps. Then it is known by ([Lu-Z]) that for any $(b, l) \in \mathcal{D}_d$, $\mathfrak{p}_d(b, l)$ is the Vergne polarization of l in \mathfrak{g}_b with respect to the basis $\{Z_1, \dots, Z_n\}$.

2.4 Pukanszky jump indices

Let's note by $\mathcal{S}(b, l)$ the set of Pukanszky jump indices for l in \mathfrak{g}_b . We have

$$j \in \mathcal{S}(b, l) \Leftrightarrow \mathfrak{g}_b(l) + \langle Z_{j+1}, \dots, Z_n \rangle \subsetneq \mathfrak{g}_b(l) + \langle Z_j, \dots, Z_n \rangle.$$

The number of Pukanszky jump indices for l in \mathfrak{g}_b is equal to the dimension of the coadjoint orbit of l in \mathfrak{g}_b^* . The elements of \mathfrak{g}_b^* in general position in the sense of Pukanszky all have the same set of Pukanszky jump indices noted by \mathcal{S}_b . See ([C-G], [Pu], [Pe], [Pe1]) for more details on the parametrization of co-adjoint orbits. The relation with the newly defined indices is given by the following proposition:

Proposition 2.5. *For every $b \in \mathcal{B}$ and $l \in \mathfrak{g}_b^*$, the set of Pukanszky jump indices equals the set of indices $j_i(b, l), k_i(b, l)$, i. e.*

$$\mathcal{S}(b, l) = \{j_i(b, l), k_i(b, l) \mid 1 \leq i \leq d\}.$$

Proof. (i) For every $i \in \{1, \dots, d\}$, let's note $\tilde{\mathfrak{p}}_i(b, l)$ for the stabilizer of $l|_{\mathfrak{p}_i(b, l)}$, i. e.

$$\tilde{\mathfrak{p}}_i(b, l) := \{X \in \mathfrak{p}_i(b, l) \mid \langle l, [X, \mathfrak{p}_i(b, l)] \rangle \equiv 0\}.$$

For $i = 0$, we put $\tilde{\mathfrak{p}}_0(b, l) := \mathfrak{g}_b(l)$. It is then easy to check that

$$\mathfrak{g}_b(l) = \tilde{\mathfrak{p}}_0(b, l) \subset \tilde{\mathfrak{p}}_1(b, l) \subset \dots \subset \tilde{\mathfrak{p}}_d(b, l).$$

(ii) Let's put $J_0 := \emptyset$ and $J_r := \{k_1, k_2, \dots, k_r\}$. We then have for all $r \in \{1, \dots, d\}$,

$$\begin{aligned} Z_{j_r}^{r-1}(b, l) &\notin \tilde{\mathfrak{p}}_{r-1}(b, l) + \langle Z_j^{r-1}(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_{r-1} \rangle \\ Z_{k_r}^{r-1}(b, l) &\notin \tilde{\mathfrak{p}}_{r-1}(b, l) + \langle Z_k^{r-1}(b, l) \mid k_r + 1 \leq k \leq n, k \notin J_{r-1} \rangle \end{aligned}$$

In fact, let's assume that

$$Z_{j_r}^{r-1}(b, l) = U + \sum_{j=j_r+1, j \notin J_{r-1}}^n c_j Z_j^{r-1}(b, l) \quad \text{with } U \in \tilde{\mathfrak{p}}_{r-1}(b, l).$$

Then

$$\begin{aligned} \langle l, [Z_{j_r}^{r-1}(b, l), Z_{k_r}^{r-1}(b, l)] \rangle &= \langle l, [U, Z_{k_r}^{r-1}(b, l)] \rangle + \sum_{j=j_r+1, j \notin J_{r-1}}^n c_j \langle l, [Z_j^{r-1}(b, l), Z_{k_r}^{r-1}(b, l)] \rangle \\ &= 0 \end{aligned}$$

as $U \in \tilde{\mathfrak{p}}_{r-1}(b, l)$, as well as $Z_j^{r-1}(b, l)$ if $j > j_r$. This is a contradiction and hence

$$Z_{j_r}^{r-1}(b, l) \notin \tilde{\mathfrak{p}}_{r-1}(b, l) + \langle Z_j^{r-1}(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_{r-1} \rangle.$$

Similarly for $Z_{k_r}^{r-1}(b, l)$.

(iii) We have

$$\begin{aligned} Z_{j_r}^s(b, l) &\notin \tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle \\ Z_{k_r}^s(b, l) &\notin \tilde{\mathfrak{p}}_s(b, l) + \langle Z_k^s(b, l) \mid k_r + 1 \leq k \leq n, k \notin J_s \rangle \end{aligned}$$

for all $r \in \{1, \dots, d\}$ and all $s < r$.

For $s = r - 1$, the result is true by (ii).

Let's assume that

$$Z_{j_r}^s(b, l) \notin \tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle$$

and

$$Z_{j_r}^{s-1}(b, l) \in \tilde{\mathfrak{p}}_{s-1}(b, l) + \langle Z_j^{s-1}(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_{s-1} \rangle.$$

As $\tilde{\mathfrak{p}}_{s-1}(b, l) \subset \tilde{\mathfrak{p}}_s(b, l)$ and as $Z_i^s(b, l) = Z_i^{s-1}(b, l) + c_{s,i} Z_{k_s}^{s-1}(b, l)$ for some constant $c_{s,i}$,

$$\begin{aligned} Z_{j_r}^{s-1}(b, l) &\in \tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^{s-1}(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle + \mathbb{R} Z_{k_s}^{s-1}(b, l) \\ &= \tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle + \mathbb{R} Z_{k_s}^{s-1}(b, l) \end{aligned}$$

and, by construction,

$$Z_{j_r}^s(b, l) \in (\tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle) \oplus \mathbb{R} Z_{k_s}^{s-1}(b, l).$$

The direct sum is justified by the fact that

$$\tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle \subset \mathfrak{p}_s(b, l)$$

and that $\mathfrak{p}_{s-1}(b, l) = \mathfrak{p}_s(b, l) \oplus \mathbb{R} Z_{k_s}^{s-1}(b, l)$. The direct sum implies that

$$Z_{j_r}^s(b, l) \in \tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle,$$

which is a contradiction. Hence

$$Z_{j_r}^s(b, l) \notin \tilde{\mathfrak{p}}_s(b, l) + \langle Z_j^s(b, l) \mid j_r + 1 \leq j \leq n, j \notin J_s \rangle$$

for all $r \in \{1, \dots, d\}$ and all $s < r$. Similarly for $Z_{k_r}^s(b, l)$.

(iv) For $s = 0$ we get

$$Z_{j_r}(b, l) \notin \mathfrak{g}_b(l) + \langle Z_{j_r+1}(b, l), \dots, Z_n(b, l) \rangle.$$

Similarly for $Z_{k_r}(b, l)$. So

$$\{j_i(b, l), k_i(b, l) \mid 1 \leq i \leq d\} \subset \mathcal{S}(b, l).$$

(v) As $\mathfrak{p}_d(b, l)$ is the Vergne polarization for l in \mathfrak{g}_b associated to the Jordan-Hölder basis Z_1, \dots, Z_n and as

$$\mathfrak{g}_b = \mathbb{R} Z_{k_1}(b, l) \oplus \dots \oplus \mathbb{R} Z_{k_d}(b, l) \oplus \mathfrak{p}_d(b, l),$$

$n - d = \dim \mathfrak{p}_d(b, l)$ and $2d$ is the dimension of the orbit of l in \mathfrak{g}_b^* , i. e. the number of Pukanszky jump indices. This proves that

$$\mathcal{S}(b, l) = \{j_i(b, l), k_i(b, l) \mid 1 \leq i \leq d\}.$$

□

2.6 Local jump indices

For any $k \in \{1, \dots, n\}$, let's put $\mathfrak{g}_{b,k}$ for the subset generated by Z_k, \dots, Z_n in \mathfrak{g}_b and $l_k = l|_{\mathfrak{g}_{b,k}}$. We say that k is a local jump index for (b, l) , if

$$\mathfrak{g}_{b,k+1} + \mathfrak{g}_{b,k}(l_k) \subsetneq \mathfrak{g}_{b,k} + \mathfrak{g}_{b,k}(l_k) = \mathfrak{g}_{b,k}$$

where

$$\mathfrak{g}_{b,k}(l_k) = \{X \in \mathfrak{g}_{b,k} \mid \langle l, [X, \mathfrak{g}_{b,k}]_b \rangle \equiv 0\}.$$

This means that k is a Pukanszky jump index for l_k in $\mathfrak{g}_{b,k}$. The local jump indices are characterized by the following proposition:

Proposition 2.7. *For every $b \in \mathcal{B}$ and every $l \in \mathfrak{g}_b^*$, the set $\mathcal{R}(b, l)$ of local jump indices equals the set of the indices $k_i(b, l)$ constructed in (2.3), i. e.*

$$\mathcal{R}(b, l) = \{k_i(b, l) \mid 1 \leq i \leq d\}.$$

Proof. The proof is done in several steps.

(i) Every local jump index is a Pukanszky jump index. In fact, assume k is a local jump index for (b, l) , i. e.

$$\mathfrak{g}_{b,k+1} + \mathfrak{g}_{b,k}(l_k) \subsetneq \mathfrak{g}_{b,k} + \mathfrak{g}_{b,k}(l_k) = \mathfrak{g}_{b,k}.$$

Then $Z_k \notin \mathfrak{g}_{b,k+1} + \mathfrak{g}_{b,k}(l_k)$ and, for all $U \in \mathfrak{g}_{b,k+1}$, $Z_k + U \notin \mathfrak{g}_{b,k}(l_k)$. Given such a U , there exists $V = V(U) \in \mathfrak{g}_{b,k}$ such that $\langle l, [Z_k + U, V]_b \rangle = \langle l_k, [Z_k + U, V]_b \rangle \neq 0$, which means that $Z_k + U \notin \mathfrak{g}_b(l)$, for $U \in \mathfrak{g}_{b,k+1}$ arbitrary. Hence $Z_k \notin \mathfrak{g}_{b,k+1} + \mathfrak{g}_b(l)$ and k is a Pukanszky jump index.

(ii) Let $\mathfrak{p}_b(l) = \sum_{k=0}^n \mathfrak{g}_{b,k}(l_k)$ be the Vergne polarization of l associated to the Jordan-Hölder basis $\{Z_1, \dots, Z_n\}$. Let $\mathcal{R} = \{i_1, \dots, i_s\}$ be the set of local jump indices for (b, l) . Then $\{Z_{i_1}, \dots, Z_{i_s}\}$ is a complementary basis to $\mathfrak{p}_b(l)$ in \mathfrak{g}_b . In fact, if k is a local jump index, $\mathfrak{g}_{b,k+1} + \mathfrak{g}_{b,k}(l_k) \subsetneq \mathfrak{g}_{b,k}$. In that case, $\mathfrak{g}_{b,k}(l_k) \subset \mathfrak{g}_{b,k+1}$ and hence $\mathfrak{g}_{b,k}(l_k) \subset \mathfrak{g}_{b,k+1}(l_{k+1})$. If we write $\mathfrak{p}_{b,k+1}(l_{k+1})$ and $\mathfrak{p}_{b,k}(l_k)$ for the corresponding Vergne polarizations in \mathfrak{g}_{k+1} and \mathfrak{g}_k respectively, we hence have $\mathfrak{p}_{b,k+1}(l_{k+1}) = \mathfrak{p}_{b,k}(l_k)$ and we may add Z_k to a complementary basis to $\mathfrak{p}_{b,k+1}(l_{k+1})$ in $\mathfrak{g}_{b,k+1}$ to get a complementary basis to $\mathfrak{p}_{b,k}(l_k)$ in $\mathfrak{g}_{b,k}$. Moreover, it is easy to see that $Z_k \notin \mathfrak{p}_{b,r}(l_r) = \sum_{j=r}^n \mathfrak{g}_{b,j}(l_j)$ for all $r \leq k$. In fact, by assumption,

$$Z_k \notin \mathfrak{p}_{b,k}(l_k) = \mathfrak{g}_{b,k}(l_k) + \sum_{j=k+1}^n \mathfrak{g}_{b,j}(l_j) \subset \mathfrak{g}_{b,k+1} + \mathfrak{g}_{b,k}(l_k).$$

Hence $\langle l_k, [Z_k, \mathfrak{p}_{b,k}(l_k)]_b \rangle \neq 0$ as otherwise $\mathfrak{p}_{b,k}(l_k) + \mathbb{R}Z_k$ would also be a polarization for l_k in $\mathfrak{g}_{b,k}$, which is impossible, by the maximality of the polarization $\mathfrak{p}_{b,k}$. So there exists $Y_k \in \mathfrak{p}_{b,k}(l_k) \subset \mathfrak{p}_{b,r}(l_r)$ such that $\langle l, [Z_k, Y_k]_b \rangle \neq 0$ and $Z_k \notin \mathfrak{p}_{b,r}(l_r)$, for any $r \leq k$. On the other hand, if k is not a local jump index, $Z_k \in \mathfrak{g}_{b,k+1} + \mathfrak{g}_{b,k}(l_k)$, $\dim \mathfrak{p}_{b,k}(l_k) = \dim \mathfrak{p}_{b,k+1}(l_{k+1}) + 1$ (by [C-G]) and no vector has to be added to the complementary basis. This shows that $\{Z_{i_1}, \dots, Z_{i_s}\}$ is a complementary basis to $\mathfrak{p}_b(l)$ in \mathfrak{g}_b .

(iii) The point (ii) of the proof shows that the number of local jump indices for (b, l) is equal to $\dim(\mathfrak{g}_b/\mathfrak{p}_b(l)) = d$, i. e. is equal to half the dimension of the coadjoint orbit. But the dimension of the coadjoint orbit is equal to the number of Pukanszky jump indices. So, by (i), half of the Pukanszky jump indices are local jump indices.

(iv) The $k_i(b, l)$'s are local jump indices for l in \mathfrak{g}_b . In fact, let's first notice that by the successive construction steps, we have for fixed i ,

$$Z_{k_i(b,l)}^{i-1} = Z_{k_i(b,l)} + \sum_{1 \leq r \leq i-1; k_i(b,l) < k_r(b,l)} c_r(b,l) Z_{k_r(b,l)}$$

and

$$Z_{j_i(b,l)}^{i-1} = Z_{j_i(b,l)} + \sum_{1 \leq r \leq i-1; k_i(b,l) < k_r(b,l)} d_r(b,l) Z_{k_r(b,l)}$$

for certain constants $c_r(b,l)$ and $d_r(b,l)$, with $d_r(b,l) = 0$ if $k_r(b,l) < j_i(b,l)$. Let's assume that $k_i(b,l)$ is not a local jump index, i. e. that

$$Z_{k_i(b,l)} \in \mathfrak{g}_{k_i(b,l)+1} + \mathfrak{g}_{k_i(b,l)}(l_{k_i(b,l)}).$$

As

$$\{Z_{k_r(b,l)} \mid 1 \leq r \leq i-1; k_i(b,l) < k_r(b,l)\} \subset \mathfrak{g}_{k_i(b,l)+1},$$

we also have

$$Z_{k_i(b,l)}^{i-1} \in \mathfrak{g}_{k_i(b,l)+1} + \mathfrak{g}_{k_i(b,l)}(l_{k_i(b,l)}),$$

i. e. we may write $Z_{k_i(b,l)}^{i-1} = U + V$ with $U = U(b,l) \in \mathfrak{g}_{k_i(b,l)+1}$ and $V = V(b,l) \in \mathfrak{g}_{k_i(b,l)}(l_{k_i(b,l)})$. Moreover, as $k_i(b,l) + 1 \leq j_i(b,l)$, we also have

$$Z_{j_i(b,l)}, Z_{j_i(b,l)}^{i-1} \in \mathfrak{g}_{k_i(b,l)+1} \subset \mathfrak{g}_{k_i(b,l)}.$$

Hence $\langle l, [Z_{j_i(b,l)}^{i-1}, V]_b \rangle = 0$ as $V \in \mathfrak{g}_{k_i(b,l)}(l_{k_i(b,l)})$ and $\langle l, [Z_{j_i(b,l)}^{i-1}, U]_b \rangle = 0$ by definition of $k_i(b,l)$. This gives $\langle l, [Z_{j_i(b,l)}^{i-1}, Z_{k_i(b,l)}^{i-1}]_b \rangle = 0$, which is a contradiction.

(v) The points (iii) and (iv) prove the proposition. \square

2.8 Generic elements

If \mathcal{B} is an open subset of a real finite dimensional vector space, the set

$$\mathcal{D}_{gen} := \mathcal{D}_d = \{(b,l) \in \mathcal{B} \times \mathfrak{g}^* \mid j_i(b,l) = j_i \text{ and } k_i(b,l) = k_i, \forall i\} \subset \mathcal{B} \times \mathfrak{g}^*$$

is a Zariski open subset of $\mathcal{B} \times \mathfrak{g}^*$, called the *set of generic elements* of $\mathcal{B} \times \mathfrak{g}^*$, by an argument similar to the one we used for \mathcal{D}_1 in (2.3). Hence, for $(b,l) \in \mathcal{D}_{gen} = \mathcal{D}_d$, the indices $j_i(b,l)$ and $k_i(b,l)$ are equal to j_i and k_i respectively, and are independent of $(b,l) \in \mathcal{D}_{gen}$. In particular, all the elements of \mathcal{D}_{gen} have the same set of Pukanszky jump indices which we shall denote $\mathcal{S} = \{j_i, k_i \mid 1 \leq i \leq d\}$, and the same set of local jump indices which we shall denote $\mathcal{R} = \{k_i \mid 1 \leq i \leq d\}$.

If $\mathcal{B} = \emptyset$, i. e. if there is no variable structure, then \mathcal{D}_{gen} is identified with a Zariski open subset \mathfrak{g}_{gen}^* of \mathfrak{g}^* , called the *set of generic elements* of \mathfrak{g}^* . One has

$$\mathfrak{g}_{gen}^* \subset \mathfrak{g}_{Puk}^* \subset \mathfrak{g}_{max}^* \subset \mathfrak{g}^*,$$

where \mathfrak{g}_{Puk}^* denotes the elements of \mathfrak{g}^* which are in general position in the sense of Pukanszky (with respect to the basis $(Z_i)_i$) and \mathfrak{g}_{max}^* the set of elements of \mathfrak{g}^* whose coadjoint orbit is of maximal dimension ([Lu-Z], [C-G]).

Let's notice that \mathcal{D}_r , $r \in \{1, \dots, d\}$ and \mathcal{D}_{gen} are Ad^*G -invariant, i. e. that for all $g \in G$,

$$\begin{aligned} (b,l) \in \mathcal{D}_r &\Leftrightarrow (b, (Ad^*g)(l)) \in \mathcal{D}_r \\ (b,l) \in \mathcal{D}_{gen} &\Leftrightarrow (b, (Ad^*g)(l)) \in \mathcal{D}_{gen} \end{aligned}$$

(see [Lu-Mu]). This allows us to restrict ourselves to orbit sections, which may be done in the following way:

For any fixed $b \in \mathcal{B}$, let S_b be the set of jump indices in the sense of Pukanszky for the algebra \mathfrak{g}_b and the elements in general position in \mathfrak{g}_b^* , and let $T_b = \{1, \dots, n\} \setminus S_b$. Let $V_{T_b} = \sum_{i \in T_b} \mathbb{R}Z_i^*$. It is known that each coadjoint orbit in $(\mathfrak{g}_b^*)_{Puk}$ meets $V_{T_b} \cap (\mathfrak{g}_b^*)_{Puk}$ in exactly one point and conversely. Hence $V_{T_b} \cap (\mathfrak{g}_b^*)_{Puk}$ represents a section of the coadjoint orbits of $(\mathfrak{g}_b^*)_{Puk}$. Let's also note that $l \in V_{T_b}$ if and only if $\langle l, Z_j \rangle = 0$ for all the Pukanszky jump indices $j \in S_b$ (see [C-G] for this construction). In many situations we want to restrict ourselves to these orbit sections, because this will allow us to assume that the linear forms l are zero in the coordinates corresponding to the Pukanszky jump indices. To do this, let's call \mathcal{T} the union of these orbit sections, or, more precisely,

$$\mathcal{T} = \bigcup_{b \in \mathcal{B}} \{b\} \times V_{T_b} \text{ and } \mathcal{D}_{T,gen} = \mathcal{D}_{gen} \cap \mathcal{T}.$$

Then $\mathcal{D}_{T,gen}$ is a Zariski open subset of \mathcal{T} . Let's now consider

$$\begin{aligned} \mathcal{B}_1 &:= \{b \in \mathcal{B} \mid \exists l \in \mathfrak{g}^* \text{ s. t. } (b, l) \in \mathcal{D}_{T,gen}\} \\ &= \{b \in \mathcal{B} \mid \exists l \in \mathfrak{g}^* \text{ s. t. } (b, l) \in \mathcal{D}_{gen}\}. \end{aligned}$$

Let's notice that \mathcal{B}_1 is open in \mathcal{B} . In fact, it is the projection onto the first variable of the open set \mathcal{D}_{gen} .

Let $(b, l) \in \mathcal{D}_{gen}$. One knows that this l is also in general position in \mathfrak{g}_b^* in the sense of Pukanszky (see [Lu-Z]). Moreover, the set of Pukanszky jump indices S_b of \mathfrak{g}_b^* coincides with the set

$$\{j_i(b, l), k_i(b, l) \mid 1 \leq i \leq d\} = \{j_i, k_i \mid 1 \leq i \leq d\}$$

and is hence independent of b , if $b \in \mathcal{B}_1$. Hence, for all $b \in \mathcal{B}_1$, the sets S_b , resp. T_b coincide. Let's write T for this common set and $V_T = \sum_{i \in T} \mathbb{R}Z_i^*$. It is then obvious that

$$\mathcal{D}_{T,gen} = \mathcal{D}_{gen} \cap (\mathcal{B}_1 \times V_T).$$

If $\mathcal{B} = \emptyset$, then $\mathcal{D}_{T,gen}$ is identified with

$$\mathfrak{g}_{T,gen}^* = \mathfrak{g}_{gen}^* \cap V_T,$$

the set of generic elements contained in the orbit section V_T .

In the rest of this paper, we shall restrict ourselves to \mathcal{B}_1 and we may hence assume that $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{D}_{T,gen} = \mathcal{D}_{gen} \cap (\mathcal{B} \times V_T)$.

2.9 New parameters

Let j_1 and k_1 be as in (2.3). Let's put $X := Z_{k_1}, Y := Z_{j_1}$ and $Z_b := [Z_{k_1}, Z_{j_1}]_b = [X, Y]_b$. Let's write $\mathcal{V}_{j_1} := \langle Z_{j_1+1}, \dots, Z_n \rangle$ for the vector subspace generated by $\langle Z_{j_1+1}, \dots, Z_n \rangle$. As $(Z_j)_j$ is a Jordan-Hölder basis for all the \mathfrak{g}_b 's,

$$[Z_{j_1}, U]_b \subset \mathcal{V}_{j_1} \subset \mathfrak{Z}(\mathfrak{g}_b), \forall U \in \mathfrak{g}_b,$$

where $\mathfrak{Z}(\mathfrak{g}_b)$ denotes the center of \mathfrak{g}_b , and the definitions of $\mathfrak{p}_1(b, l)$ and of the basis $(Z_i^1(b, l))_i$ depend in fact only on $l|_{\mathcal{V}_{j_1}}$, and even on its kernel $\ker(l|_{\mathcal{V}_{j_1}})$. For $\mathfrak{p}_1(b, l)$ this is clear by definition. For the basis vectors $Z_i^1(b, l)$, let's first notice that one may write

$$\mathcal{V}_{j_1} = \mathbb{R}Z_b \oplus \ker(l|_{\mathcal{V}_{j_1}}),$$

that, for fixed b , $l|_{\mathcal{V}_{j_1}}$ is completely determined by its value on Z_b and by $\ker(l|_{\mathcal{V}_{j_1}})$ and that hence $\frac{\langle l, [Z_{j_1}, Z_i]_b \rangle}{\langle l, [Z_{j_1}, Z_{k_1}]_b \rangle}$ depends only on $\ker(l|_{\mathcal{V}_{j_1}})$ (if $(b, l) \in \mathcal{D}_{T,gen}$ and so $\langle l, [Z_{j_1}, Z_{k_1}]_b \rangle \neq 0$).

From this point of view, varying l means, at this level, considering all the codimensional one subspaces $V(b)$ of \mathcal{V}_{j_1} such that $Z_b \notin V(b)$. So, instead of characterizing $\mathfrak{p}_1(b, l)$ by (b, l) , we may also characterize it by b and a subspace $V(b)$, respectively by b and a vector basis of $V(b)$. This leads to the following parametrization: Let $m := n - j_1$ and let's identify \mathcal{V}_{j_1} with \mathbb{R}^m thanks to the basis $\{Z_{j_1+1}, \dots, Z_n\}$. A codimensional one subspace of \mathcal{V}_{j_1} is hence characterized by a $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{m-1}) \in (\mathbb{R}^m)^{m-1}$ such that the \tilde{b}_i 's are independent, i. e. by a Zariski open subset of $(\mathbb{R}^m)^{m-1}$. The subspaces $V(b)$ considered here verify the supplementary condition $Z_b \notin V(b)$, which may be expressed by $\det(Z_b, \tilde{b}_1, \dots, \tilde{b}_{m-1}) \neq 0$, which again gives a Zariski open subset of $(\mathbb{R}^m)^{m-1}$.

Let

$$\begin{aligned}\mathcal{E} &:= \{(b, \tilde{b}) \in \mathcal{B} \times (\mathbb{R}^m)^{m-1} \mid \tilde{b}_i \text{'s independent}\} \\ \mathcal{E}_1 &:= \{(b, \tilde{b}) \in \mathcal{E} \mid \det(Z_b, \tilde{b}_1, \dots, \tilde{b}_{m-1}) \neq 0\}\end{aligned}$$

Similarly to what has been done in (2.3) we may then define, for all $(b, \tilde{b}) \in \mathcal{E}_1$,

$$\mathfrak{p}_1(b, \tilde{b}) := \{U \in \mathfrak{g}_b \mid [Z_{j_1}, U]_b \in \langle \tilde{b} \rangle\},$$

where $\langle \tilde{b} \rangle$ denotes the vector subspace generated by $\tilde{b}_1, \dots, \tilde{b}_{m-1}$. In order to define the basis of $\mathfrak{p}_1(b, \tilde{b})$, let's first notice that as

$$\mathcal{V}_{j_1} = \mathbb{R}Z_b \oplus \langle \tilde{b} \rangle,$$

there exist for each Z_i a unique $\alpha_i \in \mathbb{R}$ and a unique $v_i \in \langle \tilde{b} \rangle$ such that

$$[Z_{j_1}, Z_i]_b = \alpha_i Z_b + v_i = \alpha_i [Z_{k_1}, Z_{j_1}]_b + v_i.$$

The number α_i is given by

$$\alpha_i = \frac{\det([Z_{j_1}, Z_i]_b, \tilde{b}_1, \dots, \tilde{b}_{m-1})}{\det([Z_{k_1}, Z_{j_1}]_b, \tilde{b}_1, \dots, \tilde{b}_{m-1})}, \quad (2.1)$$

and is a rational function in b, \tilde{b} , if $(\mathfrak{g}, \mathcal{B})$ is a rationally variable Lie algebra.

Let's also notice that $\alpha_i = 0$ if $i > k_1$, because then $[Z_{j_1}, Z_i]_b = 0$. We now define

$$Z_i^1(b, \tilde{b}) = Z_i + \alpha_i Z_{k_1}, \quad \text{for } i \neq k_1.$$

In particular, $Z_i^1(b, \tilde{b}) = Z_i$ if $i > k_1$. It is then easy to check that $\mathfrak{p}_1(b, \tilde{b})$ is an ideal of codimension one in \mathfrak{g}_b . Let's write $[\cdot, \cdot]_{(b, \tilde{b})}$ for the Lie bracket in $\mathfrak{p}_1(b, \tilde{b})$ (which of course coincides with $[\cdot, \cdot]_b$). We have the following result:

Proposition 2.10. *$\{Z_i^1(b, \tilde{b}) \mid i \in I_1\}$ is a Jordan-Hölder basis of $\mathfrak{p}_1(b, \tilde{b})$ and the structure constants for this basis in $\mathfrak{p}_1(b, \tilde{b})$ are rational functions in b, \tilde{b} , if $(\mathfrak{g}, \mathcal{B})$ is a rationally variable Lie algebra.*

Proof. Let's first show that $Z_i^1(b, \tilde{b}) \in \mathfrak{p}_1(b, \tilde{b})$. If $i > k_1$, this is due to the fact that then $Z_i^1(b, \tilde{b}) = Z_i$ and that $[Z_{j_1}, Z_i]_b = 0$ by the definition of k_1 . If $i < k_1$,

$$[Z_{j_1}, Z_i^1(b, \tilde{b})]_b = [Z_{j_1}, Z_i]_b + \alpha_i [Z_{j_1}, Z_{k_1}]_b = v_i \in \langle \tilde{b} \rangle$$

and hence $Z_i^1(b, \tilde{b}) \in \mathfrak{p}_1(b, \tilde{b})$.

By construction the vectors $Z_i^1(b, \tilde{b})$, $i \in I_1$, are independent and hence they form a basis of $\mathfrak{p}_1(b, \tilde{b})$, as $\mathfrak{p}_1(b, \tilde{b})$ can at most be of dimension $n - 1$ because $Z_{k_1} \notin \mathfrak{p}_1(b, \tilde{b})$. So

$$\mathfrak{g}_b = \mathbb{R}Z_{k_1} \oplus \mathfrak{p}_1(b, \tilde{b}) = \mathbb{R}X \oplus \mathfrak{p}_1(b, \tilde{b}).$$

It is easy to compute the structure constants $\tilde{a}_{ij}^k(b, \tilde{b})$ for the basis $Z_i^1(b, \tilde{b})$ in $\mathfrak{p}_1(b, \tilde{b})$. One gets:

$$\begin{aligned} \tilde{a}_{ij}^k(b, \tilde{b}) &= a_{ij}^k(b) \quad \text{if } i, j > k_1 \\ \tilde{a}_{ij}^k(b, \tilde{b}) &= a_{ij}^k(b) + \alpha_i a_{k_1 j}^k(b) \quad \text{if } k > j > k_1 > i \end{aligned}$$

Similarly if $k > i > k_1 > j$. Moreover:

$$\begin{aligned} \tilde{a}_{ij}^k(b, \tilde{b}) &= a_{ij}^k(b) \quad \text{if } \max\{i, j\} < k < k_1 \\ \tilde{a}_{ij}^k(b, \tilde{b}) &= a_{ij}^k(b) + \alpha_i a_{k_1 j}^k(b) + \alpha_j a_{i k_1}^k(b) \quad \text{if } \max\{i, j\} < k_1 < k \end{aligned}$$

The other structure constants are zero.

If $(\mathfrak{g}, \mathcal{B})$ is a rationally variable Lie algebra, then α_i is a rational function in b, \tilde{b} . So the same is true for the structure constants. The formula (2.1) for α_i shows that the new structure constants depend only on b and $\langle \tilde{b} \rangle$. \square

2.11 Coexponential basis

Let's recall that for $(b, l) \in \mathcal{D}_{gen}$, $Z_{k_1} = Z_{k_1}(b, l), \dots, Z_{k_d} = Z_{k_d}(b, l)$ form a complementary basis to the polarization $\mathfrak{p}_d(b, l) = \mathfrak{p}_{(b, l)}$ in \mathfrak{g}_b . Let's also recall that the k_i 's are the local jump indices for $l \in \mathfrak{g}_b$, for all $(b, l) \in \mathcal{D}_{gen}$. This basis is a coexponential basis if the Z_{k_i} 's are correctly ordered (In fact, the indices k_i are not necessarily decreasing as defined in (2.3), whereas the indices are ordered if the Z_{k_i} 's are constructed as local jump indices!). Let's write X_1, \dots, X_d for this (ordered) basis, which is the same for all $(b, l) \in \mathcal{D}_{gen}$.

Let's now consider $\{Z_{k_1}^0(b, l) = Z_{k_1}, Z_{k_2}^1(b, l), \dots, Z_{k_d}^{d-1}(b, l)\}$ which forms also a coexponential basis to $\mathfrak{p}_d(b, l)$ in \mathfrak{g}_b , with the property that

$$Z_{k_i}^{i-1}(b, l) \in \mathfrak{p}_{i-1}(b, l) \setminus \mathfrak{p}_i(b, l).$$

Moreover, the definition of the basis in (2.3) shows that both bases are linked by relations of the form

$$\begin{aligned} Z_{k_1}^0(b, l) &= Z_{k_1} \\ Z_{k_2}^1(b, l) &= Z_{k_2} + c(2, 1)Z_{k_1} \\ Z_{k_3}^2(b, l) &= Z_{k_3} + c(3, 2)Z_{k_2} + c(3, 1)Z_{k_1} \\ &\dots = \dots \\ Z_{k_d}^{d-1}(b, l) &= Z_{k_d} + c(d, d-1)Z_{k_{d-1}} + \dots + c(d, 1)Z_{k_1} \end{aligned}$$

and

$$\begin{aligned} Z_{k_1} &= Z_{k_1}^0(b, l) \\ Z_{k_2} &= Z_{k_2}^1(b, l) + c'(2, 1)Z_{k_1}^0(b, l) \\ Z_{k_3} &= Z_{k_3}^2(b, l) + c'(3, 2)Z_{k_2}^1(b, l) + c(3, 1)Z_{k_1}^0(b, l) \\ &\dots = \dots \\ Z_{k_d} &= Z_{k_d}^{d-1}(b, l) + c'(d, d-1)Z_{k_{d-1}}^{d-2}(b, l) + \dots + c'(d, 1)Z_{k_1}^0(b, l) \end{aligned}$$

where the $c(i, j)$ and $c'(i, j)$ are rational functions in (b, l) whose denominators are not annihilated if $(b, l) \in \mathcal{D}_{gen}$. Similarly if we consider the dependence in (b, \tilde{b}) instead of (b, l) and get the basis $Z_{k_i}^{i-1}(b, \tilde{b})$. Again, the coefficients appearing in the change of basis are rational functions in (b, \tilde{b}) whose denominators are not annihilated, by the restrictions put on the parameters (b, \tilde{b}) .

2.12 New variable structures

Originally, $\langle \tilde{b} \rangle$ was taken for $\ker(l|_{\mathcal{V}_{j_1}})$. So we want to consider only those linear forms on $\mathfrak{p}_1(b, \tilde{b})$ which annihilate $\langle \tilde{b} \rangle$. To this effect, we take the quotient by $\langle \tilde{b} \rangle$ and get new variable structures. The details of these constructions are the following: Let's recall that $\mathfrak{p}_1(b, \tilde{b})$, the basis $(Z_k^1(b, \tilde{b}))_{k \in I_1}$ and the structure constants for the Lie bracket in $\mathfrak{p}_1(b, \tilde{b})$ with respect to this new basis depend only on b and $\langle \tilde{b} \rangle$. We may hence define a new rationally variable Lie algebra $(\mathfrak{q}_1, \mathcal{E}_1)$ by

$$\mathfrak{q}_1(b, \tilde{b}) := \mathfrak{p}_1(b, \tilde{b}) / \langle \tilde{b} \rangle,$$

by taking in $\mathfrak{q}_1(b, \tilde{b})$ the Jordan-Hölder basis

$$\{Z_k^1(b, \tilde{b}) \bmod \langle \tilde{b} \rangle \mid k \in I_1, k \leq j_1\} \cup \{Z_b \bmod \langle \tilde{b} \rangle\}$$

and by identifying this basis with a fixed basis in a real vector space \mathfrak{q}_1 of dimension j_1 . Then the dependence on b and \tilde{b} is entirely put into the Lie bracket $[\cdot, \cdot]_{(b, \tilde{b})}$ and $(\mathfrak{q}_1, \mathcal{E}_1)$ may be viewed as a rationally variable Lie algebra, whose associated variable Lie group will be noted (Q_1, \mathcal{E}_1) . Similarly we define a rationally variable Lie algebra $(\mathfrak{r}_1, \mathcal{E}_1)$ by

$$\mathfrak{r}_1(b, \tilde{b}) := \mathfrak{p}_1(b, \tilde{b}) / \langle Z_{j_1}, \tilde{b} \rangle = \mathfrak{q}_1(b, \tilde{b}) / \langle Z_{j_1} \rangle,$$

by taking in $\mathfrak{r}_1(b, \tilde{b})$ the Jordan-Hölder basis

$$\{Z_k^1(b, \tilde{b}) \bmod \langle Z_{j_1}, \tilde{b} \rangle \mid k \in I_1, k < j_1\} \cup \{Z_b \bmod \langle Z_{j_1}, \tilde{b} \rangle\}.$$

Let's write $\tilde{Z}_k^1(b, \tilde{b}) = Z_k^1(b, \tilde{b}) \bmod \langle Z_{j_1}, \tilde{b} \rangle$ if $k \in I_1, k < j_1$ and $\tilde{Z}_{j_1+1}^1(b, \tilde{b}) = Z_b \bmod \langle Z_{j_1}, \tilde{b} \rangle$ for this basis. We then make the same type of identifications as before and we write (R_1, \mathcal{E}_1) for the associated variable Lie group.

2.13

Let's now consider the new variable structure $(\mathfrak{r}_1, \mathcal{E}_1)$. All the preceding constructions may of course be made for this new structure. We shall write $\tilde{\mathcal{T}}, \tilde{\mathcal{T}}_{(b, \tilde{b})}, \tilde{\mathcal{T}}, V_{\tilde{\mathcal{T}}_{(b, \tilde{b})}}, V_{\tilde{\mathcal{T}}}, \tilde{\mathcal{D}}_{\tilde{\mathcal{T}}, gen}, \tilde{j}_i(b, \tilde{b}, \tilde{l}), \tilde{k}_i(b, \tilde{b}, \tilde{l}), \tilde{j}_i(b, \tilde{b}), \tilde{k}_i(b, \tilde{b}), \tilde{j}_i, \tilde{k}_i, \tilde{\mathcal{D}}_i, \tilde{I}_i$ for $i \geq 2$ (as $i = 1$ corresponds to the level of \mathfrak{g}) for the corresponding objects associated to $(\mathfrak{r}_1, \mathcal{E}_1)$. Here the indices are defined with respect to the Jordan-Hölder basis given in (2.12). In particular, $\tilde{\mathcal{T}} = \cup_{(b, \tilde{b}) \in \mathcal{E}_1} \{(b, \tilde{b})\} \times V_{\tilde{\mathcal{T}}_{(b, \tilde{b})}}$. We define

$$\Phi : \tilde{\mathcal{T}} \rightarrow \mathcal{T}$$

by

$$\Phi(b, \tilde{b}, \tilde{l}) = (b, l),$$

where $l \in \mathfrak{g}^*$ is given by

$$\begin{aligned} \langle l, U \rangle &:= \langle \tilde{l}, \tilde{U} \rangle \quad \forall U \in \mathfrak{p}_1(b, \tilde{b}) \\ \langle l, X \rangle &:= 0, \end{aligned}$$

with $\tilde{U} = U \bmod \langle Z_{j_1}, \tilde{b} \rangle$. In this manner l is completely characterized and $\langle l, X \rangle = \langle l, Y \rangle = 0$, $l|_{\langle \tilde{b} \rangle} \equiv 0$. Let's notice in particular that this l is zero on the basis vectors corresponding to the Pukanszky jump indices of \mathfrak{g}_b , i. e. that $l \in V_{T_b}$ and hence that $(b, l) \in \mathcal{T}$. We then have the following lemma:

Lemma 2.14. *The map Φ is well defined, one-to-one and continuous. The families of indices j and k satisfy*

$$j_i = \tilde{j}_i \text{ and } k_i = \tilde{k}_i \quad \forall i \in \{2, \dots, d\}.$$

Moreover,

$$\mathcal{D}_{T,gen} = \Phi(\tilde{\mathcal{D}}_{\tilde{T},gen}) \cap \mathcal{D}_1.$$

Proof. The continuity of Φ is obvious, by definition.

For any $U \in \mathfrak{p}_1(b, \tilde{b})$, let's write \tilde{U} for $U \bmod \langle Z_{j_1}, \tilde{b} \rangle \in \mathfrak{r}_1(b, \tilde{b})$. Let's first show that $\mathcal{D}_1 \cap \mathcal{T} \subset \mathfrak{Im}\Phi$. To this effect, let $(b, l) \in \mathcal{D}_1 \cap \mathcal{T}$. Hence $\langle l, [Z_{k_1}, Z_{j_1}]_b \rangle \neq 0$. Let's choose a basis \tilde{b} of $\ker(l|_{V_{j_1}})$ and let's define $\tilde{l} \in \mathfrak{r}_1(b, \tilde{b})^* \equiv \mathfrak{r}_1^*$ by $\langle \tilde{l}, \tilde{U} \rangle := \langle l, U \rangle$ for all $\tilde{U} \in \mathfrak{r}_1(b, \tilde{b})$. This \tilde{l} is well defined as $\langle l, Z_{j_1} \rangle = 0$ (as $l \in V_{T_b}$) and as $l|_{\langle \tilde{b} \rangle} \equiv 0$ (by choice of \tilde{b}). Moreover, $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{T}}$ as the Pukanszky jump indices of \tilde{l} in $\mathfrak{r}_1(b, \tilde{b})$ coincide with the Pukanszky jump indices of l in \mathfrak{g}_b that are smaller than j_1 and different from k_1 and j_1 . By the choice of l and \tilde{b} , $Z_b = [Z_{k_1}, Z_{j_1}]_b \notin \langle \tilde{b} \rangle$ and $(b, \tilde{b}) \in \mathcal{E}_1$. Hence $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{T}}$ and $\Phi(b, \tilde{b}, \tilde{l}) = (b, l)$ by construction. So $\mathcal{D}_1 \cap \mathcal{T} \subset \mathfrak{Im}\Phi$.

Let's now notice that $\tilde{\mathcal{D}}_{\tilde{T},gen} \cap \Phi^{-1}(\mathcal{D}_{T,gen}) \neq \emptyset$. In fact, $\mathcal{D}_{T,gen}$ is a nonempty open subset of \mathcal{T} contained in $\mathfrak{Im}\Phi$, as $\mathcal{D}_{T,gen} \subset \mathcal{D}_1 \cap \mathcal{T} \subset \mathfrak{Im}\Phi$. By continuity of Φ , $\Phi^{-1}(\mathcal{D}_{T,gen})$ is a nonempty open subset of $\tilde{\mathcal{T}}$. Moreover, $\tilde{\mathcal{D}}_{\tilde{T},gen}$ is a Zariski open subset of $\tilde{\mathcal{T}}$. Hence $\tilde{\mathcal{D}}_{\tilde{T},gen} \cap \Phi^{-1}(\mathcal{D}_{T,gen}) \neq \emptyset$.

We see that $\tilde{j}_2(b, \tilde{b}, \tilde{l}) = j_2(\Phi(b, \tilde{b}, \tilde{l})) = j_2(b, l)$, $\tilde{j}_2(b, \tilde{b}) = j_2(b)$, $\tilde{j}_2 = j_2$, for any $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{T}}$. In fact, the passage from $\mathfrak{p}_1(b, \tilde{b})$ to $\mathfrak{r}_1(b, \tilde{b})$ is obtained by taking the quotient with a subspace of the center of $\mathfrak{p}_1(b, \tilde{b})$. This does not effect the definition of these indices.

Let's now show that $k_2 = \tilde{k}_2$. For this purpose, let $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_{\tilde{T},gen} \cap \Phi^{-1}(\mathcal{D}_{T,gen})$ and $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$. Then

$$\langle \tilde{l}, [\tilde{Z}_{j_2}^1(b, \tilde{b}), \tilde{Z}_{\tilde{k}_2}^1(b, \tilde{b})]_{(b, \tilde{b})} \rangle \neq 0$$

(as $j_2 = \tilde{j}_2 = \tilde{j}_2(b, \tilde{b}, \tilde{l})$, $\tilde{k}_2 = \tilde{k}_2(b, \tilde{b}, \tilde{l})$) and hence

$$\langle l, [Z_{j_2}^1(b, l), Z_{k_2}^1(b, l)]_b \rangle \neq 0,$$

with $(b, l) \in \mathcal{D}_{T,gen}$. So $\tilde{k}_2 \leq k_2(b, l) = k_2$. Conversely, as $(b, l) \in \mathcal{D}_{T,gen}$ and $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$,

$$\langle l, [Z_{j_2}^1(b, l), Z_{k_2}^1(b, l)]_b \rangle \neq 0,$$

which implies that

$$\langle \tilde{l}, [\tilde{Z}_{j_2}^1(b, \tilde{b}), \tilde{Z}_{\tilde{k}_2}^1(b, \tilde{b})]_{(b, \tilde{b})} \rangle \neq 0,$$

(with $j_2 = \tilde{j}_2 = \tilde{j}_2(b, \tilde{b}, \tilde{l})$) by the definition of l . Hence $k_2 \leq \tilde{k}_2(b, \tilde{b}, \tilde{l}) = \tilde{k}_2$ as $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_{\tilde{T},gen}$. So $\tilde{k}_2 = k_2$. Let's put $\tilde{\mathcal{D}}_2 = \{(b, \tilde{b}, \tilde{l}) \in \mathcal{E}_1 \times \mathfrak{r}_1^* \mid \Phi(b, \tilde{b}, \tilde{l}) = (b, l) \in \mathcal{D}_1, \tilde{j}_2(b, \tilde{b}, \tilde{l}) = \tilde{j}_2 = j_2 \text{ and } \tilde{k}_2(b, \tilde{b}, \tilde{l}) = \tilde{k}_2 = k_2\}$. Then $\Phi(\tilde{\mathcal{D}}_2 \cap \tilde{\mathcal{T}}) = \mathcal{D}_2 \cap \mathcal{T}$. In fact, let's first notice that $\mathcal{D}_2 \cap \mathcal{T} \subset \mathcal{D}_1 \cap \mathcal{T} \subset \mathfrak{Im}\Phi$. Let then $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$ with $(b, l) \in \mathcal{D}_2 \cap \mathcal{T}$. In particular

$$\langle l, [Z_{k_2}^1(b, \tilde{b}), Z_{j_2}^1(b, \tilde{b})]_b \rangle \neq 0$$

as $j_2(b, l) = j_2$ and $k_2(b, l) = k_2$. Because of the definition of Φ ,

$$\langle \tilde{l}, [\tilde{Z}_{k_2}^1(b, \tilde{b}), \tilde{Z}_{j_2}^1(b, \tilde{b})]_{(b, \tilde{b})} \rangle \neq 0.$$

This shows first that $\tilde{Z}_{j_2}^1(b, \tilde{b}) \notin \mathfrak{r}_1(b, \tilde{b})(\tilde{l})$ (stabilizer of \tilde{l} in $r_1(b, \tilde{b})$) and hence that $\tilde{j}_2 = j_2 \leq \tilde{j}_2(b, \tilde{b}, \tilde{l})$. As we always have $\tilde{j}_2(b, \tilde{b}, \tilde{l}) \leq \tilde{j}_2$, we have in fact $\tilde{j}_2(b, \tilde{b}, \tilde{l}) = \tilde{j}_2$. But then we must have $\tilde{k}_2 = k_2 \leq \tilde{k}_2(b, \tilde{b}, \tilde{l})$. As the converse inequality is always true, we have $\tilde{k}_2(b, \tilde{b}, \tilde{l}) = \tilde{k}_2$ and $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_2$. As $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$, $\mathcal{D}_2 \cap \mathcal{T} \subset \Phi(\tilde{\mathcal{D}}_2 \cap \tilde{\mathcal{T}})$. The converse inclusion is proven similarly.

We may now proceed by induction. Let's assume that $\tilde{j}_s = j_s$ and $\tilde{k}_s = k_s$ for $s \leq i - 1$, $\Phi(\tilde{\mathcal{D}}_{i-1} \cap \tilde{\mathcal{T}}) = \mathcal{D}_{i-1} \cap \mathcal{T}$ ($\tilde{\mathcal{D}}_{i-1}$ being defined in a similar way then $\tilde{\mathcal{D}}_2$) and let's prove the same relations for i . Similar to the case $i = 2$ we see that $\tilde{j}_i(b, \tilde{b}, \tilde{l}) = j_i(\Phi(b, \tilde{b}, \tilde{l})) = j_i(b, l)$, $\tilde{j}_i(b, \tilde{b}) = j_i(b)$ and $\tilde{j}_i = j_i$ for any $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_{i-1}$. The same argument as in the case $i = 2$ shows that $\tilde{k}_i = k_i$. So, if $\tilde{\mathcal{D}}_i = \{(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_{i-1} \mid \tilde{j}_i(b, \tilde{b}, \tilde{l}) = \tilde{j}_i = j_i \text{ and } \tilde{k}_i(b, \tilde{b}, \tilde{l}) = \tilde{k}_i = k_i\}$, then $\Phi(\tilde{\mathcal{D}}_i \cap \tilde{\mathcal{T}}) = \mathcal{D}_i \cap \mathcal{T}$. For $i = d$ the procedure stops and we have the final result. \square

3 Function spaces

3.1

Let (G, \mathcal{B}) be a rationally variable Lie group. Let $\{Z_1, \dots, Z_n\}$ denote the fixed basis as in (2.2). A function h on G_b is identified with a function on \mathbb{R}^n by the formula $h(x_1, \dots, x_n) = h(\exp_b(x_1 Z_1) \cdot_b \dots \cdot_b \exp_b(x_n Z_n))$. Let $r \in \mathbb{N}$. If \mathcal{B} is a non empty open subset of a real finite dimensional vector space, we define $\mathcal{S}(G, \mathcal{B}, \mathbb{R}^r)$ to be the set of all functions

$$\begin{aligned} f : \mathcal{B} \times \mathbb{R}^r \times \mathbb{R}^n &\rightarrow \mathbb{C} \\ (b, \alpha, (x_1, \dots, x_n)) &\mapsto f(b, \alpha, \exp_b(x_1 Z_1) \cdot_b \dots \cdot_b \exp_b(x_n Z_n)) \end{aligned}$$

that are \mathcal{C}^∞ with respect to all variables and such that

$$\begin{aligned} &\|f\|_{K, A, B_1, B_2, C_1, C_2} \\ &= \sup_{b \in K; \alpha \in \mathbb{R}^r; x \in \mathbb{R}^n} \left[\sup_{|a| \leq A; |r_i| \leq B_i; |s_j| \leq C_j; i, j \in \{1, 2\}} |\alpha^{r_1} x^{s_1}| \right. \\ &\quad \left. \cdot \frac{\partial^a}{\partial b^a} \frac{\partial^{r_2}}{\partial \alpha^{r_2}} \frac{\partial^{s_2}}{\partial x^{s_2}} f(b, \alpha, \exp_b(x_1 Z_1) \cdot_b \dots \cdot_b \exp_b(x_n Z_n)) \right] \\ &< +\infty \end{aligned}$$

for any compact subset K of \mathcal{B} and $A, B_i, C_j \in \mathbb{N}$. If $\mathcal{B} = \emptyset$, we just omit the variable b as well as $\frac{\partial^a}{\partial b^a}$, $\sup_{x \in K}$ in the previous definition. Let now $\mathcal{B} = \cup_{m \in \mathbb{N}} K_m$, the K_m 's being compact sets such that $K_m \subset \text{int}(K_{m+1})$ for all m . If K runs through the K_m 's and A, B_i, C_j through \mathbb{N} , then we get a countable family of semi-norms that define the topology of $\mathcal{S}(G, \mathcal{B}, \mathbb{R}^r)$, which becomes a Fréchet space.

3.2

Let's use the notations of (2). If $\mathcal{B} \neq \emptyset$, let \mathcal{U} be a non-empty open subset of $\mathcal{D}_{T, \text{gen}}$. Hence \mathcal{U} is disjoint from $\mathcal{B} \times \{0\}$, as $(b, 0)$ never belongs to $\mathcal{D}_{T, \text{gen}}$. For any $l \in V_T = V_{T_b}$ (as $b \in \mathcal{B}_1$), let $\mathfrak{p}_{(b, l)}$ denote the Vergne polarization of l in \mathfrak{g}_b with respect to the given basis. We know that $\mathfrak{p}_{(b, l)} = \mathfrak{p}_d(b, l)$ if $(b, l) \in \mathcal{D}_{T, \text{gen}}$ (see 2.3). Let $\chi_{(b, l)}$ denote the unitary character defined on $P_{(b, l)} = \exp_b \mathfrak{p}_{(b, l)}$ by the formula $\chi_{(b, l)}(u) = e^{-i \langle l, \log_b u \rangle}$. Let i_1, \dots, i_d be the (ordered) local jump indices (for all l). Let $X_1 = Z_{i_1}, \dots, X_d = Z_{i_d}$ be the coexponential basis of \mathfrak{g}_b with respect

to $\mathfrak{p}_{(b,l)}$ as in (2.11). Thanks to this basis, $G_b/P_{(b,l)}$ may be identified with \mathbb{R}^d . Let $r \in \mathbb{N}$. The space of kernels $\mathcal{N}(G, \mathcal{U}, \mathbb{R}^r)$ is defined to be the set of all \mathcal{C}^∞ functions

$$F : \mathcal{B}_1 \times V_T \times \mathbb{R}^r \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$$

such that the following two conditions are satisfied:

(i)

$$F(b, l, \cdot, \cdot, \cdot) \equiv 0 \text{ if } (b, l) \notin \mathcal{U},$$

so in particular if $(b, l) \in (\mathcal{B}_1 \times V_T) \setminus \mathcal{D}_{T,gen}$.

(ii)

$$\begin{aligned} & \|F\|_{K,A,C,D_1,D_2,E_1,E_2,F_1,F_2} \\ &= \sup_{(b,l) \in K; \alpha \in \mathbb{R}^r; x,y \in \mathbb{R}^d} \left[\sup_{|a| \leq A; |c| \leq C; |d_1| \leq D_1; |d_2| \leq D_2; |e_1| \leq E_1; |e_2| \leq E_2; |f_1| \leq F_1; |f_2| \leq F_2} |\alpha^{d_1} x^{e_1} y^{f_1}| \right. \\ & \quad \left. \cdot \frac{\partial^a}{\partial b^a} \frac{\partial^c}{\partial l^c} \frac{\partial^{d_2}}{\partial \alpha^{d_2}} \frac{\partial^{e_2}}{\partial x^{e_2}} \frac{\partial^{f_2}}{\partial y^{f_2}} F(b, l, \alpha, x, y) \right] \\ & < +\infty \end{aligned}$$

for any compact subset K of \mathcal{U} and $A, C, D_1, D_2, E_1, E_2, F_1, F_2 \in \mathbb{N}$.

If $\mathcal{B} = \emptyset$, we again omit the variable b , as well as $\frac{\partial^a}{\partial b^a}$. In this case \mathcal{U} is taken to be an open subset of $\mathfrak{g}_{T,gen}^*$.

Let $\mathcal{U} = \cup_{s \in \mathbb{N}} K_s$, the K_s 's being compact sets in \mathcal{U} such that $K_s \subset \text{int}(K_{s+1})$ for all s . If K runs through the K_s 's and $A, C, D_1, D_2, E_1, E_2, F_1, F_2$ through \mathbb{N} , then we get a countable family of semi-norms which define the topology of $\mathcal{N}(G, \mathcal{U}, \mathbb{R}^r)$ and which turns $\mathcal{N}(G, \mathcal{U}, \mathbb{R}^r)$ into a Fréchet space.

3.3

Finally let's note $\mathcal{N}_c(G, \mathcal{U}, \mathbb{R}^r)$ for the subspace of $\mathcal{N}(G; \mathcal{U}, \mathbb{R}^r)$ formed by all the functions F defined in (3.2), which also satisfy the following two conditions:

(iii) For every $b \in \mathcal{B}_1$ (i. e. such that $(b, l) \in \mathcal{D}_{T,gen}$ for at least one l), there exists a compact set $K(b)$ in V_T such that $\{b\} \times K(b) \subset \mathcal{D}_{T,gen}$ and $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ if $l \notin K(b)$. In particular, $0 \notin K(b)$ as $(b, 0) \notin \mathcal{D}_{T,gen}$.

(iv) There exists an open neighborhood \mathcal{V} of the set $(\mathcal{B}_1 \times V_T) \setminus \mathcal{D}_{T,gen}$ in $\mathcal{B}_1 \times V_T$ such that $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ for all $(b, l) \in \mathcal{V}$ (If $(b, l) \notin \mathcal{D}_{T,gen}$, then $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ by definition of F , so that (iv) is a condition on $\mathcal{V} \cap \mathcal{D}_{T,gen}$).

The space $\mathcal{N}_c(G, \mathcal{U}, \mathbb{R}^r)$ is no longer a Fréchet space for the given semi-norms.

3.4 Remarks:

a) For $(b, l) \in \mathcal{D}_{T,gen}$, $F(b, l, \alpha, \cdot, \cdot)$ may also be considered as a function on $G_b/P_{(b,l)} \times G_b/P_{(b,l)}$ by writing

$$\begin{aligned} F(b, l, \alpha, x, y) &= F(b, l, \alpha, \exp_b(x_1 X_1) \cdot_b \cdots \cdot_b \exp_b(x_d X_d), \\ & \quad \exp_b(y_1 X_1) \cdot_b \cdots \cdot_b \exp_b(y_d X_d)) \end{aligned}$$

or even as a function on $G_b \times G_b$ if we introduce the covariance relation

$$F(b, l, \alpha, x \cdot h, y \cdot h') = \overline{\chi_{(b,l)}(h)} \chi_{(b,l)}(h') F(b, l, \alpha, x, y) \quad \forall x, y \in G_b, \forall h, h' \in P_{(b,l)}.$$

b) We may also extend F as a function in l to the whole orbit of l by setting

$$F(b, (Ad^*g)(l), \alpha, x, y) = F(b, l, \alpha, x \cdot g, y \cdot g) \quad \forall g, x, y \in G_b.$$

This is done in order to reflect correctly the unitary equivalence between π_l and $\pi_{(Ad^*g)(l)}$, and its incidence on the corresponding operator kernels.

In this paper we shall write F for any one of the preceding meanings of the function.

c) If $\mathcal{B} = \emptyset$, i. e. if there is no variable structure but a fixed Lie group, $\mathcal{D}_{gen} \equiv \mathfrak{g}_{gen}^*$ and, if \mathcal{U} is an open subset of $\mathfrak{g}_{gen}^* \cap V_T$, we define similarly $\mathcal{S}(G, \mathbb{R}^r)$ (the Schwartz space), $\mathcal{N}(G, \mathcal{U}, \mathbb{R}^r)$ and $\mathcal{N}_c(G, \mathcal{U}, \mathbb{R}^r)$, using the corresponding semi-norms to define their respective topologies.

d) If $r = 0$, we just write $\mathcal{S}(G)$, $\mathcal{N}(G, \mathcal{U})$, $\mathcal{N}_c(G, \mathcal{U})$ for the corresponding spaces.

e) We could also have defined $\mathcal{N}(G, \mathcal{U}, \mathbb{R}^r)$, $\mathcal{N}_c(G, \mathcal{U}, \mathbb{R}^r)$, $\mathcal{N}(G, \mathcal{U})$, $\mathcal{N}_c(G, \mathcal{U})$ using the coexponential basis $Z_{k_1}^0(b, l), Z_{k_2}^1(b, l), \dots, Z_{k_d}^{d-1}(b, l)$ (resp. the $Z_{k_i}^{i-1}(b, \dot{b})$'s) instead of X_1, \dots, X_d . In that case we identify $(x'_1, \dots, x'_d) \in \mathbb{R}^d$ with $\exp(x'_1 Z_{k_1}^0) \cdots \exp(x'_d Z_{k_d}^{d-1})$. As the change of bases (in both directions) is characterized by rational functions with non-vanishing denominators, if $(b, l) \in \mathcal{D}_{T,gen}$, we get the same function spaces.

4 The Fourier inversion theorem

We first prove a variable Fourier inversion theorem:

Theorem 4.1. *Let (G, \mathcal{B}) be a rationally variable Lie group with a fixed Jordan-Hölder basis Z_1, \dots, Z_n . Let's note $\mathcal{B}_1 := \{b \in \mathcal{B} \mid \exists l \in \mathfrak{g}^* \text{ s. t. } (b, l) \in \mathcal{D}_{T,gen}\}$. Let $\mathcal{R} = \{i_1, \dots, i_d\}$ be the set of local jump indices for all $(b, l) \in \mathcal{D}_{T,gen}$. For $(b, l) \in \mathcal{D}_{T,gen}$, let $\mathfrak{p}(b, l)$ be the Vergne polarization (with respect to the basis Z_1, \dots, Z_n) and $\{Z_{i_r} \mid i_r \in \mathcal{R}\}$ the coexponential basis to $\mathfrak{p}(b, l)$ in \mathfrak{g}_b defined by using the local jump indices. Let $P_{(b,l)} = \exp \mathfrak{p}(b, l)$ and $\pi_{(b,l)} = \text{ind}_{P_{(b,l)}}^{G_b} \chi_l$. With the previous choices, we have the following result: For every $r \in \mathbb{N}$ and every $F \in \mathcal{N}_c(G; \mathcal{D}_{T,gen}, \mathbb{R}^r)$, there is a unique $f \in \mathcal{S}(G, \mathcal{B}_1, \mathbb{R}^r)$ satisfying: For every $(b, l) \in \mathcal{D}_{T,gen}$, the operator $\pi_{(b,l)}(f_b(\alpha)(\cdot))$ has $F(b, l, \alpha, \cdot, \cdot)$ as a kernel (if we write $f_b(\alpha)(\cdot)$ for $f(b, \alpha, \cdot)$). Moreover, $\pi_{(b,l)}(f) = 0$ for every $b \in \mathcal{B}_1$ and every $l \in (\mathfrak{g}_b^* \setminus (\mathfrak{g}_b)_{gen}^*)$. The map $F \mapsto f$ is continuous in the given topologies.*

This theorem has of course the following consequence, obtained if $\mathcal{B} = \emptyset$, i. e. if there is no variable structure, but a fixed nilpotent Lie group.

Theorem 4.2. *Let G be a connected, simply connected, nilpotent Lie group with a fixed Jordan-Hölder basis Z_1, \dots, Z_n . Let \mathfrak{g}_{gen}^* be the set of generic elements of \mathfrak{g}^* . Let $\mathcal{R} = \{i_1, \dots, i_d\}$ be the set of local jump indices for all $l \in \mathfrak{g}_{gen}^*$. For $l \in \mathfrak{g}_{gen}^*$, let $\mathfrak{p}(l)$ be the Vergne polarization (with respect to the basis Z_1, \dots, Z_n) and $\{Z_{i_r} \mid i_r \in \mathcal{R}\}$ the coexponential basis to $\mathfrak{p}(l)$ in \mathfrak{g} defined by using local jump indices. Let $P_l = \exp \mathfrak{p}_l$ and $\pi_l = \text{ind}_{P_l}^G \chi_l$. With the previous choices, we have the following result: For every function $F \in \mathcal{N}_c(G, \mathfrak{g}_{gen}^*)$ there is a unique function $f \in \mathcal{S}(G)$ satisfying: For every $l \in \mathfrak{g}_{gen}^*$, the operator $\pi_l(f)$ has $F(l, \cdot, \cdot)$ as a kernel. Moreover, $\pi_l(f) = 0$ for every $l \in (\mathfrak{g}^* \setminus \mathfrak{g}_{gen}^*)$. The map $F \mapsto f$ is continuous in the given topologies.*

The proof of (4.1) will be given in (6). Because of remark (3.4), it will also be possible to work with the variable exponential basis $Z_{k_1}^0(b, l), \dots, Z_{k_d}^{d-1}(b, l)$ in the proof, instead of the fixed basis Z_{i_1}, \dots, Z_{i_d} . Once the results of theorem 4.1 and of theorem 4.2 are known, the observations made in the introduction allow us to compute the function $f \in \mathcal{S}(G)$ with the Plancherel formula. We get:

Proposition 4.3. *Under the hypotheses of theorem 4.2, the function $f \in \mathcal{S}(G)$ is obtained by*

$$f(x) = \left(\frac{1}{2\pi}\right)^{n-d} \int_{V_T} \left(\int_{G/P_l} F(l, xu, u) du \right) |Pf(l)| dl,$$

where $\left(\frac{1}{2\pi}\right)^{n-d} |Pf(l)| dl$ is the realization of the Plancherel measure on V_T , if $n = \dim \mathfrak{g}$ and $2d$ the number of Pukanszky jump indices.

Proof. We use the Plancherel formula

$$f(x) = \int_{\hat{G}} \text{tr}(\pi(x)^{-1} \circ \pi(f)) d\mu(\pi)$$

and notice that the Plancherel measure is given by $\left(\frac{1}{2\pi}\right)^{n-d} |Pf(l)| dl, l \in V_T$ where the Pfaffian $Pf(l)$ is defined by

$$|Pf(l)| = \left(\det(\langle l, [Z_i, Z_j] \rangle)_{i,j \in S} \right)^{\frac{1}{2}}.$$

Moreover,

$$\text{tr}(\pi_l(x)^{-1} \circ \pi_l(f)) = \int_{G/P_l} F(l, xu, u) du.$$

□

5 Results on the Radon transform

5.1

In this section we recall some well established facts on the Radon transform (see [H]), with the purpose to be able to recover a function f , if we know its integrals over all hyperplanes. For the proofs we refer to ([H]).

Let f be a function on \mathbb{R}^m , integrable on each hyperplane of \mathbb{R}^m . Let \mathbf{P}^m be the space of all hyperplanes of \mathbb{R}^m . Let $\xi \in \mathbf{P}^m$ and let $d\mu$ denote the Euclidean measure on ξ . We then define:

Definition: The Radon transform $\mathcal{R}f$ of f at ξ is defined by

$$\mathcal{R}f(\xi) := \int_{\xi} f(x) d\mu(x).$$

Let's now parametrize the hyperplanes of \mathbb{R}^m in the following way:

$$\xi(\omega, r) = \{x \in \mathbb{R}^m \mid \langle x, \omega \rangle = r\}$$

where $r \in \mathbb{R}$ and where ω is a normal unit vector to the hyperplane ξ . Note that (ω, r) and $(-\omega, -r)$ characterize the same hyperplane ξ . We may hence write

$$\mathcal{R}f(\xi) = \mathcal{R}f(\omega, r) = \int_{\langle x, \omega \rangle = r} f(x) d\mu(x)$$

and we have $\mathcal{R}f(\omega, r) = \mathcal{R}f(-\omega, -r)$, with $(\omega, r) \in \mathbf{S}^{m-1} \times \mathbb{R}$, \mathbf{S}^{m-1} being the unit sphere in \mathbb{R}^m .

We also consider the following Schwartz spaces:

Definition: (i) The Schwartz space $\mathcal{S}(\mathbf{P}^m)$ on the space of hyperplanes is defined by

$$\mathcal{S}(\mathbf{P}^m) := \{\varphi \in \mathcal{S}(\mathbf{S}^{m-1} \times \mathbb{R}) \mid \varphi(\omega, r) = \varphi(-\omega, -r), \forall (\omega, r) \in \mathbf{S}^{m-1} \times \mathbb{R}\},$$

with $\varphi \in \mathcal{S}(\mathbf{S}^{m-1} \times \mathbb{R})$ if and only if φ is a \mathcal{C}^∞ function such that

$$\sup_{(\omega, r) \in \mathbf{S}^{m-1} \times \mathbb{R}} |(1 + |r|^k) \frac{d^l}{dr^l} (D\varphi)(\omega, r)| < +\infty$$

for all $k, l \in \mathbb{N}$, for all differential operators D on \mathbf{S}^{m-1} . The Radon transform maps $\mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}(\mathbf{P}^m)$.

(ii) We denote $\mathcal{S}^*(\mathbb{R}^m)$ the space of all functions $f \in \mathcal{S}(\mathbb{R}^m)$ such that

$$\int_{\mathbb{R}^m} f(x)P(x)dx = 0 \quad \text{for all polynomials } P.$$

(iii) Similarly, we denote $\mathcal{S}^*(\mathbf{P}^m)$ the space of all functions $\varphi \in \mathcal{S}(\mathbf{P}^m)$ such that

$$\int_{\mathbb{R}} \varphi(\omega, r)p(r)dr = 0 \quad \text{for all polynomials } p \text{ on } \mathbb{R}, \forall \omega \in \mathbf{S}^{m-1}.$$

We have the following result, proven in ([H]):

Theorem 5.2. *The Radon transform is a bijection from $\mathcal{S}^*(\mathbb{R}^m)$ onto $\mathcal{S}^*(\mathbf{P}^m)$.*

5.3

The previous theorem says in particular that for every $\varphi \in \mathcal{S}^*(\mathbf{P}^m)$, there exists a unique $f \in \mathcal{S}^*(\mathbb{R}^m)$ such that $\varphi = \mathcal{R}f$. As a matter of fact, f may be computed from φ by an inversion formula (see [H]):

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^m} \int_{\mathbf{S}^{m-1}} \left[\int_0^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-isr} \mathcal{R}f(\omega, r) dr \right) e^{is\langle x, \omega \rangle} s^{m-1} ds \right] d\omega \\ &= \frac{1}{(2\pi)^m} \int_{\mathbf{S}^{m-1}} \left[\int_0^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-isr} \varphi(\omega, r) dr \right) e^{is\langle x, \omega \rangle} s^{m-1} ds \right] d\omega \\ &= \frac{1}{(2\pi)^m} \int_{\mathbf{S}^{m-1}} \left[\int_0^{+\infty} \hat{\varphi}^2(\omega, s) s^{m-1} e^{is\langle x, \omega \rangle} ds \right] d\omega \end{aligned}$$

where $\hat{\varphi}^2$ denotes the partial Fourier transform in the second variable. But, as $\varphi \in \mathcal{S}^*(\mathbf{P}^m)$, $\hat{\varphi}^2$ and all its partial derivatives with respect to the second variable are zero at $s = 0$. Hence the function ψ defined by

$$\psi(\omega, s) = \begin{cases} s^{m-1} \hat{\varphi}^2(\omega, s) & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}$$

is a \mathcal{C}^∞ function and belongs to $\mathcal{S}(\mathbf{S}^{m-1} \times \mathbb{R})$. Finally

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^m} \int_{\mathbf{S}^{m-1}} \left[\int_{-\infty}^{+\infty} \psi(\omega, s) e^{is\langle x, \omega \rangle} ds \right] d\omega \\ &= \frac{1}{(2\pi)^m} \int_{\mathbf{S}^{m-1}} \hat{\psi}^2(\omega, -\langle x, \omega \rangle) d\omega \end{aligned}$$

where $\hat{\psi}^2 \in \mathcal{S}(\mathbf{S}^{m-1} \times \mathbb{R})$.

6 Proof of the Fourier inversion theorem

6.1

If $n = 1$, $\mathfrak{g}_b \equiv \mathbb{R}$ for all b and there is no need for a variable structure. The result is just the classical Fourier inversion theorem.

6.2

Let's now proceed by recurrence on $\dim \mathfrak{g}$. Assume that $\dim \mathfrak{g} = n$. For the rest of this paper we shall just write $x \cdot y$ for the product in G_b , i. e. instead of $x \cdot_b y$. We use the construction and the definitions of (2). Let's consider in particular the new rationally variable Lie algebra $(\mathfrak{r}_1, \mathcal{E}_1)$. Let's recall that we put $X = Z_{k_1}, Y = Z_{j_1}$ (the same for all b by the choice of \mathcal{D}_{gen}) and $Z_b = [X, Y]_b$. Let's consider the function $\Phi : \tilde{\mathcal{T}} \rightarrow \mathcal{T}$ defined in (2.12) and let's recall that $\mathcal{D}_{T,gen} = \Phi(\tilde{\mathcal{D}}_{\tilde{\mathcal{T}},gen}) \cap \mathcal{D}_1$.

Let $F \in \mathcal{N}_c(G, \mathcal{D}_{T,gen}, \mathbb{R}^r)$ and let's define $\dot{F} \in \mathcal{N}_c(R_1, \tilde{\mathcal{D}}_{\tilde{\mathcal{T}},gen}, \mathbb{R}^{r+2})$ by

$$\dot{F}(b, \tilde{b}, \tilde{l}; s, t, \alpha; \dot{g}_1, \dot{g}'_1) = (2\pi)^2 | \langle l, [X, Y]_b \rangle | \cdot F(b, l; \alpha; \exp_b((s+t)X) \cdot g_1, \exp_b(tX) \cdot g'_1),$$

for $(b, \tilde{b}, \tilde{l}) \in \mathcal{E}_1 \times \mathfrak{r}_1^*, s, t \in \mathbb{R}, \alpha \in \mathbb{R}^r, \dot{g}_1, \dot{g}'_1 \in R_1(b, \tilde{b})$, where $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$. In particular, $\dot{F}(b, \tilde{b}, \tilde{l}, \cdot, \cdot, \cdot, \cdot) \equiv 0$ if $(b, \tilde{b}, \tilde{l}) \notin \tilde{\mathcal{D}}_{\tilde{\mathcal{T}},gen}$, as, in that case, $(b, l) \notin \mathcal{D}_{T,gen}$ and hence $F(b, l, \cdot, \cdot, \cdot) \equiv 0$. This function is well defined, because $l|_{\langle Y, \tilde{b} \rangle} \equiv 0$ and $\langle Y, \tilde{b} \rangle$ is contained in the center of $\mathfrak{p}_1(b, \tilde{b})$ and hence in the polarization $\mathfrak{p}_d(b, \tilde{b})$. It has the correct covariance condition as F does and it is \mathcal{C}^∞ . It is bounded for the appropriate semi-norms, as F is. For every (b, \tilde{b}) such that $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_{\tilde{\mathcal{T}},gen}$ for at least one \tilde{l} , there exists a compact set $K(b, \tilde{b})$ in $V_{\tilde{\mathcal{T}}(b, \tilde{b})} = V_{\tilde{\mathcal{T}}}$ (the section of the orbits in $\mathfrak{r}_1(b, \tilde{b})^*$) such that $0 \notin K(b, \tilde{b})$ and $\dot{F}(b, \tilde{b}, \tilde{l}; \cdot, \cdot, \cdot, \cdot) \equiv 0$ if $\tilde{l} \notin K(b, \tilde{b})$. In fact, let $K(b)$ be the compact subset of $V_{T_b} = V_T$ introduced in (3.3) and $L_{(b, \tilde{b})} = \{l \in V_T \mid l|_{\langle \tilde{b} \rangle} \equiv 0\}$, which is a closed subset of V_T . Then $A(b, \tilde{b}) = K(b) \cap L_{(b, \tilde{b})}$ is a compact subset of V_T and $L(b, \tilde{b})$. Moreover the map

$$\begin{aligned} L_{(b, \tilde{b})} &\rightarrow V_{\tilde{\mathcal{T}}(b, \tilde{b})} = V_{\tilde{\mathcal{T}}} \\ l &\mapsto \tilde{l} \end{aligned}$$

defined by $\tilde{l}(\tilde{u}) = l(u)$ for any $u \in \mathfrak{p}_1(b, \tilde{b})$ such that $\tilde{u} = u \bmod \langle Y, \tilde{b} \rangle$ is a continuous bijection from $L_{(b, \tilde{b})}$ onto $V_{\tilde{\mathcal{T}}}$. Let $K(b, \tilde{b})$ be the image of $A(b, \tilde{b})$ under this map. Hence $K(b, \tilde{b})$ is a compact subset of $V_{\tilde{\mathcal{T}}}$ such that $0 \notin K(b, \tilde{b})$. By construction, $\tilde{l} \in V_{\tilde{\mathcal{T}}} \setminus K(b, \tilde{b})$ implies that $l \in V_T \setminus K(b)$ and hence that $F(b, l; \cdot, \cdot, \cdot) \equiv 0$ and $\dot{F}(b, \tilde{b}, \tilde{l}; \cdot, \cdot, \cdot, \cdot) \equiv 0$. This proves condition (iii) of (3.3).

To prove (iv) of (3.3), let's take $(b_0, \tilde{b}_0, \tilde{l}_0) \in (\mathcal{E}_1 \times V_{\tilde{\mathcal{T}}}) \setminus \tilde{\mathcal{D}}_{\tilde{\mathcal{T}},gen}$. Let $(b_0, l_0) = \Phi(b_0, \tilde{b}_0, \tilde{l}_0)$. By (2.14), $(b_0, l_0) \notin \mathcal{D}_{T,gen}$. Hence, by (3.3), there exists $\mathcal{V}_{(b_0, l_0)}$, open neighborhood of (b_0, l_0) in $\mathcal{B} \times V_T$ such that $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ for every $(b, l) \in \mathcal{V}_{(b_0, l_0)}$. Then $\Phi^{-1}(\mathcal{V}_{(b_0, l_0)})$ is an open neighborhood of $(b_0, \tilde{b}_0, \tilde{l}_0)$ in $\mathcal{E}_1 \times V_{\tilde{\mathcal{T}}}$. Let $(b, \tilde{b}, \tilde{l}) \in \Phi^{-1}(\mathcal{V}_{(b_0, l_0)})$ and let's put $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$. Then $(b, l) \in \mathcal{V}_{(b_0, l_0)}$ and hence $F(b, l, \cdot, \cdot, \cdot) \equiv 0$. But then, by definition of \dot{F} , $\dot{F}(b, \tilde{b}, \tilde{l}, \cdot, \cdot, \cdot, \cdot) \equiv 0$.

So the function \dot{F} satisfies the hypotheses of theorem (4.1).

Let's write $\dot{\tilde{\pi}}_{(b, \tilde{b}, \tilde{l})}$ for the irreducible unitary representation of $R_1(b, \tilde{b})$, the polarization $\mathfrak{p}_{(b, \tilde{b}, \tilde{l})}$ of \tilde{l} in $\mathfrak{r}_1(b, \tilde{b})$ being given by

$$\mathfrak{p}_{(b, \tilde{b}, \tilde{l})} = \mathfrak{p}_{(b, l)} \bmod \langle Y, \tilde{b} \rangle = \mathfrak{p}_d(b, l) \bmod \langle Y, \tilde{b} \rangle,$$

where $\mathfrak{p}_{(b,l)} = \mathfrak{p}_d(b, l)$ is the Vergne polarization for l in \mathfrak{g}_b . Then $\mathfrak{p}_{(b,\tilde{b},\tilde{l})}$ is the Vergne polarization for \tilde{l} in $\mathfrak{r}_1(b, \tilde{b})$ with respect to the given basis. By the recurrence hypothesis there exists $f_0 \in \mathcal{S}(R_1, \mathcal{E}_1, \mathbb{R}^{r+2})$ such that the operator $\dot{\tilde{\pi}}_{(b,\tilde{b},\tilde{l})}(f_0(b, \tilde{b}, s, t, \alpha, \cdot))$ has $\dot{F}(b, \tilde{b}, \tilde{l}; s, t, \alpha; \cdot, \cdot)$ as a kernel for every $(b, \tilde{b}, \tilde{l}) \in \tilde{\mathcal{D}}_{\tilde{T}, gen}$. In particular, for any fixed (b, \tilde{b}) , $f_0(b, \tilde{b}, s, t, \alpha, \cdot) \in \mathcal{S}(P_1(b, \tilde{b})/\exp_b < Y, \tilde{b} >)$.

6.3

Assume that $Z_b \notin < \tilde{b} >$. Let's recall that $\mathcal{V}_{j_1} := < Z_{j_1+1}, \dots, Z_n >$. Then $\mathcal{V}_{j_1} / < \tilde{b} > \equiv \mathbb{R}Z_b \text{ mod } < \tilde{b} >$ for all (b, \tilde{b}) and $< Y, Z_{j_1+1}, \dots, Z_n > / < \tilde{b} > \equiv < Y, Z_b > \text{ mod } < \tilde{b} >$. For $l \in L_{(b,\tilde{b})}$ such that $< l, Z_b > \neq 0$, let's put $c_l = < l, Z_b > (\neq 0)$ and write $\chi_{(b,\tilde{b},l)}$ for the character defined on

$$\exp_{(b,\tilde{b})} \left(< Y, Z_{j_1+1}, \dots, Z_n > / < \tilde{b} > \right) \equiv \exp_b(< Y, Z_b >)$$

by $\chi_{(b,\tilde{b},l)}(u) = e^{-i<l, \log_{(b,\tilde{b})} u>}$. In particular, $\chi_{(b,\tilde{b},c_l)}(\exp_b(rZ_b)) := \chi_{(b,\tilde{b},l)}(\exp_b(rZ_b)) = e^{-irc_l}$. We then define a function f_1 by

$$\begin{aligned} f_1(b, \tilde{b}, s, t, \alpha, g_1, l) &:= \int_{\exp_{(b,\tilde{b})}(\mathcal{V}_{j_1}/<\tilde{b}>)} f_0(b, \tilde{b}, s, t, \alpha, g_1 \cdot z) \chi_{(b,\tilde{b},l)}(z) dz \\ &= \int_{\mathbb{R}} f_0(b, \tilde{b}, s, t, \alpha, g_1 \cdot \exp_b(vZ_b)) e^{-ivc_l} dv \quad \text{if } Z_b \notin < \tilde{b} > \end{aligned}$$

for $g_1 \in R_1(b, \tilde{b}) = \exp_{(b,\tilde{b})} \mathfrak{r}_1(b, \tilde{b})$. We write $f_1(b, \tilde{b}, s, t, \alpha; g_1, c_l)$ and notice that, for fixed $(b, \tilde{b}, s, t, \alpha)$, $f_1(b, \tilde{b}, s, t, \alpha; \cdot, c) \in \mathcal{S}(P_1(b, \tilde{b})/\exp_b < Y, Z_{j_1+1}, \dots, Z_n >, \chi_{(b,\tilde{b},c)})$, which means that it is a Schwartz function on $P_1(b, \tilde{b})/\exp_b < Y, Z_{j_1+1}, \dots, Z_n >$ (for any fixed basis), satisfying the covariance condition

$$f_1(b, \tilde{b}, s, t, \alpha; g_1 \cdot \exp_b(yY) \cdot \exp_b(zZ_b) \cdot z_{\tilde{b}}, c) = e^{icz} f_1(b, \tilde{b}, s, t, \alpha; g_1, c)$$

for any $z_{\tilde{b}} \in \exp_b(< \tilde{b} >)$ (covariance for the character $\chi_{(b,\tilde{b},c)} = \chi_{(b,\tilde{b},l_c)}$ on $< Y, Z_b, \tilde{b} > = < Y, Z_{j_1+1}, \dots, Z_n >$, where the linear form l_c satisfies $< l_c, Y > = < l_c, \tilde{b}_j > = 0$ for all j and $< l_c, Z_b > = c$). It is Schwartz in s, t, α, c and \mathcal{C}^∞ in b, \tilde{b} . Moreover, as $\dot{F}(b, \tilde{b}, \tilde{l}, \cdot, \cdot, \cdot, \cdot) \equiv 0$ if $\tilde{l} \notin K(b, \tilde{b})$, there exists a constant $C(b, \tilde{b})$ such that $\dot{\tilde{\pi}}_{(b,\tilde{b},\tilde{l})}(f_0(b, \tilde{b}, \cdot, \cdot, \cdot, \cdot)) \equiv 0$ if $|< \tilde{l}, \tilde{Z}_b >| = |c_l| < C(b, \tilde{b})$ (as $K(b, \tilde{b}) \subset \tilde{\mathcal{D}}_{\tilde{T}, gen}$ is compact and as $< \tilde{l}, \tilde{Z}_b > \neq 0$ for $\tilde{l} \in K(b, \tilde{b})$). In that case,

$$\dot{\tilde{\pi}}_{(b,\tilde{b},\tilde{l})}(f_0(b, \tilde{b}, \cdot, \cdot, \cdot, \cdot)) = \int_{R_1(b,\tilde{b})/\exp_b(\mathbb{R}Z_b)} \dot{\tilde{\pi}}_{(b,\tilde{b},\tilde{l})}(g_1) \left[\int_{\mathbb{R}} f_0(b, \tilde{b}, \cdot, \cdot, \cdot, g_1 \exp_b(zZ_b)) e^{-ic_l z} dz \right] dg_1 = 0$$

for any \tilde{l} such that $|c_l| < C(b, \tilde{b})$ and hence

$$f_1(b, \tilde{b}, \cdot, \cdot, \cdot, g_1, c_l) = \int_{\mathbb{R}} f_0(b, \tilde{b}, \cdot, \cdot, \cdot, g_1 \cdot \exp_b(zZ_b)) e^{-izc_l} dz = 0,$$

by (1.5), if $|c_l| < C(b, \tilde{b})$. This is in particular true if $< l, Z_b > = c_l = 0$. Let's now define a function $\tilde{R}F$ by

$$\tilde{R}F(b, \tilde{b}, \alpha, \exp_b(sX) \cdot g_1, l) = \tilde{R}F(b, \tilde{b}, \alpha, \exp_b(sX) \cdot g_1, c_l) := \int_{\mathbb{R}} f_1(b, \tilde{b}, s, t, \alpha, g_1^{-t}, c_l) dt$$

if $c_l = < l, Z_b >$, where $g_1^{-t} = \exp_b(-tX) \cdot g_1 \cdot \exp_b(tX)$ and $g_1 \in P_1(b, \tilde{b})/\exp_b(< \tilde{b} >)$. For $g_1 = w \cdot \exp_b(yY) \cdot \exp_b(zZ_b) \text{ mod } \exp_b(< \tilde{b} >)$, the computation shows that

$$\tilde{R}F(b, \tilde{b}, \alpha, \exp_b(sX) \cdot w \cdot \exp_b(yY) \cdot \exp_b(zZ_b), c) = \frac{1}{c} e^{icz} \int_{\mathbb{R}} f_1(b, \tilde{b}, s, \frac{t}{c}, \alpha, w^{-\frac{t}{c}}, c) e^{-ity} dt,$$

if $c \neq 0$ and $\tilde{R}F(b, \tilde{b}, \alpha, \cdot, 0) = 0$. Let's notice that for fixed b, \tilde{b} this is a Schwartz function in α, s, w, y, c , as $f_1(b, \tilde{b}, \cdot, \cdot, \cdot, c) \equiv 0$ if $|c| < C(b, \tilde{b})$ (especially if $c = 0$) and

$$\tilde{R}F(b, \tilde{b}, \alpha, \cdot, c) \in \mathcal{S}(G_b/\exp_b(Z_b, \tilde{b}), \chi_{(b, \tilde{b}, c)}) = \mathcal{S}(G_b/\exp_b \langle Z_{j_1+1}, \dots, Z_n \rangle, \chi_{(b, \tilde{b}, c)})$$

i. e. it satisfies the covariance condition

$$\tilde{R}F(b, \tilde{b}, \alpha, g \cdot \exp_b(zZ_b), c) = e^{icz} \tilde{R}F(b, \tilde{b}, \alpha, g, c)$$

as f_1 does.

Finally we get a function RF by

$$\begin{aligned} RF(b, \tilde{b}, \alpha, g) &:= \int_{\mathbb{R}} \tilde{R}F(b, \tilde{b}, \alpha, \exp_b(sX) \cdot w \cdot \exp_b(yY) \cdot \exp_b(zZ_b), c) dc \\ &= \int_{\mathbb{R}} \tilde{R}F(b, \tilde{b}, \alpha, \exp_b(sX) \cdot w \cdot \exp_b(yY), c) e^{icz} dc \end{aligned}$$

if $g = \exp_b(sX) \cdot w \cdot \exp_b(yY) \cdot \exp_b(zZ_b) \bmod \exp_b(\langle \tilde{b} \rangle)$. As $\tilde{R}F$ is Schwartz in c , RF is Schwartz in z . In particular, for fixed b, \tilde{b}, α , $RF(b, \tilde{b}, \alpha, \cdot) \in \mathcal{S}(G_b/\langle \tilde{b} \rangle)$. Let's also notice that \hat{F} depends only on $b, \langle \tilde{b} \rangle$ (instead of b, \tilde{b}). So the same is true for $f_0, f_1, \tilde{R}F, RF$. By construction, RF is \mathcal{C}^∞ in b, \tilde{b} and Schwartz in α . So, for fixed b, α , we have defined a Schwartz function on $G_b/\exp_b(\langle \tilde{b} \rangle)$ for all \tilde{b} such that $Z_b \notin \langle \tilde{b} \rangle$.

The Fourier inversion theorem implies that

$$\hat{R}F^{Z_b}(b, \tilde{b}, \alpha, g, c) := \int_{\mathbb{R}} RF(b, \tilde{b}, \alpha, g \cdot \exp_b(zZ_b)) e^{-icz} dz = \frac{1}{2\pi} \tilde{R}F(b, \tilde{b}, \alpha, g, c),$$

where $\hat{R}F^{Z_b}$ denotes the partial Fourier transform in the direction of Z_b .

6.4

Let $\mathcal{E}, \mathcal{E}_1$ be as in (2.9). Let's recall that we have defined $RF(b, \tilde{b}, \cdot, \cdot)$ for all $(b, \tilde{b}) \in \mathcal{E}_1$, i. e. for all (b, \tilde{b}) such that $Z_b \notin \langle \tilde{b} \rangle$. Let's also recall that by assumption on \mathcal{B} there exists at least one $l \in \mathfrak{g}^*$ such that $(b, l) \in \mathcal{D}_{T, gen}$. We now define

$$RF(b, \tilde{b}, \cdot, \cdot) := 0 \text{ if } Z_b \in \langle \tilde{b} \rangle.$$

We have to prove that this extension of the function RF to \mathcal{E} is \mathcal{C}^∞ in (b, \tilde{b}) . To achieve this, let $(b_0, \tilde{b}_0) \in \mathcal{E}$ such that $Z_{b_0} \in \langle \tilde{b}_0 \rangle$. We shall show that there exists an open neighborhood \mathcal{O} of (b_0, \tilde{b}_0) such that $RF(b, \tilde{b}, \cdot, \cdot) \equiv 0$ for all $(b, \tilde{b}) \in \mathcal{O} \cap \mathcal{E}_1$. This will imply that the extension of RF is \mathcal{C}^∞ in (b, \tilde{b}) at (b_0, \tilde{b}_0) .

Let $l \in \mathfrak{g}^*$ such that $l|_{\mathcal{V}_{j_1}} \neq 0$ and let's write

$$l|_{\mathcal{V}_{j_1}} = l_{j_1+1} Z_{j_1+1}^* + \dots + l_n Z_n^*.$$

Hence $\ker(l|_{\mathcal{V}_{j_1}})$ is a hyperplane of \mathcal{V}_{j_1} which admits $\frac{1}{\sqrt{l_{j_1+1}^2 + \dots + l_n^2}}(l_{j_1+1}, \dots, l_n)$ (expressed in the basis $Z_{j_1+1}^*, \dots, Z_n^*$) as a unit normal vector. In fact, $(a_{j_1+1}, \dots, a_n) = a_{j_1+1} Z_{j_1+1} + \dots + a_n Z_n \in \ker(l|_{\mathcal{V}_{j_1}})$ if and only if $a_{j_1+1} l_{j_1+1} + \dots + a_n l_n = 0$. So

$$\begin{aligned} Z_b \in \ker(l|_{\mathcal{V}_{j_1}}) &\Leftrightarrow (l_{j_1+1}, \dots, l_n) \cdot Z_b = 0 \\ &\Leftrightarrow \frac{(l_{j_1+1}, \dots, l_n) \cdot Z_b}{\sqrt{l_{j_1+1}^2 + \dots + l_n^2} \|Z_b\|} = 0 \end{aligned}$$

where \cdot denotes the scalar product and where Z_b is also expressed in the basis Z_{j_1+1}, \dots, Z_n . Let's consider

$$\mathcal{N} := \{(b, l) \in \mathcal{B}_1 \times V_T \mid \langle l, Z_b \rangle = 0\}$$

and $(b_0, l_0) \in \mathcal{N}$ such that $\ker l_0|_{\mathcal{V}_{j_1}} = \langle \tilde{b}_0 \rangle$. Let \mathcal{K} be any compact neighborhood of $(b_0, l_0) \in \mathcal{B}_1 \times V_T$ and let $\text{int}(\mathcal{K})$ be its interior. By (3.3, (iv)), there exists for all $(b_1, l_1) \in \mathcal{N}$ an open neighborhood $\mathcal{V}_{(b_1, l_1)}$ in $\mathcal{B}_1 \times V_T$ such that $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ for all $(b, l) \in \mathcal{V}_{(b_1, l_1)}$. Let's put

$$\mathcal{V} := \bigcup_{(b_1, l_1) \in \mathcal{N} \cap \mathcal{K}} \mathcal{V}_{(b_1, l_1)}.$$

Then \mathcal{V} is an open subset of $\mathcal{B}_1 \times V_T$, $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ for all $(b, l) \in \mathcal{V}$ and $\mathcal{K}_1 := \mathcal{K} \cap ((\mathcal{B}_1 \times V_T) \setminus \mathcal{V})$ is compact. By construction, $\langle l, Z_b \rangle \neq 0$ on \mathcal{K}_1 .

Let's now define

$$\mathcal{W} := \{l \in \mathfrak{g}^* \mid l|_{\mathcal{V}_{j_1}} \neq 0\}$$

(Zariski open subset of \mathfrak{g}^*) and

$$f : \mathcal{K} \cap \mathcal{W} \rightarrow \mathbb{R}$$

by

$$f(b, l) := \left| \frac{(l_{j_1+1}, \dots, l_n) \cdot Z_b}{\sqrt{l_{j_1+1}^2 + \dots + l_n^2} \|Z_b\|} \right|.$$

This function f is continuous and strictly positive on the compact set \mathcal{K}_1 . It represents of course the cosine of the angle between the directions of \mathcal{V}_{j_1} given by Z_b and by a normal vector to $\ker l|_{\mathcal{V}_{j_1}}$. Let

$$\varepsilon := \min\{f(b, l) \mid (b, l) \in \mathcal{K}_1\}.$$

Then $\varepsilon > 0$ and

$$\mathcal{U} := \{(b, l) \in \text{int}(\mathcal{K}) \cap \mathcal{W} \mid f(b, l) < \frac{\varepsilon}{2}\}$$

is a nonempty open subset of $\mathcal{B}_1 \times V_T$, contained in \mathcal{K} , containing (b_0, l_0) as $f(b_0, l_0) = 0$. We then define

$$\begin{aligned} \Psi : \mathcal{E}' &\rightarrow \mathcal{B}_1 \times V_T \\ (b, \tilde{b}) &\mapsto (b, (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^*) \end{aligned}$$

where $(\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^*$ is defined by

$$\begin{aligned} (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^* (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1}) &= 1 \\ (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^* (\tilde{b}_j) &= 0 \quad \text{for } j = 1, \dots, m-1 \\ (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^* (Z_k) &= 0 \quad \text{for } k \leq j_1 \end{aligned}$$

Obviously, $(\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^* \in V_T$, as its coordinates at the Pukanszky jump indices are zero. Of course, the coordinates of $(\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^*$ in the basis $Z_{j_1+1}^*, \dots, Z_n^*$ coincide with the coordinates of $\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1}$ in the basis Z_{j_1+1}, \dots, Z_n .

The function Ψ is a continuous map. Let's also notice that $\mathfrak{Im} \Psi \cap \mathcal{U} \neq \emptyset$. In fact, let $l \in V_T$ be such that the angle of Z_{b_0} with a normal vector to $\ker l|_{\mathcal{V}_{j_1}}$ is very close to $\frac{\pi}{2}$, but different from it, so that $0 < f(b_0, l) < \frac{\varepsilon}{2}$. Take now \tilde{b} to be an arbitrary basis of $\ker(l|_{\mathcal{V}_{j_1}})$. Then $Z_{b_0} \notin \langle \tilde{b} \rangle$, $(\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^* = kl|_{\mathcal{V}_{j_1}}$ for some constant k and $f(b_0, (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^*) = f(b_0, l) < \frac{\varepsilon}{2}$. So $(b_0, (\tilde{b}_1 \wedge \dots \wedge \tilde{b}_{m-1})^*) \in \mathcal{U} \cap \mathfrak{Im} \Psi$. Hence $\mathcal{O} = \Psi^{-1}(\mathcal{U})$ is a nonempty open subset of \mathcal{E}' , containing (b_0, \tilde{b}_0) as $f(b_0, ((\tilde{b}_0)_1 \wedge \dots \wedge (\tilde{b}_0)_{m-1})^*) = 0$.

Let now $(b, \tilde{b}) \in \mathcal{O} \cap \mathcal{E}'_1$, i. e. such that $Z_b \notin \langle \tilde{b} \rangle$. Let $(b, l) = \Phi(b, \tilde{b}, \tilde{l})$ with $l = (\tilde{b}_1 \wedge \cdots \wedge \tilde{b}_{m-1})^* \in V_T$ and \tilde{l} the corresponding element of \mathfrak{r}_1^* (see 2.13 for the definition of Φ). In particular, $(b, l) = \Psi(b, \tilde{b}) \in \mathcal{U}$ and $f(b, l) < \frac{\varepsilon}{2}$. Hence $(b, l) \in \mathcal{K} \setminus \mathcal{K}_1$, i. e. $(b, l) \in \mathcal{V}$ and $F(b, l, \cdot, \cdot, \cdot) \equiv 0$. This implies that $\dot{F}(b, \tilde{b}, \tilde{l}, \cdot, \cdot, \cdot, \cdot) \equiv 0$. So $f_0(b, \tilde{b}, \cdot, \cdot, \cdot, \cdot) \equiv 0$ and $RF(b, \tilde{b}, \cdot, \cdot) \equiv 0$, by the constructions of (6.3). This proves our claim.

6.5

Let's again assume that $Z_b \notin \langle \tilde{b} \rangle$. Let $\tilde{l} \in \mathfrak{p}_1(b, \tilde{b})^*$ (\tilde{l} being in the chosen orbit section) such that $\tilde{l}|_{\langle \tilde{b} \rangle} \equiv 0$ and $\langle \tilde{l}, Y \rangle = 0$. Then the representations of $P_1(b, \tilde{b})$ and $P_1(b, \tilde{b})/\exp_b(\langle \tilde{b} \rangle)$ induced from the character $\chi_{(b, \tilde{b}, \tilde{l})}$ act on the same L^2 -space and may be identified. Let's write $\tilde{\pi}_{(b, \tilde{b}, \tilde{l})}$ for both of them. We have, for $c = \langle \tilde{l}, Z_b \rangle$, for $RF(b, \tilde{b}, \alpha, \exp_b(sX)(\cdot))$ considered as a function on $P_1(b, \tilde{b})/\langle \tilde{b} \rangle$,

$$\begin{aligned}
\tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(RF(b, \tilde{b}, \alpha, \exp_b(sX)(\cdot))) &= \int \int \int RF(b, \tilde{b}, \alpha, \exp_b(sX) \cdot w \cdot \exp_b(yY) \\
&\quad \cdot \exp_b(zZ_b)) \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w \cdot \exp_b(yY) \cdot \exp_b(zZ_b)) d\tilde{w} d\tilde{y} d\tilde{z} \\
&= \int \int \int RF(b, \tilde{b}, \alpha, \exp_b(sX) \cdot w \cdot \exp_b(yY) \\
&\quad \cdot \exp_b(zZ_b)) \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w) e^{-icz} d\tilde{w} d\tilde{y} d\tilde{z} \\
&= \frac{1}{2\pi} \int \int \tilde{R}F(b, \tilde{b}, \alpha, \exp_b(sX) \cdot w \cdot \exp_b(yY), c) \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w) d\tilde{w} d\tilde{y} \\
&= \frac{1}{2\pi} \int \int \int f_1(b, \tilde{b}, s, t, \alpha, w^{-t} \cdot \exp_b(yY) \cdot \exp_b(-ytZ_b), c) \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w) d\tilde{w} d\tilde{y} d\tilde{t} \\
&= \frac{1}{2\pi} \int \int \int f_1(b, \tilde{b}, s, t, \alpha, w^{-t}, c) e^{-icyt} \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w) d\tilde{w} d\tilde{y} d\tilde{t}
\end{aligned}$$

Let now $l \in \mathfrak{g}^* \cap V_T$ such that $(b, l) \in \mathcal{D}_{T,gen}$ and $l|_{\langle \tilde{b} \rangle} \equiv 0$. In particular, $\langle l, X \rangle = \langle l, Y \rangle = 0$, by the choice of the orbit section, and $c = \langle l, Z_b \rangle \neq 0$ as $(b, l) \in \mathcal{D}_{T,gen}$. Let \tilde{l} be the corresponding linear form on $\mathfrak{r}_1(b, \tilde{b})$ and $\tilde{\pi}_{(b, \tilde{b}, \tilde{l})}$ the associated representation of $R_1(b, \tilde{b})$. Let's also notice that $\pi_{(b, l)}$ may be considered as a representation of G_b and of $G_b/\exp_b(\langle \tilde{b} \rangle)$, as $\langle \tilde{b} \rangle$ is included in the polarization of l and as $l|_{\langle \tilde{b} \rangle} \equiv 0$. This identification will be made in

the following computations:

$$\begin{aligned}
\left(\pi_{(b,l)}(RF(b, \tilde{b}, \alpha, \cdot))\xi\right)(s)(g_1) &= \int \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}\left(RF(b, \tilde{b}, \alpha, \exp_b(s-t)X \cdot (\cdot)^t)\right)\xi(t)(g_1)dt \\
&= \int \int \int \int RF(b, \tilde{b}, \alpha, \exp_b(s-t)X \cdot (w \exp_b y Y \exp_b z Z_b)^t) \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)e^{-icz}\xi(t)(g_1)d\dot{w}dydzdt \\
&= \frac{1}{2\pi} \int \int \int \tilde{R}F(b, \tilde{b}, \alpha, \exp_b(s-t)X \cdot (w \exp_b y Y)^t, c) \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)\xi(t)(g_1)d\dot{w}dydt \\
&= \frac{1}{2\pi} \int \int \int \left(\int f_1(b, \tilde{b}, s-t, u, \alpha, (w \exp_b y Y)^{t-u}, c)du\right) \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)\xi(t)(g_1)d\dot{w}dydt \\
&= \frac{1}{2\pi} \int \int \int \int f_1(b, \tilde{b}, s-t, u, \alpha, w^{t-u} \exp_b y Y \exp_b(t-u)y Z_b, c) \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)\xi(t)(g_1)dud\dot{w}dydt \\
&= \frac{1}{2\pi} \int \int \int \int f_1(b, \tilde{b}, s-t, u, \alpha, w^{t-u}, c)e^{ic(t-u)y} \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)\xi(t)(g_1)dud\dot{w}dydt \\
&= \frac{1}{2\pi} \int \int \int \int \left(\int f_0(b, \tilde{b}, s-t, u, \alpha, w^{t-u} \exp_b r Z_b)e^{-icr} dr\right) \\
&\quad e^{ic(t-u)y} \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)\xi(t)(g_1)dud\dot{w}dydt \\
&= \left(\frac{1}{2\pi}\right)^2 \frac{1}{|c|} \int \int \int f_0(b, \tilde{b}, s-t, t, \alpha, w \exp_b r Z_b)e^{-icr} \\
&\quad \tilde{\pi}_{(b, \tilde{b}, \tilde{l})}(w)\xi(t)(g_1)d\dot{w}tdr \\
&= \left(\frac{1}{2\pi}\right)^2 \frac{1}{|c|} \int \int \int f_0(b, \tilde{b}, s-t, t, \alpha, w \exp_b r Z_b) \\
&\quad \dot{\tilde{\pi}}_{(b, \tilde{b}, \tilde{l})}(w \exp_b r Z_b)\xi(t)(g_1)d\dot{w}tdr \\
&= \left(\frac{1}{2\pi}\right)^2 \frac{1}{|c|} \int \dot{\tilde{\pi}}_{(b, \tilde{b}, \tilde{l})}\left(f_0(b, \tilde{b}, s-t, t, \alpha, (\cdot))\right)\xi(t)(\dot{g}_1)dt \\
&= \left(\frac{1}{2\pi}\right)^2 \frac{1}{|c|} \int \int_{R_1(b, \tilde{b})/P_{(b, \tilde{b}, \tilde{l})}} \dot{F}(b, \tilde{b}, \tilde{l}, s-t, t, \alpha, \dot{g}_1, \dot{g}'_1)\xi(t)(\dot{g}'_1)d\dot{g}'_1 dt \\
&= \int \int_{P_1(b, \tilde{b})/P_{(b, l)}} F(b, l, \alpha, \exp_b s X \cdot g_1, \exp_b t X \cdot g'_1) \\
&\quad \xi(\exp_b t X \cdot g'_1)d\dot{g}'_1 dt
\end{aligned}$$

where $P_{(b, \tilde{b}, \tilde{l})} = \exp_{(b, \tilde{b})} \mathfrak{p}_{(b, \tilde{b}, \tilde{l})}$, $P_{(b, l)} = \exp_b \mathfrak{p}_{(b, l)}$ and where the respective polarizations $\mathfrak{p}_{(b, \tilde{b}, \tilde{l})}$ and $\mathfrak{p}_{(b, l)}$ satisfy $\mathfrak{p}_{(b, \tilde{b}, \tilde{l})} = \mathfrak{p}_{(b, l)} \bmod \langle Y, \tilde{b} \rangle$. We hence have constructed a function $RF(b, \tilde{b}, \alpha, \dot{g})$, with $\dot{g} \in G_b/\exp_b(\langle \tilde{b} \rangle)$ for fixed b, \tilde{b} with $Z_b \notin \langle \tilde{b} \rangle$, such that the operator $\pi_{(l, b)}(RF(b, \tilde{b}, \alpha, \cdot))$ has $F(b, l, \alpha, \cdot, \cdot)$ as a kernel for every l such that $l|_{\langle \tilde{b} \rangle} \equiv 0$, if $\pi_{(l, b)}$ is considered as a representation of $G_b/\exp_b(\langle \tilde{b} \rangle)$.

6.6

Now we have to construct a function $f(b, \alpha, \cdot) = f_b(\alpha)(\cdot)$ on all of G_b such that the operator $\pi_{(l,b)}(f_b(\alpha)(\cdot))$ has $F(b, l, \alpha, \cdot, \cdot)$ as a kernel for $(b, l) \in \mathcal{D}_{T,gen}$, if $\pi_{(l,b)}$ is considered as a representation of G_b . This will be done via the Radon transform. For this purpose, let's first change the parametrization, in order to meet the hypothesis of the theory of the Radon transform. Let's note $\varepsilon(\tilde{b})$ for the sign of $Z_b \cdot (\tilde{b}_1 \wedge \cdots \wedge \tilde{b}_{m-1}) = \det(Z_b, \tilde{b}_1, \dots, \tilde{b}_{m-1})$, in the case where $Z_b \notin \langle \tilde{b} \rangle$. Let's then define

$$\omega_{\tilde{b}} = \varepsilon(\tilde{b}) \frac{\tilde{b}_1 \wedge \cdots \wedge \tilde{b}_{m-1}}{\|\tilde{b}_1 \wedge \cdots \wedge \tilde{b}_{m-1}\|},$$

i. e. $\omega_{\tilde{b}}$ is the normal unit vector to the hyperplane $\langle \tilde{b} \rangle$ pointing to the same halfspace than Z_b . Let's put $k(b, \tilde{b}) = Z_b \cdot \omega_{\tilde{b}}$ (\cdot being the scalar product). Then $k(b, \tilde{b})$ is a \mathcal{C}^∞ function in b, \tilde{b} which is strictly positive if $Z_b \notin \langle \tilde{b} \rangle$. Each element u of \mathcal{V}_{j_1} admits then two different decompositions

$$u = \alpha_1 \tilde{b}_1 + \cdots + \alpha_{m-1} \tilde{b}_{m-1} + \alpha_m Z_b = \beta_1 \tilde{b}_1 + \cdots + \beta_{m-1} \tilde{b}_{m-1} + \beta_m \omega_{\tilde{b}}.$$

Then

$$\beta_m = u \cdot \omega_{\tilde{b}} = \alpha_m (Z_b \cdot \omega_{\tilde{b}}) = \alpha_m k(b, \tilde{b})$$

and

$$Z_b = k(b, \tilde{b}) \omega_{\tilde{b}} + a(b, \tilde{b})$$

for some $a(b, \tilde{b}) \in \langle \tilde{b} \rangle$, the coordinates of $a(b, \tilde{b})$ being \mathcal{C}^∞ functions of b, \tilde{b} .

Let's now define a function

$$g_{(b,\tilde{b})}(\alpha)(\dot{w}; (\omega_{\tilde{b}}, r)) := \frac{1}{k(b, \tilde{b})} RF(b, \tilde{b}, \alpha, \dot{w} \exp_b(r \omega_{\tilde{b}})) = \frac{1}{k(b, \tilde{b})} RF(b, \tilde{b}, \alpha, \dot{w} \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)),$$

with $\dot{w} \in G_b/\mathcal{V}_{j_1}$ (expressed in a fixed basis) and where the last equality is justified by the fact that RF is defined mod $\exp_b(\langle \tilde{b} \rangle)$ and that $a(b, \tilde{b}) \in \langle \tilde{b} \rangle$. Moreover, let's put

$$g_{(b,\tilde{b})}(\alpha)(\cdot) \equiv 0, \quad \text{if } Z_b \in \langle \tilde{b} \rangle.$$

The function g is defined mod $\langle \tilde{b} \rangle$, \mathcal{C}^∞ in b, \tilde{b} by (6.4), Schwartz in $\omega_{\tilde{b}}, \dot{w}, r$ (as it is \mathcal{C}^∞ in $\omega_{\tilde{b}}$ and as $\omega_{\tilde{b}}$ is bounded). Moreover, as $r \omega_{\tilde{b}} = (-r)(-\omega_{\tilde{b}})$, we have

$$g_{(b,\tilde{b})}(\alpha)(\dot{w}; (\omega_{\tilde{b}}, r)) = g_{(b,\tilde{b})}(\alpha)(\dot{w}; (-\omega_{\tilde{b}}, -r)).$$

In order to apply the special result of the Radon transform (see 5.2), we still have to check that

$$\int_{\mathbb{R}} r^k g_{(b,\tilde{b})}(\alpha)(\dot{w}; (\omega_{\tilde{b}}, r)) dr = \int_{\mathbb{R}} r^k \frac{1}{k(b, \tilde{b})} RF(b, \tilde{b}, \alpha, \dot{w} \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) dr = 0, \quad \forall k \in \mathbb{N}.$$

Let's notice that this is equivalent to the fact that

$$\frac{\partial^k}{\partial c^k} \hat{R}F^{Z_b}(b, \tilde{b}, \alpha, \dot{w}, c)|_{c=0} = 0, \quad \forall k \in \mathbb{N},$$

where $\hat{R}F^{Z_b}$ denotes the partial Fourier transform in the Z_b direction. In order to check that this is true, let's recall that $F(b, l, \cdot, \cdot, \cdot) \equiv 0$ if $l \notin K(b)$, where $K(b)$ is a compact subset of V_T not containing 0 and hence, that $\hat{F}(b, \tilde{l}, \cdot, \cdot, \cdot, \cdot) \equiv 0$ if $\tilde{l} \notin K(b, \tilde{b})$ and in particular if \tilde{l} is in a certain neighborhood of 0. As in (6.3) this implies that there exists a constant $C(b, \tilde{b})$ such

that $f_1(b, \tilde{b}, \cdot, \cdot, \cdot, c) \equiv 0$ and $\tilde{R}F(b, \tilde{b}, \cdot, \cdot, c) \equiv 0$ if $|c| < C(b, \tilde{b})$. Finally, as $\tilde{R}F(b, \tilde{b}, \cdot, \cdot, c) = 2\pi \hat{R}F^{Z_b}(b, \tilde{b}, \cdot, \cdot, c)$, this proves that

$$\frac{\partial^k}{\partial c^k} \hat{R}F^{Z_b}(b, \tilde{b}, \alpha, \dot{w}, c)|_{c=0} = 0, \quad \forall k \in \mathbb{N}.$$

By the results on the Radon transform (5.2), there exists a unique $f \in \mathcal{S}(G, \mathcal{B}_1, \mathbb{R}^r)$ such that

$$\begin{aligned} \int_{\langle \tilde{b} \rangle} f(b, \alpha, \dot{w} \exp_b(r\omega_{\tilde{b}}) \exp_b v) dv &= \int_{r\omega_{\tilde{b}} + \langle \tilde{b} \rangle} f(b, \alpha, \dot{w} \exp_b u) du \\ &= g_{(b, \tilde{b})}(\alpha)(\dot{w}; (\omega_{\tilde{b}}, r)) \\ &= \frac{1}{k(b, \tilde{b})} RF(b, \tilde{b}, \alpha, \dot{w} \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) \end{aligned}$$

Finally, if $\ker(l|_{\mathcal{V}_{j_1}}) = \langle \tilde{b} \rangle$ and $Z_b \notin \langle \tilde{b} \rangle$, in particular if $(b, l) \in \mathcal{D}_{T, gen}$,

$$\begin{aligned} &\pi_{(b, l)}(f(b, \alpha, \cdot)) \xi(g) \\ &= \int_{G/\langle \tilde{b} \rangle} \int_{\langle \tilde{b} \rangle} f(b, \alpha, g_1 v) \pi_{(b, l)}(g_1) \xi(g) dv d\dot{g}_1 \quad \text{as } \pi_{(b, l)}(v) = 1 \\ &= \int_{G/\mathcal{V}_{j_1}} \int_{\mathbb{R}} \int_{\langle \tilde{b} \rangle} f(b, \alpha, g_2 \exp_b(r\omega_{\tilde{b}}) v) \pi_{(b, l)}(g_2 \exp_b(r\omega_{\tilde{b}})) \xi(g) dv dr d\dot{g}_2 \\ &= \int_{G/\mathcal{V}_{j_1}} \int_{\mathbb{R}} \int_{\langle \tilde{b} \rangle} f(b, \alpha, g_2 \exp_b(r\omega_{\tilde{b}}) v) \pi_{(b, l)}(g_2 \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) \\ &\quad \exp_b(-\frac{1}{k(b, \tilde{b})} a(b, \tilde{b})) \xi(g) dv dr d\dot{g}_2 \\ &= \int_{G/\mathcal{V}_{j_1}} \int_{\mathbb{R}} \left[\int_{r\omega_{\tilde{b}} + \langle \tilde{b} \rangle} f(b, \alpha, g_2 \exp_b u) du \right] \\ &\quad \pi_{(b, l)}(g_2 \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) \xi(g) dr d\dot{g}_2 \quad \text{as } \pi_{(b, l)}(\exp_b(-\frac{1}{k(b, \tilde{b})} a(b, \tilde{b}))) = 1 \\ &= \int_{G/\mathcal{V}_{j_1}} \int_{\mathbb{R}} g_{(b, \tilde{b})}(\alpha)(\dot{g}_2, (\omega_{\tilde{b}}, r)) \pi_{(b, l)}(g_2 \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) \xi(g) dr d\dot{g}_2 \\ &= \int_{G/\mathcal{V}_{j_1}} \int_{\mathbb{R}} \frac{1}{k(b, \tilde{b})} RF(b, \tilde{b}, \alpha, \dot{g}_2 \exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) \pi_{(b, l)}(g_2) \\ &\quad \pi_{(b, l)}(\exp_b(\frac{1}{k(b, \tilde{b})} r Z_b)) \xi(g) dr d\dot{g}_2 \\ &= \pi_{(b, l)}(RF(b, \tilde{b}, \alpha, \cdot)) \xi(g) \\ &= \int_{G/P(b, l)} F(b, l, \alpha, g, g') \xi(g') d\dot{g}' \end{aligned}$$

This proves that $f(b, \alpha, \cdot)$ is the function we were looking for. The uniqueness of the function f comes from the fact that if (b, l) runs through $\mathcal{D}_{T, gen}$, then, for fixed b , l runs through a dense open set of $\mathfrak{g}_b^* \cap V_T$. So, if we assume that there are two different functions f_1 and f_2 such that $\pi_{(b, l)}(f_j)(\alpha)$ has $F(b, l, \alpha, \cdot, \cdot)$ as a kernel for $j = 1, 2$, for all $(b, l) \in \mathcal{D}_{T, gen}$ and all α , $\pi_{(b, l)}(f_1(b, \alpha, \cdot)) = \pi_{(b, l)}(f_2(b, \alpha, \cdot))$ on a dense subset of \mathfrak{g}_b^* , and this implies that $f_1(b, \alpha, \cdot) = f_2(b, \alpha, \cdot)$ (everywhere by continuity). Let's finally notice that all the constructed functions are \mathcal{C}^∞ in all the variables, because, in particular, this property is respected by the construction of the inversion of the Radon transform. In fact, $f \in \mathcal{S}(G, \mathcal{B}_1, \mathbb{R}^r)$.

Let's now prove that $\pi_{(b,l)}(f) = 0$ for every $b \in \mathcal{B}_1$ and every $l \in \mathfrak{g}_b^* \setminus (\mathfrak{g}_b^*)_{gen}$. Let $b_0 \in \mathcal{B}_1$ be fixed and $l_0 \in \mathfrak{g}_{b_0}^* \setminus (\mathfrak{g}_{b_0}^*)_{gen}$. Let's take a sequence $(l_n)_{n \in \mathbb{N}^*}$ which converges to l_0 in $\mathfrak{g}_{b_0}^*$ and $l_n \in (\mathfrak{g}_{b_0}^*)_{gen}$ for all n . Let $\{l'_n\} = \Omega_{l_n} \cap V_T$ be the intersection of the orbit with the section $(\mathfrak{g}_{b_0}^*)_{gen} \cap V_T$. Let $K(b_0)$ be the compact subset of V_T , given by 3.3 (iii), such that $F(b_0, l, \cdot, \cdot, \cdot) \equiv 0$ if $l \notin K(b_0)$. We shall show that there exists N_0 such that $n \geq N_0$ implies that $l'_n \notin K(b_0)$. This will have as a consequence that $F(b_0, l'_n, \cdot, \cdot, \cdot) \equiv 0$, $\pi_{(b_0, l'_n)}(f) = 0$ and $\pi_{(b_0, l_n)}(f) = 0$ by conjugation. But we may identify the topological spaces $\mathfrak{g}_{b_0}^*/\text{Ad}^*G_{b_0}$ and $\text{Prim}_*L^1(G_{b_0})$ (with the hull-kernel topology), as the group is $*$ -regular (see [Bo-L-Sch-V]). So, as $(l_n)_n$ converges to l_0 , $\bigcap_{n \geq n_0} \text{Ker} \pi_{(b_0, l_n)} \subset \text{Ker} \pi_{(b_0, l_0)}$. Hence $\pi_{(b_0, l_0)}(f) = 0$ and it remains to prove that $l'_n \notin K(b_0)$ for all $n \geq N_0$, for some N_0 . To do that, we have to distinguish two cases: If $l_0 \in (\mathfrak{g}_{b_0}^*)_{Puk} \setminus (\mathfrak{g}_{b_0}^*)_{gen}$, then $(l'_n)_n$ converges to l'_0 in $(\mathfrak{g}_{b_0}^*)_{Puk} \cap V_T$. So, if infinitely many l'_n 's belong to the compact $K(b_0)$, $l'_0 \in K(b_0) \subset (\mathfrak{g}_{b_0}^*)_{gen}$, as $K(b_0)$ is closed in $(\mathfrak{g}_{b_0}^*)_{Puk} \cap V_T$ (which is Hausdorff). So, as $(\mathfrak{g}_{b_0}^*)_{gen}$ is G_{b_0} -invariant (see [Lu-Mu]), $l_0 \in (\mathfrak{g}_{b_0}^*)_{gen}$, which is a contradiction. Let's now assume that $l_0 \in \mathfrak{g}_{b_0}^* \setminus (\mathfrak{g}_{b_0}^*)_{Puk}$. Let's recall that there exists a G_{b_0} -invariant polynomial P such that

$$(\mathfrak{g}_{b_0}^*)_{Puk} = \{l \in \mathfrak{g}_{b_0}^* \mid P(l) \neq 0\}$$

(see [C-G]). By continuity of P and because $K(b_0)$ is compact in the Hausdorff space V_T ,

$$\delta = \min_{l \in K(b_0)} |P(l)| > 0.$$

Let's again assume that infinitely many l'_n belong to $K(b_0)$ and hence satisfy $|P(l'_n)| \geq \delta$. These l'_n converge to l'_0 in V_T and hence, by continuity of P ,

$$|P(l'_0)| = |P(l_0)| \geq \delta > 0.$$

This proves that $l_0 \in (\mathfrak{g}_{b_0}^*)_{Puk}$, contrary to our assumption.

So we have shown that $\pi_{(b_0, l_0)}(f) = 0$, if $l_0 \in \mathfrak{g}_{b_0}^* \setminus (\mathfrak{g}_{b_0}^*)_{gen}$.

Finally, by construction, the map $F \rightarrow f$ is one-to-one and continuous with respect to the given topologies.

7 Applications

7.1

The result on the inverse Fourier transform makes it possible to construct functions f whose associated operators $\pi_l(f)$ have certain prescribed properties almost everywhere. This is useful in some analysis problems.

7.2 Rank one operators

Let Z_1, \dots, Z_n be a fixed Jordan-Hölder basis of \mathfrak{g} . Let \mathfrak{p}_l be the Vergne polarization with respect to that basis and let X_1, \dots, X_d be the coexponential basis to \mathfrak{p}_l as in (2.11). Let's recall that we may take X_1, \dots, X_d fixed for all $l \in \mathfrak{g}_{gen}^*$. Thus we identify G/P_l with \mathbb{R}^d thanks to this fixed basis.

Let $\xi(l, x), \eta(l, x) \in \mathcal{S}(\mathfrak{g}_{T, gen}^* \times \mathbb{R}^d) \equiv \mathcal{S}(\mathfrak{g}_{T, gen}^*, G/P_l, \chi_l)$ such that $\xi(l, \cdot) \equiv 0$ if l is outside a compact subset of $\mathfrak{g}_{T, gen}^*$. Let's write $\xi_l(\cdot) := \xi(l, \cdot)$ and $\eta_l(\cdot) := \eta(l, \cdot)$. Then there exists a unique $f \in \mathcal{S}(G)$ such that $\pi_l(f)$ has $\xi_l \otimes \bar{\eta}_l$ as an operator kernel, for every $l \in \mathfrak{g}_{T, gen}^*$. This means that $\pi_l(f)$ is a rank one operator or 0 for all $l \in \mathfrak{g}^*$, as $\pi_l(f)\xi = \langle \xi, \eta \rangle \xi_l$ if $l \in \mathfrak{g}_{T, gen}^* \cap V_T$ and $\pi_l(f) = 0$ if $l \in \mathfrak{g}^* \setminus \mathfrak{g}_{gen}^*$.

7.3 The Heisenberg group

The case of the Heisenberg group $H_n = \exp \mathfrak{h}_n$ with $\mathfrak{h}_n = \langle X_1, \dots, X_n, Y_1, \dots, Y_n, Z \rangle$ is treated in ([Lu-M1]). For $\lambda \neq 0$, let's put $l_\lambda(x, y, z) = \lambda z$, $\chi_\lambda(x, y, z) = e^{-i\lambda z}$ and $\pi_\lambda = \text{ind}_{P_\lambda}^G \chi_\lambda$, where $P_\lambda = \exp \mathfrak{p}_\lambda = \exp \langle Y_1, \dots, Y_n, Z \rangle$. These linear forms l_λ form an orbit section of $(\mathfrak{h}_n^*)_{gen}$. If $\lambda < 0$, the function $\alpha_\lambda(s) = e^{\frac{\lambda}{2} \sum_{k=1}^n s_k^2}$ in $\mathfrak{H}_{\pi_\lambda} = L^2(H_n/P_\lambda, \chi_\lambda)$ satisfies

$$d\pi_\lambda(X_k - iY_k)\alpha_\lambda = 0, \quad k = 1, \dots, n.$$

Let now $\gamma \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ such that $\text{supp} \gamma(\lambda, s) \subset [-K, -\varepsilon] \times \mathbb{R}^n$ for arbitrary fixed $0 < \varepsilon < K$ and let's consider $F(\lambda, s, t) := \alpha_\lambda(s) \overline{\gamma(\lambda, t)}$ for $s, t \in \mathbb{R}^n$. The function F satisfies the hypotheses of theorem (4.2). So there exists $f \in \mathcal{S}(H_n)$ such that $\pi_\lambda(f^*)$ has $F(\lambda, \cdot, \cdot)$ as a kernel, i. e. such that $\pi_\lambda(f^*) = P_{\alpha_\lambda, \gamma_\lambda}$ (where $\gamma_\lambda(\cdot) = \gamma(\lambda, \cdot)$). Hence

$$\begin{aligned} d\pi_\lambda(X_k - iY_k)\pi_\lambda(f^*) &= 0 & k = 1, \dots, n; \lambda \neq 0 \\ \pi_\lambda(f^* (X_k + iY_k)) &= 0 & k = 1, \dots, n; \lambda \neq 0 \end{aligned}$$

and hence

$$f^* (X_k + iY_k) = 0 \quad k = 1, \dots, n.$$

In ([Lu-M1]), this retract f has been computed explicitly.

7.4

In ([Lu-M2]) the work on the Heisenberg group is generalized to arbitrary connected, simply connected, nilpotent Lie groups. If

$$d = \max\{\dim(\mathfrak{g}/\mathfrak{p}) \mid \exists l \in \mathfrak{g}^* \text{ s. t. } \mathfrak{p} = \mathfrak{p}(l) \text{ is a polarization for } l \text{ in } \mathfrak{g}\},$$

one constructs families of elements of the enveloping algebra $V_{1,\varepsilon}, \dots, V_{d,\varepsilon} \in \mathfrak{U}(\mathfrak{g})$ and maps $\xi_\varepsilon(l, x) \in \mathcal{S}(V_T \times \mathbb{R}^d)$ where V_T is the usual orbit section, such that $\xi_\varepsilon(l, \cdot) = \xi_{\varepsilon,l}(\cdot) \equiv 0$ outside an open subset \mathcal{A}_ε of $\mathfrak{g}_{gen}^* \cap V_T$ and such that

$$d\pi_l(V_{k,\varepsilon})\xi_{\varepsilon,l} \equiv 0.$$

One then concludes as for the Heisenberg group by applying the inverse Fourier theorem: Given any $\eta_{\varepsilon,l}(s) = \eta_\varepsilon(l, s) \in \mathcal{S}(V_T \times \mathbb{R}^d)$ such that $\eta_\varepsilon(l, \cdot) \equiv 0$ outside a compact subset of \mathcal{A}_ε , there exists $0 \neq f \in \mathcal{S}(G)$ such that $\pi_l(f)$ has $\xi_{\varepsilon,l} \otimes \bar{\eta}_{\varepsilon,l}$ as a kernel and hence that

$$\begin{aligned} d\pi_l(V_{k,\varepsilon})\pi_l(f) &= 0 & k = 1, \dots, d; \forall l \in V_T \cap \mathfrak{g}_{gen}^* \\ \pi_l(f^* * V_{k,\varepsilon}^*) &= 0 & k = 1, \dots, d; \forall l \in V_T \cap \mathfrak{g}_{gen}^*. \end{aligned}$$

So

$$f^* * V_{k,\varepsilon}^* = 0 \quad k = 1, \dots, d.$$

7.5

Let d be as in (2.3). Let B be any operator on \mathbb{R}^d defined by a Schwartz kernel $G(x, y)$, i. e. for any $\xi \in L^2(\mathbb{R}^d)$, $B\xi(x) := \int_{\mathbb{R}^d} G(x, y)\xi(y)dy$. Let now C be any compact subset of $\mathfrak{g}_{T,gen}^*$ and N an open neighborhood of C with compact closure contained in $\mathfrak{g}_{T,gen}^*$. Choose $\phi \in \mathcal{S}(V_T)$ such that $\text{supp}\phi \subset N$ and $\phi \equiv 1$ on C . Let's define $F \in \mathcal{N}_c(G, \mathfrak{g}_{T,gen}^*)$ by $F(l, x, y) := \phi(l)G(x, y)$. By (4.2) there exists $f \in \mathcal{S}(G)$ such that $\pi_l(f) = B$ for all $l \in C$ and $\pi_l(f) = 0$ if $l \in \mathfrak{g}_{T,gen}^* \setminus N$ or $l \in \mathfrak{g}^* \setminus \mathfrak{g}_{gen}^*$, provided the polarizations and coexponential bases are chosen as previously. This may be considered as a generalized Domar's property (see [D]).

7.6

The result of (7.5) may for instance be used to construct functions $f, g \in \mathcal{S}(G)$ such that $f * g = f$. In fact, take any $\xi, \eta \in \mathcal{S}(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathfrak{g}_{T,gen}^*)$ such that $\phi(l) \equiv 0$ outside a compact subset of $\mathfrak{g}_{T,gen}^*$. Let's define $F(l, x, y) := \phi(l)\xi(x)\overline{\eta(y)}$ and $G(l, x, y) := \frac{1}{\|\eta\|_2}\phi(l)\eta(x)\overline{\eta(y)}$. Let $f, g \in \mathcal{S}(G)$ be the functions given by (4.2) such that $\pi_l(f)$ and $\pi_l(g)$ have $F(l, \cdot, \cdot)$ and $G(l, \cdot, \cdot)$ as operator kernels respectively, for every $l \in \mathfrak{g}_{T,gen}^*$. As

$$\int_{\mathbb{R}^d} F(x, y)G(y, z)dy = F(x, z),$$

we have

$$\pi_l(f * g) = \pi_l(f)\pi_l(g) = \pi_l(f), \quad \forall l \in \mathfrak{g}_{T,gen}^*.$$

This implies that $f * g = f$.

7.7 Minimal ideals:

As in ([Lu2]), the construction of (7.6) allows us to prove the existence of minimal ideals of a given hull, provided this hull contains $\hat{G}_{sing} := \hat{G} \setminus \hat{G}_{gen}$, where \hat{G}_{sing} and \hat{G}_{gen} are defined in the following way: Let $\mathfrak{g}_{sing}^* := \mathfrak{g}^* \setminus \mathfrak{g}_{gen}^*$. Let $K : \mathfrak{g}^* \rightarrow \hat{G}$ be defined by $K(l) := [\pi_l]$ (the equivalence class of π_l) and $\hat{G}_{sing} := K(\mathfrak{g}_{sing}^*)$, $\hat{G}_{gen} := K(\mathfrak{g}_{gen}^*)$. Again, it is sufficient to work in the orbit section. We have the following result:

Theorem 7.8. *Let C be a closed subset of \hat{G} such that $\hat{G}_{sing} \subset C$. Let $m(C)$ be the set of all Schwartz functions $f \in \mathcal{S}(G)$ such that for all $l \in \mathfrak{g}_{gen,T}^* := \mathfrak{g}_{gen}^* \cap V_T$, the operator $\pi_l(f)$ has an operator kernel of the form $\Phi(l)\xi \otimes \bar{\eta}$, where $\Phi \in C_c^\infty(\mathfrak{g}_{gen,T}^*)$ with $\Phi|_{K^{-1}(C) \cap \mathfrak{g}_{gen,T}^*} \equiv 0$ and $0 \neq \xi, \eta \in \mathcal{S}(\mathbb{R}^d)$ ($d = \dim(\mathfrak{g}/\mathfrak{p}_l)$). Let $j(C)$ be the closed ideal of $L^1(G)$ generated by $m(C)$. Then $j(C)$ is the minimal closed ideal of $L^1(G)$ whose hull, in \hat{G} , is C , i. e. such that*

$$h(j(C)) := \{[\pi_l] \in \hat{G} \mid \pi_l(j(C)) \equiv 0\} = C.$$

Proof. By (4.2), $m(C) \neq \emptyset$. Let now $[\pi_l] \in C$ and $f \in m(C)$. If $l \in \mathfrak{g}_{gen,T}^* \cap K^{-1}(C)$, then $\pi_l(f)$ has an operator kernel of the form $\Phi(l)\xi \otimes \bar{\eta}$ and $\pi_l(f) = 0$, as $\Phi(l) = 0$. By conjugation, this remains true for every $l \in \mathfrak{g}_{gen}^* \cap K^{-1}(C)$. If $l \in \mathfrak{g}_{sing}^* \cap K^{-1}(C)$, then $\pi_l(f) = 0$, by (4.2). Hence $C \subset h(j(C))$.

Conversely, let $l_0 \in \mathfrak{g}^* \setminus K^{-1}(C)$. Then $l_0 \in \mathfrak{g}_{gen}^*$, by assumption on C . We may even assume that $l_0 \in \mathfrak{g}_{gen,T}^*$ (by replacing, if necessary, l_0 by the element of Ω_{l_0} in the orbit section). So we may choose $\Phi \in C_c^\infty(\mathfrak{g}_{gen}^* \cap V_T)$ such that $\Phi(l_0) \neq 0$ and $\Phi|_{K^{-1}(C) \cap \mathfrak{g}_{gen,T}^*} \equiv 0$, and $0 \neq \xi, \eta \in \mathcal{S}(\mathbb{R}^d)$ arbitrary. We construct f such that $\pi_l(f)$ has $\Phi(l)\xi \otimes \bar{\eta}$ as an operator kernel. Then $f \in m(C) \subset j(C)$, but $\pi_{l_0}(f) \neq 0$, as $\Phi(l_0) \neq 0$. So $l_0 \notin h(j(C))$ and $h(j(C)) = C$.

Let now $f \in m(C)$ be arbitrary. Let Φ, ξ, η be such that $\pi_l(f)$ has $\Phi(l)\xi \otimes \bar{\eta}$ as an operator kernel for all $l \in \mathfrak{g}_{gen}^* \cap V_T$. Let $\Psi \in C_c^\infty(\mathfrak{g}_{gen}^* \cap V_T)$ such that $\Psi \equiv 0$ on $K^{-1}(C) \cap \mathfrak{g}_{gen,T}^*$ and

$\Psi \equiv 1$ on $\text{supp}\Phi$. Let's put $G(l, x, y) := \frac{1}{\|\xi\|_2^2} \Psi(l)\xi(x)\overline{\xi(y)}$ and let $g \in \mathcal{S}(G)$ such that $\pi_l(g)$ has $G(l, \cdot, \cdot)$ as an operator kernel for all $l \in \mathfrak{g}_{gen}^* \cap V_T$. By construction,

$$\int_{G/P_l} G(l, x, y)F(l, y, z)dy = F(l, x, z), \quad \forall l \in \mathfrak{g}_{gen}^* \cap V_T,$$

i. e.

$$\pi_l(g) \circ \pi_l(f) = \pi_l(f) \quad \forall l \in \mathfrak{g}_{gen}^* \cap V_T,$$

and hence $g * f = f$. Moreover $C = h(j(C)) = h(m(C)) \subset h(\{g\})$. So, by a result of Ludwig ([Lu], Lemma 2), $j(C) \subset J$ for any closed two-sided ideal J of $L^1(G)$ such that $h(J) \subset C$. Hence $j(C)$ is minimal. \square

In particular, if $C = \hat{G}_{sing}$, then the minimal closed ideal whose hull in \hat{G} is \hat{G}_{sing} , is generated by the functions f obtained in the following way: There exists $\Phi \in \mathcal{C}_c^\infty(\mathfrak{g}_{gen}^* \cap V_T)$ (in particular $\Phi|_{\mathfrak{g}_{sing}^* \cap V_T} \equiv 0$), $0 \neq \xi, \eta, \in \mathcal{S}(\mathbb{R}^d)$ such that $\pi_l(f)$ has $\Phi(l)\xi \otimes \bar{\eta}$ as an operator kernel.

This result appears already in ([Lu2]), although there the Fourier inversion theorem has not yet been proven explicitly. In that sense, our result fills the gap in ([Lu2]). Another method, using functional calculus, gives the same result on the existence of minimal ideals of a given hull without the restriction $\hat{G}_{sing} \subset C$. But the retract method of this paper yields a better characterization of these minimal ideals.

7.9 Final remarks:

The results of (7.2, 7.6, 7.8) have already been obtained through the use of functional calculus (see for instance [Lu]). But the method of the Fourier inversion theorem seems more natural and elegant. Unfortunately it has one big disadvantage. The Fourier inversion theorem as presented in this paper is restricted to kernel functions whose support in l is contained in the set of generic elements \mathfrak{g}_{gen}^* (a Zariski open dense subset of \mathfrak{g}^*). It is still an open question how to extend the Fourier inversion theorem to all of \mathfrak{g}^* , as for singular elements of \mathfrak{g}^* , the orbit dimension is very often not maximal. For the results of (7.3) and (7.4), the Fourier inversion theorem is absolutely necessary. Moreover, the Fourier inversion theorem as presented in this paper is necessary in all the situation where one wants to make sure that the function obtained by the Plancherel formula is a Schwartz function (see introduction).

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