SEPARATION OF UNITARY REPRESENTATIONS OF CONNECTED LIE GROUPS BY THEIR MOMENT SETS

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Abstract. We show that every unitary representation π of a connected Lie group G is characterized up to quasi equivalence by its complete moment set.

Moreover, irreducible unitary representations π of G are characterized by their moment sets.

1. Introduction

Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and π a unitary representation of G on a separable Hilbert space $\mathcal{H}_\pi$. Note $\mathcal{H}_\pi^\infty$ the space of $\mathcal{C}^\infty$ vectors for π. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. In [Wi], N. Wildberger defined the moment map ψ_π of π as follows.

For all ξ of $\mathcal{H}_\pi^\infty \setminus \{0\}$, the element $\psi_\pi(\xi)$ in $\mathfrak{g}^*$ is defined by :

$$\psi_\pi(\xi)(X) := \frac{1}{i}\frac{\langle d\pi(X)\xi,\xi\rangle_{\mathcal{H}_\pi}}{\langle \xi,\xi\rangle_{\mathcal{H}_\pi}}, \quad X \in \mathfrak{g}. \tag{1.1}$$

The moment set I_π of the representation π is by definition the closure in $\mathfrak{g}^*$ of the image of the moment map:

$$\psi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \longrightarrow \mathfrak{g}^*.$$

Let us suppose π irreducible. Generally, the moment set is the closed convex hull of a co-adjoint orbit in $\mathfrak{g}^*$ (see [Wi] and [AL]). For instance, if G is an exponential Lie group, I_π is the convex hull of the co-adjoint orbit associated through the Kirillov map to π. Nevertheless, as shown in [Wi], the moment set does not characterize the representation even for nilpotent connected and simply connected Lie groups.

In [BLS], A. Baklouti, J. Ludwig and M. Selmi extended the moment map to the dual of the (complex) universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ as follows:

For all A in $\mathcal{U}(\mathfrak{g})$ and ξ in $\mathcal{H}_\pi^\infty \setminus \{0\}$,

$$\Psi_\pi(\xi)(A) := \mathfrak{Re}\left(\frac{1}{i}\frac{\langle d\pi(A)\xi,\xi\rangle_{\mathcal{H}_\pi}}{\langle \xi,\xi\rangle_{\mathcal{H}_\pi}}\right), \tag{1.2}$$

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and define the generalized moment set of π as the convex hull $J(\pi)$ of the image of this generalized moment map Ψ_π.

In [ABLS], it is shown that for all exponential Lie group G, the generalized moment set characterizes the unitary irreducible representations of G. More precisely, we have:

Theorem 1.1. *Let $G = \exp\mathfrak{g}$ be an exponential Lie group. Let π and ρ be two unitary irreducible representations of G. Then:*

$$\pi \simeq \rho \quad \text{if and only if} \quad J(\pi) = J(\rho).$$

Later on, L. Abdelmoula, D. Arnal and M. Selmi [AAS] extended this result for connected and simply connected type I solvable Lie group of the form $\mathbb{R} \ltimes \mathbb{R}^d$.

In this paper, we introduce the so-called complete moment set $\tilde{J}(\pi)$ (see Definition 2.2) and we show that $\tilde{J}(\pi)$ is the generalized moment set of the representation $\tilde{\pi} = \aleph_0\pi$. Moreover, if ρ and π are two unitary representations of G having the same generalized moment set ($J(\rho) = J(\pi)$), then their complete moment sets coincide.

Let us say that two unitary representations π and ρ of G are quasi equivalent if their multiples $\tilde{\pi}$ and $\tilde{\rho}$ are equivalent. Then, we show that the complete moment set $\tilde{J}(\pi)$ characterizes the unitary representation of a connected Lie group G up to quasi equivalence.

Theorem 1.2. *Let G be a connected Lie group. Let π and ρ be two unitary representations of G in separable Hilbert spaces $\mathcal{H}_\pi$, $\mathcal{H}_\rho$. Then the following are equivalent:*
i) π and ρ are quasi equivalent,
ii) $\tilde{J}(\pi) = \tilde{J}(\rho)$.

Finally, in the irreducible case, we get the following characterization:

Theorem 1.3. *Let G be a connected Lie group. Let π and ρ be two irreducible unitary representations of G. Then the following are equivalent:*
i) π and ρ are equivalent,
ii) $\tilde{J}(\pi) = \tilde{J}(\rho)$,
iii) $J(\pi) = J(\rho)$.

In these proofs we use the fact that the spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$ contain many analytic vectors. The property that $\tilde{J}(\pi) = \tilde{J}(\rho)$ allows us to write a coefficient function $\mathfrak{g}amma^\pi_\xi$ associated to π for an analytic vector ξ as a converging sum of coefficients $\sum_{i\in\mathbb{N}} \mathfrak{g}amma^\rho_{\eta_i}$ with analytic vectors η_i associated to $\mathcal{H}_\rho$. This then implies the existence of a non trivial intertwining operator between $\tilde{\pi}$ and $\tilde{\rho}$. The quasi equivalence of π and ρ follows. Moreover, if π and ρ are irreducible, we can directly define with this method an intertwining operator between $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$.

2. Generalized and complete Moment sets

In this paper, G is a connected Lie group and every unitary representation π of G is supposed to be carried by a separable Hilbert space $\mathcal{H}_\pi$. Let π such a representation of G.

Definition 2.1. *The generalized moment map Ψ_π of π associates to each C^∞ vector ξ for π the real linear form on the complex enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ defined by:*

$$\Psi_\pi(\xi)(A) := \mathfrak{Re}\left(\frac{1}{i}\frac{\langle d\pi(A)\xi,\xi\rangle_{\mathcal{H}_\pi}}{\langle \xi,\xi\rangle_{\mathcal{H}_\pi}}\right),$$

The generalized moment set $J(\pi)$ of π is then the convex hull of the image of Ψ_π in the real dual $\mathcal{U}(\mathfrak{g})^*_\mathbb{R}$ of $\mathcal{U}(\mathfrak{g})$.

In this paper we shall study a new moment set for π, called the complete moment set.

Definition 2.2. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of C^∞ vectors for π such that:*
1) $\sum_{n\in\mathbb{N}}\|\xi_n\|^2_{\mathcal{H}_\pi}=1$,
2) $\sum_{n\in\mathbb{N}}\|d\pi(A)\xi_n\|^2_{\mathcal{H}_\pi}<\infty$, for all A in $\mathcal{U}(\mathfrak{g})$.

The complete moment mapping $\tilde{\Psi}_\pi$ associates to the sequence (ξ_n) the $\mathbb{R}$-linear form:

$$\tilde{\Psi}_\pi\left((\xi_n)\right)(A) = \mathfrak{Re}\left(\frac{1}{i}\sum_{n\in\mathbb{N}}\langle d\pi(A)\xi_n,\xi_n\rangle_{\mathcal{H}_\pi}\right).$$

The complete moment set $\tilde{J}(\pi)$ of π is then the image of $\tilde{\Psi}_\pi$ in the real dual $\mathcal{U}(\mathfrak{g})^*_\mathbb{R}$ of $\mathcal{U}(\mathfrak{g})$.

The hypothesis *2)* in our definition guarantees the convergence of the series defining $\tilde{\Psi}_\pi\left((\xi)_n\right)(A)$. We shall prove the convexity of the complete moment set later on (see Proposition 3.2).

Now, if f is a $\mathbb{C}$-linear form on $\mathcal{U}(\mathfrak{g})$, and $g=\mathfrak{Re}(f)$ its real part, then

$$f(A)=\mathfrak{Re}(f)(A)-i\mathfrak{Re}(f)(iA)=g(A)-ig(iA).$$

Conversely, if g is a $\mathbb{R}$-linear form on $\mathcal{U}(\mathfrak{g})$, it defines with the above formula a unique $\mathbb{C}$-linear mapping f such that $\mathfrak{Re}(f)=g$.

Let us define the complex complete moment set $\tilde{J}(\pi)_\mathbb{C}$ of the representation $(\pi,\mathcal{H}_\pi)$ of G in the same way as we did for $\tilde{J}(\pi)$, except that the elements f of $\tilde{J}(\pi)_\mathbb{C}$ are now complex valued linear functionals for which there is a sequence $(\xi_n)\in\mathcal{H}_\pi^\infty$, such that $\sum_{n\in\mathbb{N}}\|\xi_n\|^2_{\mathcal{H}_\pi}=1$ and $\sum_{n\in\mathbb{N}}\|d\pi(A)\xi_n\|^2_{\mathcal{H}_\pi}<\infty$, $(A\in\mathcal{U}(\mathfrak{g}))$. This sequence determines f by:

$$f(A):=\tilde{\Psi}_{\pi,\mathbb{C}}((\xi_n))(A):=\sum_{n\in\mathbb{N}}\langle d\pi(A)\xi_n,\xi_n\rangle_{\mathcal{H}_\pi},\quad \forall A\in\mathcal{U}(\mathfrak{g}). \tag{2.1}$$

Obviously the equality of $\tilde{J}(\pi)$ and $\tilde{J}(\rho)$ is equivalent to $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$.

Let us also remark that:

Lemma 2.1. *Let π be a unitary representation of the connected Lie group G then*

$$J(\pi) \subset \tilde{J}(\pi) \subset \overline{J(\pi)}, \tag{2.2}$$

where $^{-}$ denotes the closure in $\mathcal{U}(\mathfrak{g})^*$ for the topology of pointwise convergence.

Proof. $J(\pi) \subset \tilde{J}(\pi)$:

Indeed if $\lambda_1, \dots, \lambda_k$ are positive numbers such that $\lambda_1 + \cdots + \lambda_k = 1$ and $\xi_1, \dots, \xi_n$ are C^∞ vectors with norm 1, we put $\xi'_n = \sqrt{\lambda_n}\xi_n$, for the (finite) sequence (ξ'_n), $\sum_{n=1}^k \|\xi'_n\|^2_{\mathcal{H}_\pi} = 1$ and:

$$\begin{aligned}\tilde{\Psi}_\pi\left((\xi'_n)\right)(A) &= \mathfrak{Re}\left(\frac{1}{i}\sum_{n=1}^k \langle d\pi(A)\xi'_n, \xi'_n\rangle_{\mathcal{H}_\pi}\right)\\ &= \mathfrak{Re}\left(\frac{1}{i}\sum_{n=1}^k \lambda_n\langle d\pi(A)\xi_n, \xi_n\rangle_{\mathcal{H}_\pi}\right) = \left(\sum_{n=1}^k \lambda_n \Psi_\pi(\xi_n)\right)(A).\end{aligned}$$

$\tilde{J}(\pi) \subset \overline{J(\pi)}$:

If f belongs to $\tilde{J}(\pi)$, then for all A in $\mathcal{U}(\mathfrak{g})$,

$$f(A) = \sum_{n=0}^\infty \mathfrak{Re}\left(\frac{1}{i}\langle d\pi(A)\xi_n, \xi_n\rangle_{\mathcal{H}_\pi}\right); \quad \left(\sum_{n=0}^\infty \|\xi_n\|^2_{\mathcal{H}_\pi} = 1\right).$$

For any N in $\mathbb{N}$, we put:

$$f_N(A) = \sum_{n=0}^N \mathfrak{Re}\left(\frac{1}{i}\frac{\|\xi_n\|^2_{\mathcal{H}_\pi}}{\sum_{k=0}^N \|\xi_k\|^2_{\mathcal{H}_\pi}}\frac{\langle d\pi(A)\xi_n, \xi_n\rangle_{\mathcal{H}_\pi}}{\|\xi_n\|^2_{\mathcal{H}_\pi}}\right).$$

Then f_N is in $J(\pi)$ and

$$f_N(A) = \frac{1}{\sum_{k=0}^N \|\xi_k\|^2_{\mathcal{H}_\pi}}\sum_{n=0}^N \mathfrak{Re}\left(\frac{1}{i}\langle d\pi(A)\xi_n, \xi_n\rangle_{\mathcal{H}_\pi}\right) \to f(A) \quad \text{as } N \text{ goes to } \infty.$$

Hence f is in $\overline{J(\pi)}$. □

Let us now prove:

Lemma 2.2. *Let π and ρ be two unitary representations of G. If $J(\pi) = J(\rho)$, then $\tilde{J}(\pi) = \tilde{J}(\rho)$.*

Proof. Let f be in $\tilde{J}(\pi)$ and (ξ_n) a (finite or infinite) sequence in $\mathcal{H}_\pi^\infty \setminus \{0\}$ such that:

$$\sum_n \|\xi_n\|^2_{\mathcal{H}_\pi} = 1, \quad f(A) = \sum_n \mathfrak{Re}\left(\frac{1}{i}\langle d\pi(A)\xi_n, \xi_n\rangle_{\mathcal{H}_\pi}\right), \quad A \in \mathcal{U}(\mathfrak{g}).$$

Let us put $\xi'_n = \frac{1}{\|\xi_n\|_{\mathcal{H}_\pi}}\xi_n$. The linear form f_n defined by

$$f_n(A) = \mathfrak{Re}\left(\frac{1}{i}\langle d\pi(A)\xi'_n, \xi'_n\rangle_{\mathcal{H}_\pi}\right)$$

is for any n in $J(\pi) = J(\rho)$. There exists a finite sequence $\eta'_{K_{n-1}+1}, \dots, \eta'_{K_n}$ of norm 1 vectors in $\mathcal{H}_\rho^\infty$ ($K_0 = 0$) and positive numbers $\mu_{K_{n-1}+1}, \dots, \mu_{K_n}$ such that:

$$\sum_{j=K_{n-1}+1}^{K_n} \mu_j = 1, \quad f_n(A) = \mathfrak{Re}\left(\frac{1}{i}\sum_{j=K_{n-1}+1}^{K_n} \mu_j\langle d\rho(A)\eta'_j, \eta'_j\rangle_{\mathcal{H}_\rho}\right).$$

Put $\eta_j = \|\xi_n\|_{\mathcal{H}_\pi}\sqrt{\mu_j}\eta'_j$, ($j = 0, 1, \dots$) we get:

$$\|\xi_n\|^2_{\mathcal{H}_\pi} f_n(A) = \mathfrak{Re}\left(\frac{1}{i}\langle d\pi(A)\xi_n, \xi_n\rangle_{\mathcal{H}_\pi}\right) = \mathfrak{Re}\left(\frac{1}{i}\sum_{j=K_{n-1}+1}^{K_n} \langle d\rho(A)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}\right)$$

and

$$\sum_{j=0}^{\infty} \|\eta_j\|^2_{\mathcal{H}_\rho} = 1, \quad f(A) = \sum_{j=0}^{\infty} \mathfrak{Re}\left(\frac{1}{i}\langle d\rho(A)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}\right).$$

This implies that for any $A \in \mathcal{U}(\mathfrak{g})$, if $A^+ \in \mathcal{U}(\mathfrak{g})$ is the formal adjoint of A,

$$\sum_{j=0}^{\infty} \|d\rho(A)\eta_j\|^2_{\mathcal{H}_\rho} = \sum_{j=0}^{\infty}\langle d\rho(A)\eta_j, d\rho(A)\eta_j\rangle_{\mathcal{H}_\rho} = f(iA^+A) < \infty.$$

This proves that $\tilde{J}(\pi) \subset \tilde{J}(\rho)$. The same argument gives the other inclusion. □

Let us remark that we can have $J(\pi) = J(\rho)$ or $\tilde{J}(\pi) = \tilde{J}(\rho)$ for two non equivalent unitary representations π and ρ. The simplest example is probably given by the trivial representation π and $\rho = \pi \oplus \pi$. For these two representations, all our moment sets coincide with $\{\varepsilon\}$ where ε is the augmentation map on $\mathcal{U}(\mathfrak{g})$: if $(X_1, \dots, X_n)$ is a basis of $\mathfrak{g}$,

$$\varepsilon(1) = 1, \quad \varepsilon(A) = 0, \quad \forall A \in Span\left(X_1^{i_1}\cdots X_n^{i_n},\ |i| = \sum_j i_j > 0\right).$$

3. Moment sets and multiple of a representation

Let π be a unitary representation of the connected Lie group G on a separable Hilbert space $\mathcal{H}_\pi$. We define now the representation $\tilde{\pi} = \aleph_0 \pi$ (see [D] for instance).

We consider a sequence $\mathcal{H}_n$ of Hilbert spaces unitary equivalent to $\mathcal{H}_\pi$ through a unitary transformation $\psi_n : \mathcal{H}_\pi \longrightarrow \mathcal{H}_n$. We define $\mathcal{H}_{\tilde{\pi}}$ as the Hilbert sum of the spaces $\mathcal{H}_n$

$$\mathcal{H}_{\tilde{\pi}} = \oplus_{n\in\mathbb{N}} \mathcal{H}_n = \left\{ \tilde{\xi} = (\xi_n),\ \xi_n \in \mathcal{H}_n,\ \|\tilde{\xi}\|^2_{\mathcal{H}_{\tilde{\pi}}} = \sum_n \|\xi_n\|^2_{\mathcal{H}_n} < \infty \right\}$$

and the representation $\tilde{\pi}$ by

$$\tilde{\pi}(g)\left(\sum_{n\in\mathbb{N}} \xi_n\right) = \sum_{n\in\mathbb{N}} \psi_n \circ \pi(g) \circ \psi_n^{-1}(\xi_n), g \in G.$$

It is easy to verify that $\tilde{\pi}$ is a continuous unitary representation of G.

Lemma 3.1. *Let π be a representation of G. Then $\tilde{\pi}$ and $\tilde{\tilde{\pi}}$ are unitary equivalent.*

The proof is an easy consequence of the well known relation $\aleph_0^2 = \aleph_0$.

Now, following [D], we define the quasi equivalence of representations.

Definition 3.1. *Two unitary representations π and ρ of the Lie group G are quasi equivalent (and we will write $\pi \sim \rho$) if the two representations $\tilde{\pi}$ and $\tilde{\rho}$ are equivalent.*

Proposition 3.2. *Let π be a unitary representation of the connected Lie group G on a separable Hilbert space $\mathcal{H}_\pi$. Then*

$$\tilde{J}(\pi) = J(\tilde{\pi}).$$

Proof. First if a vector $\tilde{\xi} = \sum_n \psi_n(\xi_n)$ is a C^∞ vector for $\tilde{\pi}$, since the projection $p_n(\tilde{\xi}) = \xi_n$ is a continuous intertwining operator, ξ_n is a C^∞ vector for π. Moreover, if A is in $\mathcal{U}(\mathfrak{g})$, we get:

$$d\tilde{\pi}(A)\tilde{\xi} = \sum_n \psi_n\left(d\pi(A)(\xi_n)\right) \quad \text{and} \quad \|d\tilde{\pi}(A)\tilde{\xi}\|^2_{\mathcal{H}_{\tilde{\pi}}} = \sum_n \|d\pi(A)\xi_n\|^2_{\mathcal{H}_\pi} < \infty.$$

Conversely, if (ξ_n) is a sequence of C^∞ vectors for π, such that, for any A in $\mathcal{U}(\mathfrak{g})$,

$$\sum_n \|d\pi(A)\xi_n\|^2_{\mathcal{H}_\pi} < \infty,$$

then the vector $\tilde{\xi} = \sum_n \psi_n(\xi_n)$ is a C^∞ vector for $\tilde{\pi}$.

Indeed, if X is in $\mathfrak{g}$, the vector $\tilde{\eta} = \sum_n \psi_n(d\pi(X)\xi_n)$ is well defined in $\tilde{\mathcal{H}}_\pi$ and we can compute the derivative in 0 of:

$$t \mapsto \tilde{\pi}(\exp tX)(\tilde{\xi}) = \sum_n \psi_n(\pi(\exp tX)\eta_n).$$

For any real numbers λ, t,

$$\left|\frac{e^{it\lambda}-1}{t} - i\lambda\right|^2 = \left(\frac{cos(t\lambda)-1}{t}\right)^2 + \left(\frac{sin(t\lambda)}{t} - \lambda\right)^2 \leq 2\lambda^2.$$

Thus if $d\pi(X) = \int_{\mathbb{R}} i\lambda \; dE_\lambda$ is the spectral decomposition of $d\pi(X)$, we can write for any n:

$$\begin{aligned}\left\|\frac{\pi(\exp tX)(\xi_n)-\xi_n}{t} - d\pi(X)\xi_n\right\|^2_{\mathcal{H}_\pi} &= \int_{\mathbb{R}} \left|\frac{e^{it\lambda}-1}{t} - i\lambda\right|^2 \; d\langle E_\lambda(\xi_n), \xi_n\rangle_{\mathcal{H}_\pi} \\ &\leq \int_{\mathbb{R}} 2\lambda^2 \; d\langle E_\lambda(\xi_n), \xi_n\rangle_{\mathcal{H}_\pi} = 2\left\|d\pi(X)\xi_n\right\|^2_{\mathcal{H}_\pi}.\end{aligned}$$

Then Lebesgue's dominated convergence theorem yields to:

$$\lim_{t\to 0}\left\|\frac{\tilde{\pi}(\exp tX)(\tilde{\xi})-\tilde{\xi}}{t} - \tilde{\eta}\right\|^2_{\mathcal{H}_{\tilde{\pi}}} = \lim_{t\to 0}\sum_n \left\|\frac{\pi(\exp tX)(\xi_n)-\xi_n}{t} - d\pi(X)\xi_n\right\|^2_{\mathcal{H}_\pi} = 0.$$

So $\tilde{\xi}$ is in the domain of $d\tilde{\pi}(X)$. By induction, we prove similarly, that it is in the domain of $d\tilde{\pi}(X_k \cdots X_1))$ for any X_i in $\mathfrak{g}$ and

$$d\tilde{\pi}(X_k \cdots X_1)(\tilde{\xi}) = \sum_n \psi_n(d\pi(X_k \cdots X_1)(\xi_n)).$$

Hence

$$\mathcal{H}^\infty_{\tilde{\pi}} = \left\{\tilde{\xi} = \sum_n \psi_n(\xi_n),\ \xi_n \in \mathcal{H}^\infty_\pi \text{ such that } \sum_n \left\|d\pi(A)\xi_n\right\|^2_{\mathcal{H}_\pi} < \infty,\ (A \in \mathcal{U}(\mathfrak{g}))\right\}.$$

Now if f is in $\tilde{J}(\pi)_{\mathbb{C}}$, by definition, there is a sequence (ξ_n) of C^∞ vectors for π such that $f = \tilde{\Psi}_{\pi,\mathbb{C}}((\xi_n))$. We saw that $\tilde{\xi} = \sum_n \psi_n(\xi_n)$ is a C^∞ vector for $\tilde{\pi}$, with norm 1 and

$$\Psi_{\tilde{\pi},\mathbb{C}}(\tilde{\xi})(A) = \langle d\tilde{\pi}(A)(\tilde{\xi}), \tilde{\xi}\rangle_{\mathcal{H}_{\tilde{\pi}}} = \sum_n \langle d\pi(A)(\xi_n), \xi_n\rangle_{\mathcal{H}_\pi} = \tilde{\Psi}_{\pi,\mathbb{C}}((\eta_n))(A).$$

Thus f belongs to $J(\tilde{\pi})_{\mathbb{C}}$.

Conversely, if f is in $J(\tilde{\pi})_{\mathbb{C}}$, there exists a finite sequence $\tilde{\xi}^0, \ldots, \tilde{\xi}^{k-1}$ of C^∞ vectors for $\tilde{\pi}$ with norm 1, positive real numbers λ^j such that $\sum_{j=0}^{k-1} \lambda^j = 1$ and

$$f(A) = \sum_{j=0}^{k-1} \lambda^j \langle d\tilde{\pi}(A)\tilde{\xi}^j, \tilde{\xi}^j\rangle_{\mathcal{H}_{\tilde{\pi}}}.$$

Now, each vector $\tilde{\xi}^j$ can be written as $\sum_n \psi_n(\xi_n^j)$ with $\sum_n \|\xi_n^j\|^2_{\mathcal{H}_\pi} = 1$ and $\sum_n \|d\pi(A)\xi_n^j\|^2_{\mathcal{H}_\pi} < \infty$. Let us thus put: $\eta_{nk+r} = \sqrt{\lambda^r}\eta_n^r$ $(n = 0, 1, \dots$ and $0 \le r < k)$. Then:

$$\sum_{p=0}^{\infty} \|\eta_p\|^2_{\mathcal{H}_\pi} = \sum_{j=0}^{k-1} \lambda^j \sum_{n=0}^{\infty} \|\xi_n^j\|^2_{\mathcal{H}_\pi} = 1$$

and

$$f(A) = \sum_{j=0}^{k-1} \lambda^j \langle d\tilde{\pi}(A)\tilde{\xi}^j, \tilde{\xi}^j\rangle_{\mathcal{H}_{\tilde{\pi}}} = \sum_{p=0}^{\infty} \frac{1}{i}\langle d\pi(A)\eta_p, \eta_p\rangle_{\mathcal{H}_\pi},$$

and f belongs to $\tilde{J}(\pi)_{\mathbb{C}}$. □

4. Analytic vectors

Let us now recall briefly some well known facts about analytic vectors for a unitary representation π of a Lie group G.

The reference for analytic vectors is [Wa]. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of the connected Lie group G. We say that a vector $\xi \in \mathcal{H}_\pi$ is analytic if the mapping

$$a_\xi : G \longrightarrow \mathcal{H}_\pi; \qquad g \mapsto \pi(g)\xi$$

is analytic. Let $\mathcal{H}_\pi^\omega$ denote the (linear subspace) of analytic vectors of $\mathcal{H}_\pi$. By a result of Nelson (see [N]), the analytic vectors are dense in $\mathcal{H}_\pi$.

For any $\xi \in \mathcal{H}_\pi^\omega$, there exists a neighborhood U of 0 in the Lie algebra $\mathfrak{g}$ of G, such that for any $X \in U$,

$$\pi(\exp X)\xi = \sum_{m=0}^{\infty} \frac{1}{m!} d\pi(X)^m \xi. \tag{4.1}$$

More precisely, if $\{X_1, \cdots, X_p\}$ is a basis of $\mathfrak{g}$, then we have the following necessary and sufficient condition for a C^∞-vector ξ in $\mathcal{H}_\pi$ to be analytic. There exists a constant $M = M(\xi) > 0$, such that

$$\|(d\pi(X_{i_1}) \cdots d\pi(X_{i_m}))\xi\|_{\mathcal{H}_\pi} \le M^m m! \quad \text{for all indices } i_1, \cdots, i_m,\ m \in \mathbb{N}. \tag{4.2}$$

This implies that if $\xi \in \mathcal{H}_\pi$ is analytic, then also the vector $\pi(g)\xi$ is analytic, since

$$\|(d\pi(X_{i_1}) \cdots d\pi(X_{i_m}))\pi(g)\xi\|_{\mathcal{H}_\pi} = \|(d\pi(\mathrm{Ad}(g^{-1})X_{i_1}) \cdots d\pi(\mathrm{Ad}(g^{-1})X_{i_m}))\xi\|_{\mathcal{H}_\pi}.$$

5. Proof of theorem 1.2

First, if π and ρ are quasi equivalent, $\tilde{\pi}$ and $\tilde{\rho}$ are unitarily equivalent, they thus have the same generalized moment set and by proposition 3.2:

$$\tilde{J}(\pi) = J(\tilde{\pi}) = J(\tilde{\rho}) = \tilde{J}(\rho).$$

Conversely let us suppose that $\tilde{J}(\pi) = \tilde{J}(\rho)$, which implies that $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$.

We first prove:

Lemma 5.1. *Suppose that π and ρ are two unitary representations of the connected Lie group G such that $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$. Let ξ be an analytic vector for the representation π with norm 1. Then there exists a sequence of analytic vectors $(\eta_n)_{n\in\mathbb{N}}$ for ρ such that for any g in G,*

$$\langle \pi(g)\xi, \xi\rangle_{\mathcal{H}_\pi} = \sum_n \langle \rho(g)\eta_n, \eta_n\rangle_{\mathcal{H}_\rho}.$$

(Let us remark that if $g = 1$ in G, the relation means $\sum_n \|\eta_n\|^2_{\mathcal{H}_\rho} = 1$ and thus the series converges absolutely for any g in G).

Proof. Let us choose an analytic vector ξ in the space $\mathcal{H}_\pi$, with norm 1. Then the complex linear functional

$$f(A) := \langle d\pi(A)\xi, \xi\rangle_{\mathcal{H}_\pi}, \quad (A \in \mathcal{U}(\mathfrak{g})),$$

is an element of $\tilde{J}(\pi)_{\mathbb{C}}$. Hence, since $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$, there exists a sequence of C^∞-vectors $(\eta_j)_j$ in $\mathcal{H}^\infty_\rho$, such that

$$\sum_{j\in\mathbb{N}} \|\eta_j\|^2_{\mathcal{H}_\rho} = 1 \quad \text{and} \quad f(A) = \sum_{j\in\mathbb{N}} \langle d\rho(A)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}, \quad \forall A \in \mathcal{U}(\mathfrak{g}). \tag{5.1}$$

¿From this, if we take m in $\mathbb{N}$ and $A = X_{i_1}\cdots X_{i_m}$ as in 4.2, it follows that

$$\begin{aligned}(M^m m!)^2 \quad \mathfrak{g}eq \quad & \|d\pi(A)\xi\|^2_{\mathcal{H}_\pi} = \langle d\pi(A^+A)\xi, \xi\rangle_{\mathcal{H}_\pi} \\ = \quad & f(A^+A) = \sum_{j\in\mathbb{N}} \langle d\rho(A^+A)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho} = \sum_{j\in\mathbb{N}} \|d\rho(A)\eta_j\|^2_{\mathcal{H}_\rho}.\end{aligned}$$

Hence for every $j \in \mathbb{N}$, we have

$$\|d\rho(X_{i_1}\cdots X_{i_m})\eta_j\|_{\mathcal{H}_\rho} \leq M^m m! \tag{5.2}$$

and so η_j is an analytic vector for every $j \in \mathbb{N}$, moreover the estimation (5.2) is independent of j.

Fix now $Y = \sum_{j=1}^p y_j X_j$ in $\mathfrak{g}$, define the norm of Y by:

$$\|Y\| := \sup_{j\in\{1,\cdots,p\}} |y_j|.$$

It follows from 4.2 that for $Y \in \mathfrak{g}$ and an analytic vector ξ in $\mathcal{H}_\pi$ we have that

$$|d\pi(Y)^m \xi\|_{\mathcal{H}_\pi} \leq p^m M(\xi)^m \|Y\|^m, \quad m \in \mathbb{N}. \tag{5.3}$$

Define the mapping $\varphi : \mathbb{R} \longrightarrow \mathcal{H}_\pi$ by

$$\varphi(t) := \pi(\exp tY)\xi.$$

Then:

$$\varphi(t) = \sum_{k=0}^{n} \frac{t^k}{k!} d\pi(Y^k)\xi + R_n(t),$$

where

$$R_n(t) = \frac{1}{n!} \int_0^t (t-s)^n d\pi(Y)^{n+1} \varphi(s) ds$$

and so by by 5.3, for $|t| < (p\|Y\| M(\xi))^{-1}$,

$$\|R_n(t)\|_{\mathcal{H}_\pi} \leq \frac{1}{n!} \left| \int_0^t (t-s)^n \|d\pi(Y^{n+1})\varphi(s)\|_{\mathcal{H}_\pi} \, ds \right| \leq \|Y\|^{n+1} p^{n+1} M(\xi)^{n+1} |t|^{n+1} \to 0$$

as n tends to ∞.

Hence for all small t, the series $\sum_{k=0}^\infty \frac{1}{k!} \langle d\pi(tY)^k \xi, \xi \rangle_{\mathcal{H}_\pi}$ converges to $\langle \pi(\exp tY)\xi, \xi\rangle_{\mathcal{H}_\pi}$.

Furthermore, by 5.2, for every Y in the open ball $U := \left\{ Z \in \mathfrak{g}, \ \|Z\| < \frac{1}{pM(\xi)} \right\}$,

$$\rho(\exp Y)\eta_j = \sum_{k=0}^\infty \frac{1}{k!} d\rho(Y)^k \eta_j \tag{5.4}$$

converges absolutely and uniformly in j.

We also have absolute convergence for the double series $\sum_{k=0}^\infty \sum_j \frac{t^k}{k!} \langle d\rho(Y^k)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}$. Indeed, by 5.1 and 5.2, for Y in U and $|t| < 1$,

$$\sum_{k=0}^\infty \sum_{j=0}^\infty \frac{|t^k|}{k!} |\langle d\rho(Y^k)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}| \leq$$

$$\sum_{k=0}^\infty \frac{|t^k|}{k!} \left(\sum_{n=0}^\infty \|d\rho(Y^k)\eta_j\|^2_{\mathcal{H}_\rho} \right)^{\frac{1}{2}} \left(\sum_{j=0}^\infty \|\eta_j\|^2_{\mathcal{H}_\rho} \right)^{\frac{1}{2}} = \sum_{k=0}^\infty \frac{|t^k|}{k!} \|d\pi(Y^k)\xi\|_{\mathcal{H}_\pi} < \infty.$$

Thus,

$$\begin{aligned} \sum_{j=0}^\infty \langle \rho(\exp tY)\eta_n, \eta_n\rangle_{\mathcal{H}_\rho} &= \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{t^k}{k!} \langle d\rho(Y^k)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho} \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} \langle d\pi(Y^k)\xi, \xi\rangle_{\mathcal{H}_\pi} = \langle \pi(\exp tY)\xi, \xi\rangle_{\mathcal{H}_\pi} \end{aligned}$$

Let us define now the subset $\mathcal{U}$ of G by,

$$\mathcal{U} = \{g \in G, \text{ such that } \langle d\pi(A)\pi(g)\xi, \xi\rangle_{\mathcal{H}_\pi} = \sum_{j=0}^{\infty} \langle d\rho(A)\rho(g)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}, \ \forall A \in \mathcal{U}(\mathfrak{g}))\}.$$

We shall now prove that $\mathcal{U} = G$. First, we show that $\mathcal{U}$ is a closed subset of G. Let $(g_n)_n$ be a sequence of $\mathcal{U}$, which converges to g. Then, we have for all A in $\mathcal{U}(\mathfrak{g})$:

$$\begin{aligned}\langle d\pi(A)\pi(g)\xi, \xi\rangle_{\mathcal{H}_\pi} &= \langle \pi(g)\xi, d\pi(A^+)\xi\rangle_{\mathcal{H}_\pi} \\ &= \lim_{n\to\infty} \langle \pi(g_n)\xi, d\pi(A^+)\xi\rangle_{\mathcal{H}_\pi} \\ &= \lim_{n\to\infty} \sum_{j=0}^{\infty} \langle \rho(g_n)\eta_j, d\rho(A^+)\eta_j\rangle_{\mathcal{H}_\rho}.\end{aligned}$$

But:

$$\begin{aligned}\sum_{j=0}^{\infty} |\langle \rho(g_n)\eta_j, d\rho(A^+)\eta_j\rangle_{\mathcal{H}_\rho}| &\leq \sum_{j=0}^{\infty} \|\rho(g_n)\eta_j\|_{\mathcal{H}_\rho} \|d\rho(A^+)\eta_j\|_{\mathcal{H}_\rho} \\ &\leq \left(\sum_j \|\rho(g_n)\eta_j\|^2_{\mathcal{H}_\rho}\right)^{\frac{1}{2}} \left(\sum_j \|d\rho(A^+)\eta_j\|^2_{\mathcal{H}_\rho}\right)^{\frac{1}{2}} \\ &= \left(\sum_j \|\eta_j\|^2_{\mathcal{H}_\rho}\right)^{\frac{1}{2}} \|d\pi(A^+)\xi\|_{\mathcal{H}_\pi} = \|d\pi(A^+)\xi\|_{\mathcal{H}_\pi} < \infty\end{aligned}$$

Thus, by Lebesgue's theorem of dominated convergence,

$$\langle d\pi(A)\pi(g)\xi, \xi\rangle_{\mathcal{H}_\pi} = \sum_j \langle d\rho(A)\rho(g)\eta_j, \eta_j\rangle_{\mathcal{H}_\rho}.$$

Hence $\mathcal{U}$ is closed.

Let us show that $\mathcal{U}$ is open. Let g be in $\mathcal{U}$ and A in $\mathcal{U}(\mathfrak{g})$. Hence, by 5.2, for any vector $Z = \sum_i z_i X_i$ in $\mathfrak{g}$ such that $\|Z\| < 1$, we have:

$$\|d\pi(Z)^k\xi\|_{\mathcal{H}_\pi} \leq (pM(\xi))^k k!, \quad k \in \mathbb{N}, \tag{5.5}$$

and so for all these Z's and $|t| < (pM(\xi))^{-1}$, we obtain

$$\begin{aligned} I &:= \sum_{k=0}^{\infty}\sum_j \frac{|t^k|}{k!}|\langle d\rho(AZ^k)\rho(g)\eta_j,\eta_j\rangle_{\mathcal{H}_\rho}| \\ &= \sum_{k=0}^{\infty}\sum_j \frac{|t^k|}{k!}|\langle d\rho(Z^k)\rho(g)\eta_j, d\rho(A^+)\eta_j\rangle_{\mathcal{H}_\rho}| \\ &\leq \sum_{k=0}^{\infty}\frac{|t^k|}{k!}\left(\sum_j \|d\rho(Z^k)\rho(g)\eta_j\|^2_{\mathcal{H}_\rho}\right)^{\frac{1}{2}}\left(\sum_j \|d\rho(A^+)\eta_j\|^2_{\mathcal{H}_\rho}\right)^{\frac{1}{2}} \\ &\leq \sum_{k=0}^{\infty}\frac{|t^k|}{k!}\|d\pi(Z^k)\pi(g)\xi\|_{\mathcal{H}_\pi}\|d\pi(A^+)\xi\|_{\mathcal{H}_\pi} < \infty \end{aligned}$$

This shows that for our Z, for $|t| < (pM(\xi))^{-1}$, since g is in $\mathcal{U}$, we have by 5.4:

$$\begin{aligned} \langle d\pi(A)\pi(\exp tZ)\pi(g)\xi,\xi\rangle_{\mathcal{H}_\pi} &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\langle d\pi(AZ^k)\pi(g)\xi,\xi\rangle_{\mathcal{H}_\pi} \\ &= \sum_{k=0}^{\infty}\sum_j \frac{t^k}{k!}\langle d\rho(AZ^k)\rho(g)\eta_j,\eta_j\rangle_{\mathcal{H}_\rho}. \\ &= \sum_j \langle d\rho(A)\rho(\exp tZ)\rho(g)\eta_j,\eta_j\rangle_{\mathcal{H}_\rho}. \end{aligned}$$

or

$$\langle d\pi(A)\pi(\exp tZg)\xi,\xi\rangle_{\mathcal{H}_\pi} = \sum_j \langle d\rho(A)\rho(\exp tZg)\eta_j,\eta_j\rangle_{\mathcal{H}_\rho}.$$

This shows that $\mathcal{U}$ is an open set. Hence, since G is connected, $G = \mathcal{U}$. Especially, for every g in G, we have:

$$\langle \pi(g)\xi,\xi\rangle_{\mathcal{H}_\pi} = \sum_j \langle \rho(g)\eta_j,\eta_j\rangle_{\mathcal{H}_\rho},$$

the series converging absolutely. □

Let us present now the proof of Theorem 1.2

Let us consider the set $\mathcal{V}$ of all the families $V = (\xi_i)_{i\in I}$ of analytic vectors for π, with norm 1 and such that, if $i \neq j$, then $\langle \pi(g)\xi_i,\xi_j\rangle_{\mathcal{H}_\pi} = 0$ for any g in G. On $\mathcal{V}$, we consider the ordering given by inclusion. We get an inductive set $(\mathcal{V},\subset)$. By Zorn's lemma, there is a family W in $\mathcal{V}$ which is maximal.

Since the vectors ξ_i in W form an orthogonal system in the separable space $\mathcal{H}_\pi$, W is finite or countable. We suppose now the vectors in W labelled by integers $W = (\xi_0,\xi_1,\dots)$.

For each n in $\mathbb{N}$, we note $\mathcal{K}_n$ the closure of the vector space spanned by all the $\pi(g)\xi_n$ (g in G). The spaces $\mathcal{K}_n$ are mutually orthogonal subspaces of $\mathcal{H}_\pi$, since, if $n \neq p$,

$$\left\langle \sum_{i=1}^{k} \lambda_i \pi(g_i)\xi_n, \sum_{\ell=1}^{q} \mu_\ell \pi(g'_\ell)\xi_p \right\rangle_{\mathcal{H}_\pi} = \sum_{i,\ell} \overline{\lambda_i}\mu_\ell \left\langle \pi((g'_\ell)^{-1}g_i)\xi_n, \xi_p \right\rangle_{\mathcal{H}_\pi} = 0.$$

We now consider the Hilbert sum $\mathcal{K}$ of the $\mathcal{K}_n$ as a closed subspace of $\mathcal{H}_\pi$. In fact this subspace coincides with $\mathcal{H}_\pi$. Indeed, each $\mathcal{K}_n$ being a G-invariant subspace of $\mathcal{H}_\pi$, $\mathcal{K}$ itself is invariant. If $\mathcal{K} \neq \mathcal{H}_\pi$, the non-trivial invariant subspace $\mathcal{K}^\perp$ contains an analytic vector η with norm 1, $W \cup \{\eta\}$ is in $\mathcal{V}$ and larger than W. This is impossible and so $\oplus_n \mathcal{K}_n = \mathcal{H}_\pi$.

For each ξ_n, according to Lemma 5.1, there exist analytic vectors $\eta_{k,n}$ for ρ such that:

$$\langle \pi(g)\xi_n, \xi_n \rangle_{\mathcal{H}_\pi} = \sum_{k=0}^{\infty} \langle \rho(g)\eta_{n,k}, \eta_{n,k} \rangle_{\mathcal{H}_\rho} \quad (g \in G).$$

Now we can build an intertwining operator T_n between $\pi|_{\mathcal{K}_n}$ and $\tilde{\rho}$.

First, we note $\mathcal{H}_{\tilde{\rho}} = \oplus_k \theta_k(\mathcal{H}_\rho)$ and define T_n on linear combinations $\sum_{i=1}^{\ell} \lambda_i \pi(g_i)\xi_n$ by:

$$T_n\left(\sum_{i=1}^{\ell} \lambda_i \pi(g_i)\xi_n\right) = \sum_{k=0}^{\infty} \sum_{i=1}^{\ell} \lambda_i \theta_k(\rho(g_i)\eta_{n,k}).$$

For the moment, T_n is not well defined with this formula, but we have:

$$\begin{aligned}
\left\| \sum_{k=0}^{\infty} \sum_{i=1}^{\ell} \theta_k(\rho(g_i)\eta_{n,k}) \right\|_{\mathcal{H}_\rho}^2 &= \sum_k \langle \sum_i \lambda_i \rho(g_i)\eta_{n,k}, \sum_j \lambda_j \rho(g_j)\eta_{n,k} \rangle_{\mathcal{H}_\rho} \\
&= \sum_k \sum_{i,j} \overline{\lambda_i}\lambda_j \langle \rho(g_j^{-1}g_i)\eta_{n,k}, \eta_{n,k} \rangle_{\mathcal{H}_\rho} \\
&= \sum_{i,j} \overline{\lambda_i}\lambda_j \sum_k \langle \rho(g_j^{-1}g_i)\eta_{n,k}, \eta_{n,k} \rangle_{\mathcal{H}_\rho} \\
&= \sum_{i,j} \overline{\lambda_i}\lambda_j \langle \pi(g_j^{-1}g_i)\xi_n, \xi_n \rangle_{\mathcal{H}_\pi} \\
&= \left\| \sum_i \lambda_i \pi(g_i)\xi_n \right\|_{\mathcal{H}_\pi}^2 .
\end{aligned}$$

Note that the permutation of the sums on k and i, j are allowed since all the series in k are absolutely convergent.

This relation proves that T_n is well defined since if $\sum_i \lambda_i \pi(g_i)\xi_n = 0$, then

$$T_n\left(\sum_i \lambda_i \pi(g_i)\xi_n\right) = 0,$$

moreover T_n is isometric. Thus T_n can be uniquely extended to $\mathcal{K}_n$ by continuity.

By definition, T_n is an intertwining operator between $\pi|_{\mathcal{K}_n}$ and $\tilde{\rho}$:

$$\begin{aligned}(T_n \circ \pi(g))\left(\sum_i \lambda_i \pi(g_i)\xi_n\right) &= T_n\left(\sum_i \lambda_i \pi(gg_i)\xi_n\right)\\ &= \sum_k \sum_i \lambda_i \theta_k(\rho(gg_i)\eta_{n,k})\\ &= (\tilde{\rho}(g) \circ T_n)\left(\sum_i \lambda_i \pi(g_i)\xi_n\right).\end{aligned}$$

Consider now the mapping $T = \oplus_n T_n$. By construction, it is a partial isometry between $\mathcal{H}_\pi$ and $\mathcal{H}_{\tilde{\tilde{\rho}}}$. With this construction, we get an intertwining operator between π and $\tilde{\tilde{\rho}}$. But Lemma 3.1 gives us a unitary intertwining U between $\tilde{\tilde{\rho}}$ and $\tilde{\rho}$ and $S = U \circ T$ is a partial isometry intertwining π and $\tilde{\rho}$.

Now we can define $\tilde{S} : \mathcal{H}_{\tilde{\pi}} = \oplus \psi_n(\mathcal{H}_\pi) \longrightarrow \mathcal{H}_{\tilde{\tilde{\rho}}} = \oplus \tilde{\theta}_n(\mathcal{H}_{\tilde{\rho}})$ by:

$$\tilde{S}\left(\sum_{n=0}^{\infty} \psi_n(v_n)\right) = \sum_{n=0}^{\infty} \tilde{\theta}_n\left(S(v_n)\right), v_n \in \mathcal{H}_\pi, n \in \mathbb{N}.$$

Finally the partial isometry $\tilde{V} = U \circ \tilde{S}$ intertwines $\tilde{\pi}$ and $\tilde{\rho}$.

Exchanging the roles of π and ρ yields to a partial isometry intertwining $\tilde{\rho}$ and $\tilde{\pi}$.

With the notations of [D], we thus have $\tilde{\pi} \leq \tilde{\rho}$ and $\tilde{\rho} \leq \tilde{\pi}$. Hence, with Corollary 5.1.5 of [D], we conclude that $\tilde{\pi}$ and $\tilde{\rho}$ are unitarily equivalent.□

6. Proof of theorem 1.3

We are now going to prove theorem 1.3. Let π and ρ be two irreducible unitary representations of the connected Lie group G. It is clear that if π and ρ are unitarily equivalent ($\pi \simeq \rho$), then every coefficient of π is also a coefficient of ρ and so $J(\pi)$ and $J(\rho)$ coincide, we saw that $J(\pi) = J(\rho)$ implies $\tilde{J}(\pi) = \tilde{J}(\rho)$ and $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$.

Hence it suffices to show that $\tilde{J}(\pi)_{\mathbb{C}} = \tilde{J}(\rho)_{\mathbb{C}}$ implies that π and ρ are equivalent.

We use Lemma 5.1. Let ξ be an analytic vector for π with norm 1. There is a sequence (η_n) of analytic vectors for ρ such that

$$\langle \pi(g)\xi, \xi\rangle_{\mathcal{H}_\pi} = \sum_n \langle \rho(g)\eta_n, \eta_n\rangle_{\mathcal{H}_\rho} \quad (g \in G).$$

¿From now one, we suppose, without loss of generality, that $\eta_0 \neq 0$.

We saw that $\tilde{\rho}$ is a unitary representation of G and the linear mapping P from $\mathcal{H}_{\tilde{\rho}}$ to $\mathcal{H}_\rho$, defined by

$$P\left(\sum_n \theta_n(\zeta_n)\right) = \zeta_0$$

is a non trivial intertwining operator between $\tilde{\rho}$ and ρ:

$$(P \circ \tilde{\rho}(g))\left(\sum_n \theta_n(\zeta_n)\right) = \rho(g)\zeta_0 = (\rho(g) \circ P)(g)\left(\sum_n \theta_n(\zeta_n)\right).$$

Now as in the proof of theorem 1.2, there is an isometric intertwining operator T between π and $\tilde{\rho}$ defined by:

$$T\left(\sum_{i=1}^k \lambda_i \pi(g_i)\xi\right) = \sum_{n=0}^\infty \sum_{i=1}^k \theta_n(\rho(g_i)\eta_n).$$

The representation π being irreducible and the linear space V generated by the vectors $\pi(g)\xi$ invariant, this space is dense in $\mathcal{H}_\pi$. Thus T can be uniquely extended to $\mathcal{H}_\pi$ by continuity. By definition, T is an intertwining between π and $\tilde{\rho}$, since for vectors in V,

$$\begin{aligned}(T \circ \pi(g))\left(\sum_i \lambda_i \pi(g_i)\xi\right) &= T\left(\sum_i \lambda_i \pi(gg_i)\Phi\right)\\ &= \sum_n \sum_i \lambda_i \theta_n(\rho(gg_i)\eta_n)\\ &= (\tilde{\rho}(g) \circ T)\left(\sum_i \lambda_i \pi(g_i)\xi\right).\end{aligned}$$

Now $A = P \circ T$ is an intertwining operator between π and ρ, since $A\xi = \eta_0 \neq 0$, it is non trivial. Hence by [Wa] Corollary 4.3.1.3, the irreducible representations π and ρ are unitarily equivalent.

This proves our result.

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