Degenerate parabolic operators of Kolmogorov type
with a geometric control condition

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We consider Kolmogorov-type equations on a rectangle domain \((x,v) \in \Omega = \mathbb{T} \times (-1,1)\), that combine diffusion in variable \(v\) and transport in variable \(x\) at speed \(v^n, \gamma \in \mathbb{N}^*\), with Dirichlet boundary conditions in \(v\). We study the null controllability of this equation with a distributed control as source term, localized on a subset \(\omega \subset \Omega\).

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\begin{align*}
\left\{ \begin{array}{l}
\left( \partial_t - v^n \partial_x - \partial_x^2 \right) f(t,x,v) = u(t,x,v)1_\omega(x,v), \\
f(t,x,\pm 1) = 0,
\end{array} \right. & \quad (t,x,v) \in (0,T) \times \mathbb{T} \times (-1,1), \\
& \quad (t,x) \in (0,T) \times \mathbb{T}.
\end{align*}
\]

Thanks to the interplay between diffusion in \(v\) and transport in \(x\), the equation diffuses both in variables \(v\) and \(x\) (contrarily to equation \((\partial_t - \partial_x^2)g(t,x,v) = 0\) but, in a weaker way than the 2D heat equation (i.e. \((\partial_t - \partial_x^2 - \partial_x^2)g(t,x,v) = 0\)). Thus, natural questions are the following ones.

**Question 1:** Is the diffusion in variable \(v\) strong enough for observability to hold when the control acts on a horizontal strip \(\omega = \mathbb{T} \times (a,b)\) with \(0 < a < b < 1\), whatever \(\gamma \in \mathbb{N}^*\) is? (i.e. as for equation \((\partial_t - \partial_x^2)g = 0, (t,x,v) \in (0,T) \times \mathbb{T} \times (-1,1)\))

**Question 2:** Is the diffusion in variable \(x\) sufficient for null controllability to hold when the control acts on a vertical strip \(\omega = \omega_1 \times (-1,1)\) where \(\omega_1 \subset \subset \mathbb{T}\)? (i.e. as for the 2D heat equation)

When the control acts on a horizontal strip \(\omega = \mathbb{T} \times (a,b)\) with \(0 < a < b < 1\), then the system is null controllable in any time \(T > 0\) when \(\gamma = 1\), and only in large time \(T > T_{\min} > 0\) when \(\gamma = 2\): a finite speed of propagation occurs (see [1]). When \(\gamma > 3\), the system is not null controllable (whatever \(T\) is) in this configuration, even if unique continuation holds (see [2]). Thus, the first order term \(v^n \partial_x\) weakens strongly the diffusion in variable \(v\) when \(\gamma \geq 3\). These results answer **Question 1**.

When the control acts on a vertical strip \(\omega = \omega_1 \times (-1,1)\) with \(\overline{\omega_1} \subset \mathbb{T}\), we investigate the null controllability on a toy model, where \((\partial_x, x \in \mathbb{T})\) is replaced by \((-\Delta)^{1/2}, x \in \Omega_1)\), and \(\Omega_1\) is an open subset of \(\mathbb{R}^N\). As the original system, this toy model satisfies the controllability properties listed above. We prove that, for \(\gamma = 1, 2\) and for appropriate domains \((\Omega_1, \omega_1)\), then null controllability does not hold (whatever \(T > 0\) is), when the control acts on a vertical strip \(\omega = \omega_1 \times (-1,1)\) with \(\overline{\omega_1} \subset \Omega_1\) (see [2]). Thus, a geometric control condition is required for the null controllability of this toy model. It indicates that a geometric control condition may be necessary for the original model too. This is a conjecture about the answer of **Question 2**.

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References
